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# $\mathfrak{K}$-FAMILIES AND CPD-H-EXTENDABLE FAMILIES

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ABSTRACT. We introduce, for any set $S$, the concept of a $\mathfrak{K}$-family between two Hilbert $C^*$-modules over two $C^*$-algebras, for a given completely positive definite (CPD-) kernel $\mathfrak{K}$ over $S$ between those $C^*$-algebras, and we obtain a factorization theorem for such $\mathfrak{K}$-families. If $\mathfrak{K}$ is a CPD-kernel and $E$ is a full Hilbert $C^*$-module, then any $\mathfrak{K}$-family which is covariant with respect to a dynamical system $(G, \eta, E)$ on $E$, extends to a $\widetilde{\mathfrak{K}}$-family on the crossed product $E \times_\eta G$, where $\widetilde{\mathfrak{K}}$ is a CPD-kernel. Several characterizations of $\mathfrak{K}$-families, under the assumption that $E$ is full, are obtained, and covariant versions of these results are also given. One of these characterizations says that such $\mathfrak{K}$-families extend as CPD-kernels, between associated (extended) linking algebras, whose $(2,2)$-corner is a homomorphism and vice versa. We discuss a dilation theory of CPD-kernels in relation to $\mathfrak{K}$-families.

**1. Introduction.** Let $\mathcal{B}$ be a $C^*$-algebra and $E$ a vector space which is a right $\mathcal{B}$-module satisfying $\alpha(xb) = (\alpha x)b = x(\alpha b)$ for $x \in E$, $b \in \mathcal{B}$, $\alpha \in \mathbb{C}$. The space $E$ is called an *inner-product $\mathcal{B}$-module* if there exists a mapping $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{B}$ such that

(i) $\langle x, x \rangle \geq 0$ for $x \in E$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
(ii) $\langle x, yb \rangle = \langle x, y \rangle b$ for $x, y \in E$ and for $b \in \mathcal{B}$,
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for $x, y \in E$,
(iv) $\langle x, \mu y + \nu z \rangle = \mu \langle x, y \rangle + \nu \langle x, z \rangle$ for $x, y, z \in E$ and for $\mu, \nu \in \mathbb{C}$.

An inner-product $\mathcal{B}$-module $E$ which is complete with respect to the norm

$$\|x\| := \|\langle x, x \rangle\|^{1/2} \quad \text{for } x \in E,$$

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is called a *Hilbert $\mathcal{B}$-module* or *Hilbert $C^*$-module over $\mathcal{B}$*. It is said to be *full* if the closure of the linear span of $\{\langle x, y\rangle : x, y \in E\} = \mathcal{B}$. Also, for each $x \in E$, we use the term $|x|$ to denote $\langle x, x\rangle^{1/2}$. Paschke and Rieffel, cf., [**10, 11**], contributed immensely to the theory of Hilbert $C^*$-modules in the early 1970s. Applications may be found in the classification of $C^*$-algebras, the dilation theory of semigroups of completely positive maps, the theory of quantum groups, etc.

Apart from the notion of the Hilbert $C^*$-module, the property of complete positivity is a key concept needed in this article. A linear mapping $\tau$ from a $C^*$-algebra $\mathcal{B}$ to a $C^*$-algebra $\mathcal{C}$ is called *completely positive* if, for each $n \in \mathbb{N}$,

$$\sum_{i,j=1}^{n} c_j^* \tau(b_j^* b_i) c_i \geq 0$$

where $b_1, b_2, \ldots, b_n$ are from $\mathcal{B}$ and $c_1, c_2, \ldots, c_n$ are from $\mathcal{C}$. The theory of completely positive maps plays an important role in operator algebras, quantum statistical mechanics, quantum information theory, etc. Completely positive maps between unital $C^*$-algebras are characterized by Paschke's GNS construction, cf., [**10,** Theorem 5.2].

Let $E$ be a Hilbert $\mathcal{B}$-module, $F$ a Hilbert $\mathcal{C}$-module and $\tau$ a linear map from $\mathcal{B}$ to $\mathcal{C}$. A map $T : E \to F$ is called a *$\tau$-map* if

$$\langle T(x), T(y)\rangle = \tau(\langle x, y\rangle) \quad \text{for all } x, y \in E.$$

Skeide [**14**] developed a factorization theorem for $\tau$-maps when $\tau$ is completely positive based on Paschke's GNS construction. This theorem generalizes the Stinespring type theorem for Hilbert $C^*$-modules due to Bhat, Ramesh and Sumesh, cf., [**3**]. Certain related covariant versions of this theorem have been explored in [**5, 6**].

The next definition of completely positive definite (CPD-) kernels on arbitrary set $S$ plays a crucial role in exploring the theory of CPD-semigroups over $S$ [**2**].

**Definition 1.1.** Let $\mathcal{B}$ and $\mathcal{C}$ be $C^*$-algebras. By $\mathfrak{B}(\mathcal{B}, \mathcal{C})$, we denote the set of all bounded linear maps from $\mathcal{B}$ to $\mathcal{C}$. For a set $S$, we say that a mapping $\mathfrak{K} : S \times S \to \mathfrak{B}(\mathcal{B}, \mathcal{C})$ is a *completely positive definite* *kernel* or a CPD-*kernel* over $S$ from $\mathcal{B}$ to $\mathcal{C}$ if

$$\sum_{i,j} c_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(b_i^* b_j) c_j \geq 0$$

for all finite choices of $\sigma_i \in S$, $b_i \in \mathcal{B}$, $c_i \in \mathcal{C}$.

The notion of a completely multi-positive map, introduced in [**5**], is an example of a CPD-kernel over the finite set $S = \{1, \ldots, n\}$. CPD-kernels over the set $S = \{0, 1\}$ and semigroups of CPD-kernels were first studied by Accardi and Kozyrev [**1**]. Motivated by the definition of a $\tau$-map, we define the $\mathfrak{K}$-family, where $\mathfrak{K}$ is a CPD-kernel, in Section 2. Some of the results concerning $\tau$-maps from [**14, 15**] are extended to $\mathfrak{K}$-families in this article.

In Section 2, for a CPD-kernel $\mathfrak{K}$, we show that any $\mathfrak{K}$-family $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ factorizes in terms of a $C^*$-correspondence $\mathcal{F}$, a mapping from the set $S$ to $\mathcal{F}$ and an isometry, if the corresponding $C^*$-algebras are assumed to be unital. The factorization result is a Stinespring-type theorem. Further, we prove a covariant version of this theorem in terms of the following notions. Let $G$ be a locally compact group, and let $\mathcal{B}$ be a $C^*$-algebra. We call a group homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{B})$ an *action of* $G$ *on* $\mathcal{B}$ and use symbol $\alpha_t$ for the image of $t \in G$ under $\alpha$. If $t \mapsto \alpha_t(b)$ is continuous for all $b \in \mathcal{B}$, then we call $(G, \alpha, \mathcal{B})$ a $C^*$-*dynamical system.* We denote by $\mathcal{UB}$ the group of all unitary elements of the $C^*$-algebra $\mathcal{B}$.

**Definition 1.2.** Let $S$ be a set, and let $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ be a kernel over $S$ with values in the set of bounded linear maps from a $C^*$-algebra $\mathcal{B}$ to a unital $C^*$-algebra $\mathcal{C}$. Let $u : G \to \mathcal{UC}$ be a unitary representation of a locally compact group $G$. The kernel $\mathfrak{K}$ is called *u-covariant* with respect to the $(G, \alpha, \mathcal{B})$ if, for all $\sigma, \sigma' \in S$,

$$\mathfrak{K}^{\sigma,\sigma'}(\alpha_t(b)) = u_t \mathfrak{K}^{\sigma,\sigma'}(b) u_t^* \quad \text{for } b \in \mathcal{B},\ t \in G.$$

Let $E$ and $F$ be Hilbert $C^*$-modules over a $C^*$-algebra $\mathcal{B}$. A map $T : E \to F$ is called *adjointable* if there exists a map $T' : F \to E$ such that

$$\langle T(x), y \rangle = \langle x, T'(y) \rangle \quad \text{for all } x \in E,\ y \in F.$$

The map $T'$ is unique for each $T$, and we denote it by $T^*$. We denote the set of all adjointable maps from $E$ to $F$ by $\mathcal{B}^a(E,F)$, and, if $E=F$, then we denote by $\mathcal{B}^a(E)$ the space $\mathcal{B}^a(E,E)$. The set of all bounded right linear maps from $E$ into $F$ will be denoted by $\mathcal{B}^r(E,F)$. Let $E$ be a Hilbert $\mathcal{B}$-module, and let $F$ be a Hilbert $\mathcal{C}$-module. A map $\Psi : E \to F$ is said to be a *morphism of Hilbert $C^*$-modules* if a $C^*$-algebra homomorphism $\psi : \mathcal{B} \to \mathcal{C}$ exists such that

$$\langle \Psi(x), \Psi(y) \rangle = \psi(\langle x, y \rangle) \quad \text{for all } x, y \in E.$$

If $E$ is full, then $\psi$ is unique for $\Psi$. A bijective map $\Psi : E \to F$ is called an *isomorphism of Hilbert $C^*$-modules* if $\Psi$ and $\Psi^{-1}$ are morphisms of Hilbert $C^*$-modules. We denote the group of all isomorphisms of Hilbert $C^*$-modules from $E$ to itself by $\mathrm{Aut}(E)$.

**Definition 1.3.** Let $G$ be a locally compact group, and let $\mathcal{B}$ be a $C^*$-algebra. Let $E$ be a full Hilbert $\mathcal{B}$-module. A group homomorphism $t \mapsto \eta_t$ from $G$ to $\mathrm{Aut}(E)$ is called a *continuous action of $G$ on $E$* if $t \mapsto \eta_t(x)$ from $G$ to $E$ is continuous for each $x \in E$. In this case, we call the triple $(G, \eta, E)$ a *dynamical system on the Hilbert $\mathcal{B}$-module $E$*.

Any $C^*$-dynamical system $(G, \alpha, \mathcal{B})$ may be regarded as a dynamical system on the Hilbert $\mathcal{B}$-module $\mathcal{B}$. In Section 2, we also examine the extendability of covariant $\mathfrak{K}$-families with respect to any dynamical system $(G, \eta, E)$ on a Hilbert $C^*$-module $E$ to the crossed product Hilbert $C^*$-module $E \times_\eta G$. For any Hilbert $C^*$-module $E$ on $\mathcal{B}$ let $E^* := \{x^* : x \in E\} \subset \mathcal{B}^a(E, \mathcal{B})$ where $x^*y := \langle x, y \rangle$ for all $x, y \in E$. Then $\mathcal{K}(E) := \overline{\mathrm{span}}\{xy : x \in E, y \in E^*\}$ is a $C^*$-subalgebra of $\mathcal{B}^a(E)$. Indeed, $E^*$ is a Hilbert $\mathcal{K}(E)$-module where $\langle x^*, y^* \rangle := xy^*$ for all $x, y \in E$. The (extended) *linking algebra* of $E$ is defined by

$$\mathcal{L}_E := \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix} \subset \mathcal{B}^a(\mathcal{B} \oplus E),$$

cf., [**12**].

It is shown in Section 3 that, for any CPD-kernel $\mathfrak{K}$, the $\mathfrak{K}$-family on full Hilbert $C^*$-modules is the same as the set of maps defined on the Hilbert $C^*$-modules which extend as a CPD-kernel between their linking algebras. A characterization of such $\mathfrak{K}$-families is obtained in terms of completely bounded maps between certain Hilbert $C^*$-modules. We derive the covariant versions of the above results as well.

In Section 4, as an application of our theory, we propose and explore a new dilation theory of any CPD-kernel $\mathfrak{K}$ associated to a family of maps between certain Hilbert $C^*$-modules. This dilation is called a CPDH-*dilation* and, under additional assumptions, the family of maps between the Hilbert $C^*$-modules becomes a $\mathfrak{K}$-family.

## 2. $\mathfrak{K}$-families and crossed products of Hilbert $C^*$-modules.

**Definition 2.1.** Let $E$ and $F$ be Hilbert $C^*$-modules over $C^*$-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Let $S$ be a set, and let $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ be a kernel. Let $\mathcal{K}^\sigma$ be a map from $E$ to $F$ for each $\sigma \in S$. The family $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is called a $\mathfrak{K}$-*family* if

$$\langle \mathcal{K}^\sigma(x), \mathcal{K}^{\sigma'}(x') \rangle = \mathfrak{K}^{\sigma,\sigma'}(\langle x, x' \rangle), \quad \text{for } x, x' \in E;\ \sigma, \sigma' \in S.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras. A $C^*$-*correspondence* from $\mathcal{A}$ to $\mathcal{B}$ is defined as a right Hilbert $\mathcal{B}$-module $E$ together with a $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}^a(E)$, where $\mathcal{B}^a(E)$ is the set of all adjointable operators on $E$. The left action of $\mathcal{A}$ on $E$ given by $\phi$ is defined as

$$ay := \phi(a)y \quad \text{for all } a \in \mathcal{A},\ y \in E.$$

The next theorem deals with the factorization of $\mathfrak{K}$-families:

**Theorem 2.2.** *Let $\mathcal{B}$ and $\mathcal{C}$ be $C^*$-algebras where $\mathcal{B}$ is unital. Let $E$ and $F$ be Hilbert $C^*$-modules over $\mathcal{B}$ and $\mathcal{C}$, respectively, and let $S$ be a set. If $\mathcal{K}^\sigma$ is a map from $E$ to $F$ for each $\sigma \in S$, then the following conditions are equivalent*:

(i) $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is a $\mathfrak{K}$-family where $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ is a* CPD-*kernel.*

(ii) *There exists a pair $(\mathcal{F}, \mathfrak{i})$ consisting of a $C^*$-correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$ and a map $\mathfrak{i} : S \to \mathcal{F}$, and there exists an isometry $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F} \to F$ such that*

$$\nu(x \otimes \mathfrak{i}(\sigma)) = \mathcal{K}^\sigma(x), \quad \textit{for all } x \in E,\ \sigma \in S. \tag{2.1}$$

*($E \bigotimes_{\mathcal{B}} \mathcal{F}$ denotes the interior tensor product of $E$ and $\mathcal{F}$.)*

*Proof.* Suppose (ii) is given. For each $\sigma, \sigma' \in S$, we define $\mathfrak{K}^{\sigma,\sigma'} : \mathcal{B} \to \mathcal{C}$ by $\mathfrak{K}^{\sigma,\sigma'}(b) := \langle \mathfrak{i}(\sigma), b\mathfrak{i}(\sigma')\rangle$ for $b \in \mathcal{B}$. The mapping $\mathfrak{K}$ is a CPD-kernel, for

$$\begin{aligned}\sum_{i,j} c_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(b_i^* b_j) c_j &= \sum_{i,j} c_i^* \langle \mathfrak{i}(\sigma_i), b_i^* b_j \mathfrak{i}(\sigma_j)\rangle c_j \\ &= \left\langle \sum_i b_i \mathfrak{i}(\sigma_i) c_i, \sum_j b_j \mathfrak{i}(\sigma_j) c_j \right\rangle \\ &\geq 0,\end{aligned}$$

for all finite choices of $\sigma_i \in S$, $b_i \in \mathcal{B}$ and $c_i \in \mathcal{C}$. Further, for $x, x' \in E$; $\sigma, \sigma' \in S$, we have

$$\langle \mathcal{K}^\sigma(x), \mathcal{K}^{\sigma'}(x')\rangle = \langle \nu(x \otimes \mathfrak{i}(\sigma)), \nu(x' \otimes \mathfrak{i}(\sigma'))\rangle = \mathfrak{K}^{\sigma,\sigma'}(\langle x, x'\rangle).$$

Thus, $\{\mathcal{K}^\sigma\}_{\sigma\in S}$ is a $\mathfrak{K}$-family, i.e., (i) holds.

Conversely, suppose (i) is given. By Kolmogorov decomposition for $\mathfrak{K}$, cf., [**2,** Theorem 3.2.3] and [**13,** Theorem 4.2], we obtain a pair $(\mathcal{F}, \mathfrak{i})$ consisting of a $C^*$-correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$ and a map $\mathfrak{i} : S \to \mathcal{F}$ such that $\mathcal{F} = \overline{\operatorname{span}}\{b\mathfrak{i}(\sigma)c : b \in \mathcal{B}, c \in \mathcal{C}, \sigma \in S\}$ satisfying

$$\mathfrak{K}^{\sigma,\sigma'}(b) = \langle \mathfrak{i}(\sigma), b\mathfrak{i}(\sigma')\rangle \quad \text{for } b \in \mathcal{B};\ \sigma, \sigma' \in S.$$

Define a linear map $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F} \to F$ by $\nu(x \otimes b\mathfrak{i}(\sigma)c) := \mathcal{K}^\sigma(xb)c$ for all $x \in E$, $b \in \mathcal{B}$, $c \in \mathcal{C}$ and $\sigma \in S$. We have

$$\begin{aligned}\langle \nu(x \otimes b\mathfrak{i}(\sigma)c), \nu(x' \otimes b'\mathfrak{i}(\sigma')c')\rangle &= \langle \mathcal{K}^\sigma(xb)c, \mathcal{K}^{\sigma'}(x'b')c'\rangle \\ &= c^* \mathfrak{K}^{\sigma,\sigma'}(\langle xb, x'b'\rangle)c' \\ &= \langle \mathfrak{i}(\sigma)c, (\langle xb, x'b'\rangle)\mathfrak{i}(\sigma')c'\rangle \\ &= \langle x \otimes b\mathfrak{i}(\sigma)c, x' \otimes b'\mathfrak{i}(\sigma')c'\rangle,\end{aligned}$$

for all $x, x' \in E$; $b, b' \in \mathcal{B}$; $c, c' \in \mathcal{C}$ and $\sigma, \sigma' \in S$. Hence, $\nu$ is an isometry satisfying equation (2.1). This proves (i) $\Rightarrow$ (ii). $\square$

We now examine the covariant version of the above theorem. If $(G, \eta, E)$ is a dynamical system on a full Hilbert $\mathcal{B}$-module $E$, then there exists a unique $C^*$-dynamical system $(G, \alpha^\eta, \mathcal{B})$, cf., [**6,** page 806]) such that

$$\alpha_t^\eta(\langle x, y\rangle) = \langle \eta_t(x), \eta_t(y)\rangle \quad \text{for all } x, y \in E \text{ and } t \in G.$$

Moreover, for all $x \in E$ and $b \in \mathcal{B}$, we have $\eta_t(xb) = \eta_t(x)\alpha_t^\eta(b)$.

**Definition 2.3.** Let $\mathcal{C}$ and $\mathcal{D}$ be unital $C^*$-algebras, and let $u : G \to \mathcal{UC}$ and $u' : G \to \mathcal{UD}$ be unitary representations on a locally compact group $G$. Let $E$ be a full Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{B}$, and let $F$ be a $C^*$-correspondence from $\mathcal{D}$ to $\mathcal{C}$. Let $S$ be a set and $(G, \eta, E)$ be a dynamical system on $E$. Consider the bounded linear maps $\mathcal{K}^\sigma : E \to F$ for $\sigma \in S$. Then, the family $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is called $(u', u)$*-covariant with respect to the dynamical system* $(G, \eta, E)$ if

$$\mathcal{K}^\sigma(\eta_t(x)) = u'_t \mathcal{K}^\sigma(x) u_t^* \quad \text{for each } t \in G,\ \sigma \in S \text{ and } x \in E.$$

**Theorem 2.4.** *Let $u : G \to \mathcal{UC}$ and $u' : G \to \mathcal{UD}$ be unitary representations of a locally compact group $G$ on unital $C^*$-algebras $\mathcal{C}$ and $\mathcal{D}$, respectively. Let $E$ be a full Hilbert $C^*$-module over a unital $C^*$-algebra $\mathcal{B}$, $F$ a $C^*$-correspondence from $\mathcal{D}$ to $\mathcal{C}$ and $S$ a set. Let $\mathcal{K}^\sigma$ be a map from $E$ to $F$ for each $\sigma \in S$. If $(G, \eta, E)$ is a dynamical system on $E$, then the following conditions are equivalent*:

(i) *$\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is a $(u', u)$-covariant $\mathfrak{K}$-family with respect to the dynamical system $(G, \eta, E)$ where $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ is a* CPD*-kernel.*

(ii) *There exists a pair $(\mathcal{F}, \mathfrak{i})$ consisting of a $C^*$-correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$ with the left action $\pi$ and a map $\mathfrak{i} : S \to \mathcal{F}$, an isometry $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F} \to F$ such that*

$$\nu(x \otimes \mathfrak{i}(\sigma)) = \mathcal{K}^\sigma(x) \quad \text{for all } x \in E,\ \sigma \in S,$$

*and unitary representations $v : G \to \mathcal{UB}^a(\mathcal{F})$ and $w' : G \to \mathcal{UB}^a(E \bigotimes_{\mathcal{B}} \mathcal{F})$ such that*

(a) *$\pi(\alpha_t^\eta(b)) = v_t \pi(b) v_t^*$ for all $b \in \mathcal{B}$, $t \in G$,*

(b) *$v_t \mathfrak{i}(\sigma) = \mathfrak{i}(\sigma) u_t$ for all $t \in G$ and $\sigma \in S$,*

(c) *$w'_t(x \otimes b\mathfrak{i}(\sigma)c) := \eta_t(x) \otimes v_t(b\mathfrak{i}(\sigma)c)$ for all $b \in \mathcal{B}$, $c \in \mathcal{C}$, $x \in E$, $\sigma \in S$ and $t \in G$,*

(d) *$\nu w'_t = u'_t \nu$ for all $t \in G$.*

*Proof.* Suppose that statement (ii) is given. The collection $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is a $\mathfrak{K}$-family where $\mathfrak{K}^{\sigma,\sigma'} : \mathcal{B} \to \mathcal{C}$ is defined by $\mathfrak{K}^{\sigma,\sigma'}(b) := \langle \mathfrak{i}(\sigma), b\mathfrak{i}(\sigma') \rangle$ for $b \in \mathcal{B}$ and $\sigma, \sigma' \in S$. Also,

$$\begin{aligned}\mathcal{K}^\sigma(\eta_t(x)) &= \nu(\eta_t(x) \otimes \mathfrak{i}(\sigma)) \\ &= \nu(\eta_t(x) \otimes v_t v_{t^{-1}} \mathfrak{i}(\sigma)) = \nu w'_t(x \otimes v_{t^{-1}} \mathfrak{i}(\sigma))\end{aligned}$$

$$= u'_t \nu(x \otimes v_{t^{-1}} \mathfrak{i}(\sigma)) = u'_t \nu(x \otimes \mathfrak{i}(\sigma) u_{t^{-1}})$$
$$= u'_t \nu(x \otimes \mathfrak{i}(\sigma)) u_{t^{-1}} = u'_t \mathcal{K}^\sigma(x) u_{t^{-1}}$$

for all $x \in E$, $\sigma \in S$ and $t \in G$. Hence, statement (i) holds.

Conversely, let us assume that (i) holds. The kernel $\mathfrak{K}$ is $u$-covariant because, for $\sigma, \sigma' \in S$; $x, x' \in E$, $t \in G$,

$$\begin{aligned}
\mathfrak{K}^{\sigma,\sigma'}(\alpha_t^\eta(\langle x, x'\rangle)) &= \mathfrak{K}^{\sigma,\sigma'}(\langle \eta_t(x), \eta_t(x')\rangle) \\
&= \langle \mathcal{K}^\sigma(\eta_t(x)), \mathcal{K}^{\sigma'}(\eta_t(x'))\rangle \\
&= \langle u'_t \mathcal{K}^\sigma(x) u_t^*, u'_t \mathcal{K}^{\sigma'}(x') u_t^*\rangle \\
&= u_t \langle \mathcal{K}^\sigma(x), \mathcal{K}^{\sigma'}(x')\rangle u_t^* \\
&= u_t \mathfrak{K}^{\sigma,\sigma'}(\langle x, x'\rangle) u_t^*.
\end{aligned}$$

By Theorem 2.2 or Kolmogorov decomposition we get a pair $(\mathcal{F}, \mathfrak{i})$ consisting of a $C^*$-correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$ where the left action is given by a $*$-homomorphism $\pi : \mathcal{B} \to \mathcal{B}^a(\mathcal{F})$ and a map $\mathfrak{i} : S \to \mathcal{F}$ such that $\overline{\operatorname{span}}\{b\mathfrak{i}(\sigma)c : b \in \mathcal{B}, c \in \mathcal{C}, \sigma \in S\} = \mathcal{F}$, satisfying

$$\mathfrak{K}^{\sigma,\sigma'}(b) = \langle \mathfrak{i}(\sigma), b\mathfrak{i}(\sigma')\rangle \quad \text{for } b \in \mathcal{B};\ \sigma, \sigma' \in S.$$

Further, we have an isometry $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F} \to F$ defined by

$$\nu(x \otimes b\mathfrak{i}(\sigma)c) := \mathcal{K}^\sigma(xb)c \quad \text{for all } x \in E,\ b \in \mathcal{B},\ c \in \mathcal{C},\ \sigma \in S.$$

For each $t \in G$, set $v_t(b\mathfrak{i}(\sigma)c) := \alpha_t^\eta(b)\mathfrak{i}(\sigma)u_t c$ for all $t \in G$, $b \in \mathcal{B}$, $c \in \mathcal{C}$ and $\sigma \in S$. Observe that

$$\begin{aligned}
&\langle v_t(b\mathfrak{i}(\sigma)c), v_t(b'\mathfrak{i}(\sigma')c')\rangle \\
&\quad = \langle \alpha_t^\eta(b)\mathfrak{i}(\sigma)u_t c, \alpha_t^\eta(b')\mathfrak{i}(\sigma')u_t c'\rangle \\
&\quad = (u_t c)^* \mathfrak{K}^{\sigma,\sigma'}(\alpha_t^\eta(b)^* \alpha_t^\eta(b')) u_t c' \\
&\quad = c^* u_t^* u_t \mathfrak{K}^{\sigma,\sigma'}(b^* b') u_t^* u_t c' \\
&\quad = \langle b\mathfrak{i}(\sigma)c, b'\mathfrak{i}(\sigma')c'\rangle
\end{aligned}$$

for all $b, b' \in \mathcal{B}$; $\sigma, \sigma' \in S$ and $c, c' \in \mathcal{C}$. Since $\alpha_t^\eta$ is an automorphism and $u_t$ is unitary for each $t \in G$, it is immediate that $v_t$ uniquely extends to a unitary $v_t : \mathcal{F} \to \mathcal{F}$ for each $t \in G$. Because of the continuity of $t \mapsto \alpha_t^\eta(b)$ for each $b \in \mathcal{B}$, the continuity of $u$ and the fact that $v_t$ is unitary for each $t \in G$, it follows that $t \mapsto v_t f$ is continuous for each $f \in \mathcal{F}$. Hence, $v : G \to \mathcal{U}\mathcal{B}^a(\mathcal{F})$ is a unitary representation. For all $b, b' \in \mathcal{B}$; $t \in G$ and $c \in \mathcal{C}$ we get

$$\begin{aligned}\pi(\alpha_t^\eta(b'))(b\mathfrak{i}(\sigma)c) &= (\alpha_t^\eta(b')b)\mathfrak{i}(\sigma)c \\ &= v_t(b'\alpha_{t^{-1}}^\eta(b)\mathfrak{i}(\sigma)u_{t^{-1}}c) \\ &= v_t\pi(b')(\alpha_{t^{-1}}^\eta(b)\mathfrak{i}(\sigma)u_{t^{-1}}c) \\ &= v_t\pi(b')v_{t^{-1}}(b\mathfrak{i}(\sigma)c).\end{aligned}$$

Thus, $v$ satisfies conditions (a) and (b).

For each $t \in G$, define $w'_t : E \bigotimes_{\mathcal{B}} \mathcal{F} \to E \bigotimes_{\mathcal{B}} \mathcal{F}$ by

$$w'_t(x \otimes b\mathfrak{i}(\sigma)c) := \eta_t(x) \otimes v_t b\mathfrak{i}(\sigma)c$$

for all $b \in \mathcal{B}$, $c \in \mathcal{C}$, $\sigma \in S$, $x \in E$. We get

$$\begin{aligned}&\langle w'_t(x \otimes b\mathfrak{i}(\sigma)c), w'_t(x' \otimes b'\mathfrak{i}(\sigma')c')\rangle \\ &= \langle v_t(b\mathfrak{i}(\sigma)c), \langle \eta_t(x), \eta_t(x')\rangle v_t(b'\mathfrak{i}(\sigma')c')\rangle \\ &= \langle v_t(b\mathfrak{i}(\sigma)c), \alpha_t^\eta(\langle x, x'\rangle)v_t(b'\mathfrak{i}(\sigma')c')\rangle \\ &= \langle v_t(b\mathfrak{i}(\sigma)c), v_t(\langle x, x'\rangle)b'\mathfrak{i}(\sigma')c')\rangle \\ &= \langle b\mathfrak{i}(\sigma)c, \langle x, x'\rangle b'\mathfrak{i}(\sigma')c'\rangle \\ &= \langle x \otimes b\mathfrak{i}(\sigma)c, x' \otimes b'\mathfrak{i}(\sigma')c'\rangle\end{aligned}$$

for all $b, b' \in \mathcal{B}$; $c, c' \in \mathcal{C}$; $x, x' \in E$ and $\sigma, \sigma \in S$. Using the strict continuity of $v$ and the continuity of $t \mapsto \eta_t(x)$ for all $x \in E$ we obtain that the map $t \mapsto w'_t z$ is continuous on finite sums of elementary tensors $z \in E \bigotimes_{\mathcal{B}} \mathcal{F}$. Now $\|w'_t\| \leq 1$ implies $w'$ is strictly continuous and therefore a unitary representation. Moreover, we have

$$\begin{aligned}\nu w'_t(x \otimes b\mathfrak{i}(\sigma)c) &= \nu(\eta_t(x) \otimes v_t(b\mathfrak{i}(\sigma)c)) \\ &= \nu(\eta_t(x) \otimes \alpha_t^\eta(b)\mathfrak{i}(\sigma)u_t c) \\ &= \mathcal{K}^\sigma(\eta_t(x)\alpha_t^\eta(b))u_t c \\ &= \mathcal{K}^\sigma(\eta_t(xb))u_t c \\ &= u'_t\mathcal{K}^\sigma(xb)u_t^* u_t c \\ &= u'_t\mathcal{K}^\sigma(xb)c \\ &= u'_t\nu(x \otimes b\mathfrak{i}(\sigma)c)\end{aligned}$$

for all $b \in \mathcal{B}$, $c \in \mathcal{C}$, $x \in E$, $\sigma \in S$ and $t \in G$. $\square$

The next corollary proves the uniqueness of Theorem 2.4.

**Corollary 2.5.** *Let $\mathcal{E}$ be another $C^*$-correspondence from $\mathcal{D}$ to $\mathcal{C}$. For $\sigma \in S$, let $\widetilde{\mu}^\sigma : E \to \mathcal{E}$ be maps such that $\overline{\operatorname{span}}\{\widetilde{\mu}^\sigma(E)\mathcal{C} : \sigma \in S\} = \mathcal{E}$, and let $\widetilde{\nu} : \mathcal{E} \to F$ be an isometry such that $\widetilde{\nu}\widetilde{\mu}^\sigma = \mathcal{K}^\sigma$. Then there exists a unitary representation $w''_t : G \to \mathcal{U}\mathcal{B}^a(\mathcal{E})$, defined by*

$$w''_t(\widetilde{\mu}^\sigma(x)c) = \widetilde{\mu}^\sigma(\eta_t(x))u_t c$$
$$\text{for } x \in E,\ t \in G,\ \sigma \in S \text{ and } c \in \mathcal{C}$$

*and a unitary $u : \mathcal{E} \to E \bigotimes_{\mathcal{B}} \mathcal{F}$ defined by $u : \widetilde{\mu}^\sigma(x) \mapsto x \otimes \mathfrak{i}(\sigma)$, where $\sigma \in S$ and $(\mathcal{F}, \mathfrak{i})$ is the Kolmogorov decomposition for kernel $\mathfrak{K}$ such that*

(a) *$\nu u = \widetilde{\nu}$, $uw''_t = w'_t u$ for all $t \in G$ and*
(b) *$u\widetilde{\mu}^\sigma = \mu^\sigma$ where, for $\sigma \in S$, the mapping $\mu^\sigma : E \to E \bigotimes_{\mathcal{B}} \mathcal{F}$ is defined by $x \mapsto x \otimes \mathfrak{i}(\sigma)$.*

*Proof.* For all $x, x' \in E$; $c, c' \in \mathcal{C}$ and $\sigma, \sigma' \in S$, we have

$$\begin{aligned}
&\langle \widetilde{\mu}^\sigma(\eta_t(x))u_t c, \widetilde{\mu}^{\sigma'}(\eta_t(x'))u_t c'\rangle \\
&\quad = \langle \mathcal{K}^\sigma(\eta_t(x))u_t c, \mathcal{K}^{\sigma'}(\eta_t(x'))u_t c'\rangle \\
&\quad = \langle u_t c, \mathfrak{K}^{\sigma,\sigma'}(\alpha_t(\langle x, x'\rangle))u_t c'\rangle \\
&\quad = \langle \mathcal{K}^\sigma(x)c, \mathcal{K}^{\sigma'}(x')c'\rangle \\
&\quad = \langle \widetilde{\mu}^\sigma(x)c, \widetilde{\mu}^{\sigma'}(x')c'\rangle.
\end{aligned}$$

Therefore, $w''$ is a unitary representation. □

Let $\mathcal{B}$ be a $C^*$-algebra, and let $G$ be a locally compact group. Let $(G, \eta, E)$ be a dynamical system on a full Hilbert $\mathcal{B}$-module $E$. The crossed product $E \times_\eta G$, cf., [**4, 7**], is the completion of an inner-product $\mathcal{B} \times_{\alpha^\eta} G$-module $C_c(G, E)$, where the module action and the $\mathcal{B} \times_{\alpha^\eta} G$-valued inner product are given by

$$lg(s) = \int_G l(t)\alpha^\eta_t(g(t^{-1}s))\, dt,$$
$$\langle l, m\rangle_{\mathcal{B}\times_{\alpha^\eta} G}(s) = \int_G \alpha^\eta_{t^{-1}}(\langle l(t), m(ts)\rangle)\, dt,$$

respectively, for $g \in C_c(G, \mathcal{B})$ and $l, m \in C_c(G, E)$. We derive, for any CPD-kernel $\mathfrak{K}$, the extendability of a covariant $\mathfrak{K}$-family to that on the crossed product of the Hilbert $C^*$-module corresponding to the given dynamical system.

**Proposition 2.6.** *Let $S$ be a set, and let $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ be a* CPD*-kernel over $S$ from a unital $C^*$-algebra $\mathcal{B}$ to a unital $C^*$-algebra $\mathcal{C}$. Let $\mathcal{D}$ be a unital $C^*$-algebra, and let $u : G \to \mathcal{UC}$ and $u' : G \to \mathcal{UD}$ be unitary representations of a locally compact group $G$. Suppose that $E$ is a full Hilbert $\mathcal{B}$-module, $F$ is a $C^*$-correspondence from $\mathcal{D}$ to $\mathcal{C}$ and $\mathcal{K}^\sigma$ is a map from $E$ to $F$ for each $\sigma \in S$. If $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is a $(u', u)$ covariant $\mathfrak{K}$-family with respect to the dynamical system $(G, \eta, E)$, then there exists a family of maps $\widetilde{\mathcal{K}}^\sigma : E \times_\eta G \to F$ such that*

$$\widetilde{\mathcal{K}}^\sigma(l) = \int_G \mathcal{K}^\sigma(l(t)) u_t \, dt \quad \textit{for all } l \in C_c(G, E), \ \sigma \in S,$$

*and there exists a* CPD*-kernel $\widetilde{\mathfrak{K}}^{\sigma,\sigma'} : \mathcal{B} \times_{\alpha^\eta} G \to \mathcal{C}$, which satisfies*

$$\widetilde{\mathfrak{K}}^{\sigma,\sigma'}(f) = \int_G \mathfrak{K}^{\sigma,\sigma'}(f(t)) u_t \, dt \quad \textit{for all } f \in C_c(G, \mathcal{B}), \ \sigma, \sigma' \in S,$$

*such that $\{\widetilde{\mathcal{K}}^\sigma\}_{\sigma \in S}$ is a $\widetilde{\mathfrak{K}}$-family.*

*Proof.* Let $(\mathcal{F}, \mathfrak{i})$ be the covariant Kolmogorov decomposition associated with the CPD-kernel $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ described in Theorem 2.4, and denote the left action associated with the $C^*$-correspondence $\mathcal{F}$ by $\pi$. Consider maps $\widetilde{\mathfrak{K}}^{\sigma,\sigma'} : \mathcal{B} \times_{\alpha^\eta} G \to \mathcal{C}$ defined by

$$\widetilde{\mathfrak{K}}^{\sigma,\sigma'}(f) := \langle \mathfrak{i}(\sigma), (\pi \times v)(f) \mathfrak{i}(\sigma') \rangle$$
$$\text{for all } f \in C_c(G, \mathcal{B}), \ \sigma, \sigma' \in S.$$

Similar computations as in Theorem 2.2 prove that $\widetilde{\mathfrak{K}}$ is a CPD-kernel on $S$ from $\mathcal{B} \times_{\alpha^\eta} G$ to $\mathcal{C}$. For $\sigma, \sigma' \in S$,

$$\begin{aligned} \widetilde{\mathfrak{K}}^{\sigma,\sigma'}(f) &= \langle \mathfrak{i}(\sigma), (\pi \times v)(f) \mathfrak{i}(\sigma') \rangle \\ &= \langle \mathfrak{i}(\sigma), \int_G \pi(f(t)) v_t \mathfrak{i}(\sigma') \, dt \rangle \\ &= \int_G \langle \mathfrak{i}(\sigma), \pi(f(t)) v_t \mathfrak{i}(\sigma') \rangle \, dt \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
&= \int_G \langle \mathfrak{i}(\sigma), \pi(f(t))\mathfrak{i}(\sigma')u_t\rangle\, dt\\
&= \int_G \langle \mathfrak{i}(\sigma), \pi(f(t))\mathfrak{i}(\sigma')\rangle u_t\, dt\\
&= \int_G \mathfrak{K}^{\sigma,\sigma'}(f(t))u_t\, dt,
\end{aligned}
$$

for all $f \in C_c(G, \mathcal{B})$. The third equality in array (2.2) follows by applying [**16,** Lemma 1.91] for a bounded linear map $L : \mathcal{B}^a(\mathcal{F}) \to \mathcal{C}$, which is defined as $L(T) := \langle \mathfrak{i}(\sigma), T\mathfrak{i}(\sigma')\rangle$ for all $T \in \mathcal{B}^a(\mathcal{F})$. Define $\widetilde{\mathcal{K}}^{\sigma} : E \times_\eta G \to F$ by

$$
\widetilde{\mathcal{K}}^{\sigma}(l) := \int_G \mathcal{K}^{\sigma}(l(t))u_t\, dt \quad \text{for all } \sigma \in S,\ l \in C_c(G, E).
$$

From Theorem 2.4, we obtain an isometry $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F} \to F$ such that

$$
\nu(x \otimes \mathfrak{i}(\sigma)) = \mathcal{K}^{\sigma}(x) \quad \text{for all } x \in E,\ \sigma \in S,
$$

and unitary representations $v : G \to \mathcal{U}\mathcal{B}^a(\mathcal{F})$ and $w' : G \to \mathcal{U}\mathcal{B}^a(E \bigotimes_{\mathcal{B}} \mathcal{F})$ satisfying conditions (a)–(d) of the theorem. For all $l \in C_c(G, E)$ and $\sigma \in S$, we obtain

$$
\widetilde{\mathcal{K}}^{\sigma}(l) = \int_G \mathcal{K}^{\sigma}(l(t))u_t\, dt = \int_G \nu(l(t)\otimes\mathfrak{i}(\sigma))u_t\, dt = \int_G \nu(l(t)\otimes v_t\mathfrak{i}(\sigma))\, dt.
$$

Finally, it follows that $\{\widetilde{\mathcal{K}}^{\sigma}\}_{\sigma\in S}$ is a $\widetilde{\mathfrak{K}}$-family because, for $\sigma, \sigma' \in S$ and $l, m \in C_c(G, E)$, we have

$$
\begin{aligned}
\langle \widetilde{\mathcal{K}}^{\sigma}(l), \widetilde{\mathcal{K}}^{\sigma'}(m)\rangle &= \left\langle \int_G \nu(l(t) \otimes v_t\mathfrak{i}(\sigma))\, dt, \int_G \nu(m(s) \otimes v_s\mathfrak{i}(\sigma'))\, ds \right\rangle\\
&= \int_G \int_G \langle v_t\mathfrak{i}(\sigma), \pi(\langle l(t), m(ts)\rangle)v_{ts}\mathfrak{i}(\sigma')\rangle\, dt\, ds\\
&= \left\langle \mathfrak{i}(\sigma), \int_G \int_G v_{t^{-1}}\pi(\langle l(t), m(ts)\rangle)v_{ts}\mathfrak{i}(\sigma')\, dt\, ds \right\rangle\\
&= \left\langle \mathfrak{i}(\sigma), \int_G \int_G \pi(\alpha^{\eta}_{t^{-1}}(\langle l(t), m(ts)\rangle))v_s\mathfrak{i}(\sigma')\, dt\, ds \right\rangle\\
&= \left\langle \mathfrak{i}(\sigma), \int_G \pi(\langle l, m\rangle(s))v_s\mathfrak{i}(\sigma')\, ds \right\rangle\\
&= \widetilde{\mathfrak{K}}^{\sigma,\sigma'}(\langle l, m\rangle).
\end{aligned}
$$

□

**3. Characterizations of $\mathfrak{K}$-families.** Let $E$ be a Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{B}$. By $M_n(E)$, we denote the Hilbert $M_n(\mathcal{B})$-module where the $M_n(\mathcal{B})$-valued inner product is defined by

$$\langle [x_{ij}]_{i,j=1}^n, [x'_{ij}]_{i,j=1}^n \rangle := \left[ \sum_{k=1}^n \langle x_{ki}, x'_{kj} \rangle \right]_{i,j=1}^n$$

for all $[x_{ij}]_{i,j=1}^n,\ [x'_{ij}]_{i,j=1}^n \in M_n(E)$.

**Definition 3.1.** Let $F$ be a Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{C}$, and let $T : E \to F$ be a linear map. For each positive integer $n$, define $T_n : M_n(E) \to M_n(F)$ by

$$T_n([x_{ij}]_{i,j=1}^n) := [T(x_{ij})]_{i,j=1}^n \quad \text{for all } [x_{ij}]_{i,j=1}^n \in M_n(E).$$

We say that $T$ is *completely bounded* if, for each positive integer $n$, $T_n$ is bounded and $\|T\|_{cb} := \sup_n \|T_n\| < \infty$.

We show in this section that $\mathfrak{K}$-families, where $\mathfrak{K}$ is a CPD-kernel, are the same as certain completely bounded maps between the Hilbert $C^*$-modules. We need the following Hilbert $C^*$-modules in order to inspect the extendability of $\mathfrak{K}$- families to CPD-kernels between the (extended) linking algebras of the Hilbert $C^*$-modules:

The vector space $E_n$ consists of elements $(x_1, x_2, \ldots, x_n)$ with $x_i \in E$ for $1 \le i \le n$, where the operations are coordinate-wise. It becomes a Hilbert $M_n(\mathcal{B})$-module with respect to the inner product whose $(i,j)$-entry is given by

$$\langle (x_1, x_2, \ldots, x_n), (x'_1, x'_2, \ldots, x'_n) \rangle_{ij} := \langle x_i, x'_j \rangle$$

for $(x_1, x_2, \ldots, x_n), (x'_1, x'_2, \ldots, x'_n) \in E_n$. The symbol $E^n$ denotes the Hilbert $\mathcal{B}$-module whose elements are $(x_1, x_2, \ldots, x_n)^t$ with $x_i \in E$ for $1 \le i \le n$, where $^t$ denotes the transpose. The inner product in $E^n$ is defined by

$$\langle (x_1, x_2, \ldots, x_n)^t, (x'_1, x'_2, \ldots, x'_n)^t \rangle := \sum_{i=1}^n \langle x_i, x'_i \rangle$$

for $(x_1, x_2, \ldots, x_n)^t, (x'_1, x'_2, \ldots, x'_n)^t \in E^n$.

From [**2,** Lemma 3.2.1], we know that $\mathfrak{K}$ is a CPD-kernel over $S$ from $\mathcal{B}$ to $\mathcal{C}$ if and only if, for all $\sigma_1, \sigma_2, \ldots, \sigma_n$, $n \in \mathbb{N}$, the map

$[\mathfrak{K}^{\sigma_i,\sigma_j}]_{i,j=1}^n : M_n(\mathcal{B}) \to M_n(\mathcal{C})$ defined by

$$[\mathfrak{K}^{\sigma_i,\sigma_j}][b_{ij}] := [\mathfrak{K}^{\sigma_i,\sigma_j}(b_{ij})]_{i,j=1}^n \quad \text{for all } [b_{ij}]_{i,j=1}^n \in M_n(\mathcal{B})$$

is (completely) positive. This realization of CPD-kernels comes in handy in the proof of the next theorem.

**Theorem 3.2.** *Let $E$ be a full Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{B}$, and let $F$ be a Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{C}$. Let $S$ be a set, and let $\mathcal{K}^\sigma$ be a linear map from $E$ to $F$ for each $\sigma \in S$. Let $F_{\mathcal{K}} := \overline{\operatorname{span}}\{\mathcal{K}^\sigma(x)c : x \in E,\ c \in \mathcal{C},\ \sigma \in S\}$. Then the following statements are equivalent*:

(a) *there exists a unique* CPD-*kernel $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B},\mathcal{C})$ such that $\{\mathcal{K}^\sigma\}_{\sigma\in S}$ is a $\mathfrak{K}$-family.*

(b) *$\{\mathcal{K}^\sigma\}_{\sigma\in S}$ extends to block-wise bounded linear maps $\begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix}$, from $\mathcal{L}_E$ to $\mathcal{L}_{F_{\mathcal{K}}}$, forming a* CPD-*kernel over $S$ from $\mathcal{L}_E$ to $\mathcal{L}_{F_{\mathcal{K}}}$, where $\vartheta$ is a $*$-homomorphism. In such a case, we call $\{\mathcal{K}^\sigma\}_{\sigma\in S}$ a* CPD-H-*extendable family.*

(c) *For each finite choice $\sigma_1, \ldots, \sigma_n \in S$ the map from $E_n$ to $F_n$ defined by*

$$\mathbf{x} \longmapsto (\mathcal{K}^{\sigma_1}(x_1), \mathcal{K}^{\sigma_2}(x_2), \ldots, \mathcal{K}^{\sigma_n}(x_n)),$$

*for $\mathbf{x} = (x_1, x_2 \ldots, x_n) \in E_n$, is completely bounded. Moreover, $F_{\mathcal{K}}$ can be made into a $C^*$-correspondence from $\mathcal{B}^a(E)$ to $\mathcal{C}$ such that the action of $\mathcal{B}^a(E)$ on $F_{\mathcal{K}}$ is non-degenerate and, for each $\sigma \in S$, $\mathcal{K}^\sigma$ is a left $\mathcal{B}^a(E)$-linear map.*

(d) *For each finite choice $\sigma_1, \ldots, \sigma_n \in S$ the map from $E_n$ to $F_n$ defined by*

$$\mathbf{x} \longmapsto (\mathcal{K}^{\sigma_1}(x_1), \mathcal{K}^{\sigma_2}(x_2), \ldots, \mathcal{K}^{\sigma_n}(x_n)),$$

*for $\mathbf{x} = (x_1, x_2 \ldots, x_n) \in E_n$, is completely bounded, and $\{\mathcal{K}^\sigma\}_{\sigma\in S}$ satisfies*

$$\langle \mathcal{K}^\sigma(y), \mathcal{K}^{\sigma'}(x\langle x', y'\rangle)\rangle = \langle \mathcal{K}^\sigma(x'\langle x, y\rangle), \mathcal{K}^{\sigma'}(y')\rangle$$

*for $x, y, x', y' \in E$.*

*Proof.*

(a) $\Rightarrow$ (b). Suppose $\mathcal{B}$ is unital. Using Theorem 2.2, we obtain a pair $(\mathcal{F}, \mathfrak{i})$ consisting of a $C^*$-correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$ and a map

$\mathfrak{i} : S \to \mathcal{F}$ such that $\overline{\operatorname{span}}\{b\mathfrak{i}(\sigma)c : b \in \mathcal{B}, c \in \mathcal{C}, \sigma \in S\} = \mathcal{F}$, and an isometry $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F} \to F$, defined by

$$\nu(x \otimes b\mathfrak{i}(\sigma)c) := \mathcal{K}^{\sigma}(xb)c \quad \text{for all } x \in E,\ b \in \mathcal{B},\ c \in \mathcal{C},\ \sigma \in S.$$

We again denote the unitary obtained from $\nu$, by restricting its codomain to $F_{\mathcal{K}}$, with $\nu$. With this unitary $\nu$, define a $*$-homomorphism $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F_{\mathcal{K}})$ by $\vartheta : a \mapsto \nu(a \otimes \mathrm{id}_{\mathcal{F}})\nu^*$. Identify $\mathcal{F}$ with $\mathcal{B}^a(\mathcal{C}, \mathcal{F})$ using $f \mapsto L_f$, where $L_f : c \mapsto fc$, and identify $\mathcal{B} \bigotimes_{\mathcal{B}} \mathcal{F}$ with $\mathcal{F}$ using $b \otimes f \mapsto bf$. For each $x, x' \in E$, $f$ and $f' \in \mathcal{F}$, and $b \in \mathcal{B}$, we obtain

$$\begin{aligned}\langle (x \otimes \mathrm{id}_{\mathcal{F}})^*(x' \otimes f), b \otimes f'\rangle &= \langle x' \otimes f, xb \otimes f'\rangle\\ &= \langle f, \langle x', xb\rangle f'\rangle = \langle f, \langle x', x\rangle bf'\rangle\\ &= \langle x^*x'f, bf'\rangle = \langle x^*x' \otimes f, b \otimes f'\rangle\\ &= \langle (x^* \otimes \mathrm{id}_{\mathcal{F}})(x' \otimes f), b \otimes f'\rangle.\end{aligned}$$

Therefore, $(x \otimes \mathrm{id}_{\mathcal{F}})^* = (x^* \otimes \mathrm{id}_{\mathcal{F}})$, for $x \in E$.

For each $\sigma \in S$, the element

$$\begin{pmatrix} \mathfrak{i}(\sigma) & 0 \\ 0 & \nu^* \end{pmatrix} \in \mathcal{B}^a\left(\begin{pmatrix} \mathcal{C} \\ F_{\mathcal{K}} \end{pmatrix}, \begin{pmatrix} \mathcal{B} \\ E \end{pmatrix} \bigotimes_{\mathcal{B}} \mathcal{F}\right).$$

We have

$$\begin{aligned}&\begin{pmatrix} \mathfrak{i}(\sigma)^* & 0 \\ 0 & \nu \end{pmatrix}\left(\begin{pmatrix} b & x^* \\ y & a \end{pmatrix} \bigotimes \mathrm{id}_{\mathcal{F}}\right)\begin{pmatrix} \mathfrak{i}(\sigma') & 0 \\ 0 & \nu^* \end{pmatrix}\\ &= \begin{pmatrix} \mathfrak{i}(\sigma)^* & 0 \\ 0 & \nu \end{pmatrix}\begin{pmatrix} b \otimes \mathfrak{i}(\sigma') & (x^* \otimes \mathrm{id}_{\mathcal{F}})\nu^* \\ y \otimes \mathfrak{i}(\sigma') & (a \otimes \mathrm{id}_{\mathcal{F}})\nu^* \end{pmatrix}\\ &= \begin{pmatrix} \mathfrak{i}(\sigma)^*(b \otimes \mathfrak{i}(\sigma')) & \mathfrak{i}(\sigma)^*(x \otimes \mathrm{id}_{\mathcal{F}})^*\nu^* \\ \nu(y \otimes \mathfrak{i}(\sigma')) & \nu(a \otimes \mathrm{id}_{\mathcal{F}})\nu^* \end{pmatrix}\end{aligned}$$

for all $b \in \mathcal{B}, x,\ y \in E,\ a \in \mathcal{B}^a(E),\ \sigma$ and $\sigma' \in S$. Thus, we obtain a CPD-kernel on $S$ from $\mathcal{L}_E$ to $\mathcal{L}_{F_{\mathcal{K}}}$ formed by maps

$$\begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix} := \begin{pmatrix} \mathfrak{i}(\sigma) & 0 \\ 0 & \nu^* \end{pmatrix}^* (\bullet \otimes \mathrm{id}_{\mathcal{F}})\begin{pmatrix} \mathfrak{i}(\sigma') & 0 \\ 0 & \nu^* \end{pmatrix},$$

where $\mathcal{K}^{\sigma^*}(x^*) := \mathcal{K}^{\sigma}(x)^*$ for $\sigma \in S$, $x \in E$.

Assume that $\mathcal{B}$ is not unital. Let $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{C}}$ be the unitalizations of $\mathcal{B}$ and $\mathcal{C}$, respectively. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a contractive approximate unit for $\mathcal{B}$. Let $\delta : \widetilde{\mathcal{B}} \to \mathbb{C}$ be the unique character vanishing on $\mathcal{B}$. For each $\sigma, \sigma'$, define $\widetilde{\mathfrak{K}}^{\sigma,\sigma'} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{C}}$ by $\widetilde{\mathfrak{K}}^{\sigma,\sigma'}(b) := \mathfrak{K}^{\sigma,\sigma'}(b)$ for all $b \in \mathcal{B}$ and $\widetilde{\mathfrak{K}}^{\sigma,\sigma'}(1_{\widetilde{\mathcal{B}}}) := \|\mathfrak{K}^{\sigma,\sigma'}\| 1_{\widetilde{\mathcal{C}}}$. For each $\lambda \in \Lambda$, define $\mathfrak{K}_\lambda^{\sigma,\sigma'} := \mathfrak{K}^{\sigma,\sigma'}(e_\lambda^* \bullet e_\lambda) + (\|\mathfrak{K}^{\sigma,\sigma'}\| 1_{\tilde{\mathcal{C}}} - \mathfrak{K}^{\sigma,\sigma'}(e_\lambda^* e_\lambda))\delta$. Mappings $\mathfrak{K}_\lambda$s are CPD-kernels, and $(\mathfrak{K}_\lambda^{\sigma,\sigma'})_{\lambda \in \Lambda}$ converges pointwise to $\widetilde{\mathfrak{K}}^{\sigma,\sigma'}$. We conclude that $\widetilde{\mathfrak{K}}$ is a CPD-kernel.

Note that $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is also a $\widetilde{\mathfrak{K}}$-family, and $E$ and $F$ are also Hilbert $C^*$-modules over $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{C}}$, respectively.

Extend $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ to a CPD-kernel over $S$ from $\begin{pmatrix} \widetilde{\mathcal{B}} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}$ to $\mathcal{L}_{F_{\mathcal{K}}}$, as above. Restricting this CPD-kernel to $\begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}$ yields the required CPD-kernel.

(b) $\Rightarrow$ (c). Let $n \in \mathbb{N}$. For $\sigma_1, \dots, \sigma_n \in S$, define a linear map $\mathbf{K}$ from $E_n$ to $F_n$ by

$$\mathbf{x} \longmapsto (\mathcal{K}^{\sigma_1}(x_1), \mathcal{K}^{\sigma_2}(x_2), \dots, \mathcal{K}^{\sigma_n}(x_n)) \quad \text{for } \mathbf{x} = (x_1, x_2 \dots, x_n) \in E_n.$$

Fix $l \in \mathbb{N}$, and let $[\mathbf{x}_{ms}]_{m,s=1}^l \in M_l(E_n)$ where

$$\mathbf{x}_{ms} = (x_{ms,1}, x_{ms,2}, \dots, x_{ms,n}) \in E_n.$$

Set

$$A := \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 & 0 \\ a_n & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \vdots & \vdots & & \vdots \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix},$$

which is an $n \times n$ block matrix consisting of blocks of $2 \times 2$ matrices. Define $B_{mk}$ as the matrix $A$ where $a_i = \mathcal{K}^{\sigma_i}(x_{mk,i})$ so that blocks of $2 \times 2$ matrices are elements of $\mathcal{L}_{F_{\mathcal{K}}}$, and thus, $B_{mk}$ is identified with an element of $M_n(\mathcal{L}_{F_{\mathcal{K}}})$. Similarly, define $C_{mk}$ as the matrix $A$ where $a_i = x_{mk,i}$, and thus, $C_{mk}$ is identified with an element of $M_n(\mathcal{L}_E)$. We have

$$\begin{aligned} \|\mathbf{K}_l([\mathbf{x}_{ms}]_{m,s=1}^l)\|^2 &= \|[\mathbf{K}(\mathbf{x}_{ms})]_{m,s=1}^l\|^2 \\ &= \|\langle [\mathbf{K}(\mathbf{x}_{ms})]_{m,s=1}^l, [\mathbf{K}(\mathbf{x}_{ms})]_{m,s=1}^l \rangle\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \left[ \sum_{k=1}^{l} \langle \mathbf{K}(\mathbf{x}_{km}), \mathbf{K}(\mathbf{x}_{ks}) \rangle \right]_{m,s=1}^{l} \right\| \\
&= \left\| \left[ \sum_{k=1}^{l} \left[ \langle \mathcal{K}^{\sigma_i}(x_{km,i}), \mathcal{K}^{\sigma_j}(x_{ks,j}) \rangle \right]_{i,j=1}^{n} \right]_{m,s=1}^{l} \right\| \\
&= \left\| \left[ \sum_{k=1}^{l} B_{km}^* B_{ks} \right]_{m,s=1}^{l} \right\| = \| [B_{ms}]_{m,s=1}^{l} \|^2 \\
&= \left\| \left[ \left[ \begin{pmatrix} \mathfrak{K}^{\sigma_i,\sigma_j} & \mathcal{K}^{\sigma_i^*} \\ \mathcal{K}^{\sigma_j} & \vartheta \end{pmatrix} \right] C_{ms} \right]_{m,s=1}^{l} \right\|^2 \\
&\le \left\| \left[ \begin{pmatrix} \mathfrak{K}^{\sigma_i,\sigma_j} & \mathcal{K}^{\sigma_i^*} \\ \mathcal{K}^{\sigma_j} & \vartheta \end{pmatrix} \right]_l \right\|^2 \| [\mathbf{x}_{ms}]_{m,s=1}^{l} \|^2,
\end{aligned}
$$

where $2 \times 2$ matrices with round brackets are block-wise bounded linear maps on the linking algebra $\mathcal{L}_E$. Therefore, from [**2,** Lemma 3.2.1], it follows that $\mathbf{K}$ is completely bounded.

Let

$$
\mathcal{D} := \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{B}^a(E) \end{pmatrix}
$$

be a $C^*$-subalgebra of $\mathcal{L}_E$ with the unit

$$
1_{\mathcal{D}} := \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_E \end{pmatrix}.
$$

We denote the $*$-homomorphism, which is the restriction of $\begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix}$ to $\mathcal{D}$, by $\theta$.

Without loss of generality, we assume that $\mathcal{B}$ is unital because, if $\mathcal{B}$ is not unital, then we can unitalize it and work as in the proof of "(a) $\Rightarrow$ (b)." Let $(\mathcal{F}, \mathfrak{i})$ be the Kolmogorov decomposition for the CPD-kernel $\begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix}$ where $\sigma, \sigma' \in S$. For each $d \in \mathcal{D}$ and $\sigma \in S$,

$$
\begin{aligned}
\| d\mathfrak{i}(\sigma) - 1_{\mathcal{D}}\mathfrak{i}(\sigma)\theta(d) \|^2 &= \| \langle d\mathfrak{i}(\sigma), d\mathfrak{i}(\sigma) \rangle - \langle d\mathfrak{i}(\sigma), 1_{\mathcal{D}}\mathfrak{i}(\sigma)\theta(d) \rangle \\
&\quad - \langle 1_{\mathcal{D}}\mathfrak{i}(\sigma)\theta(d), d\mathfrak{i}(\sigma) \rangle \\
&\quad + \langle 1_{\mathcal{D}}\mathfrak{i}(\sigma)\theta(d), 1_{\mathcal{D}}\mathfrak{i}(\sigma)\theta(d) \rangle \| \\
&= \| \theta(d^*d) - \theta(d^*d) - \theta(d^*d) + \theta(d^*d) \| = 0.
\end{aligned}
$$

Therefore, for each $\sigma, \sigma' \in S$ and for all $x \in E$, $a \in \mathcal{B}^a(E)$, we have

$$\begin{aligned}\begin{pmatrix} 0 & 0 \\ \mathcal{K}^{\sigma'}(ax) & 0 \end{pmatrix} &= \begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix} \left( \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \right) \\ &= \left\langle \mathfrak{i}(\sigma), \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \mathfrak{i}(\sigma') \right\rangle \\ &= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}^* \mathfrak{i}(\sigma), \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \mathfrak{i}(\sigma') \right\rangle \\ &= \left\langle 1_{\mathcal{D}} \mathfrak{i}(\sigma) \theta \left( \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}^* \right), \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \mathfrak{i}(\sigma') \right\rangle \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \vartheta(a) \end{pmatrix} \begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \vartheta(a)\mathcal{K}^{\sigma'}(x) & 0 \end{pmatrix}.\end{aligned}$$

Hence, $\mathcal{K}^{\sigma'}$ is a left $\mathcal{B}^a(E)$-linear map for each $\sigma' \in S$, and $\vartheta$ is non-degenerate. Observe that the Hilbert $C^*$-module $F_{\mathcal{K}}$ is a $C^*$-correspondence from $\mathcal{B}^a(E)$ to $\mathcal{C}$ with the left action given by $\vartheta$.

(c) $\Leftrightarrow$ (d). If $\mathcal{K}^{\sigma}$ is a left $\mathcal{B}^a(E)$-linear map for each $\sigma \in S$, then

$$\begin{aligned}\langle \mathcal{K}^{\sigma}(y), \mathcal{K}^{\sigma'}(x\langle x', y'\rangle)\rangle &= \langle \mathcal{K}^{\sigma}(y), \mathcal{K}^{\sigma'}(x\, x'^* y')\rangle \\ &= \langle (x\, x'^*)^* \mathcal{K}^{\sigma}(y), \mathcal{K}^{\sigma'}(y')\rangle \\ &= \langle \mathcal{K}^{\sigma}(x' x^* y), \mathcal{K}^{\sigma'}(y')\rangle \\ &= \langle \mathcal{K}^{\sigma}(x'\langle x, y\rangle), \mathcal{K}^{\sigma'}(y')\rangle,\end{aligned}$$

for all $x, y, x', y' \in E$ and $\sigma, \sigma' \in S$.

Conversely, using the equation in condition (d), we define an action $\vartheta$ on $F_{\mathcal{K}}$, of the algebra $\mathcal{F}(E)$ of all finite rank operators on $E$, by

$$\vartheta(x' x^*)\mathcal{K}^{\sigma}(y) := \mathcal{K}^{\sigma}(x' x^* y) \quad \text{for all } x, x', y \in E.$$

Since $\vartheta$ is bounded on $\mathcal{F}(E)$, it naturally extends as an adjointable action of $\mathcal{K}(E)$ on $F_{\mathcal{K}}$. Since $E$ is full, we can obtain an approximate unit $(\sum_{n=1}^{k_\lambda} \langle x_n^\lambda, y_n^\lambda\rangle)_{\lambda\in\Lambda}$ for $\mathcal{B}$ where $x_n^\lambda,\ y_n^\lambda \in E$. Using this approximate unit, it follows that $\vartheta$ is non-degenerate.

We can further extend this action to an action of $\mathcal{B}^a(E)$ on $F_{\mathcal{K}}$, cf., [**8,** Proposition 2.1]).

(c) $\Rightarrow$ (a). Let $n \in \mathbb{N}$. The algebraic tensor product ${E_n}^* \bigotimes_{\mathrm{alg}} E_n = \mathrm{span}\langle E_n, E_n\rangle$, cf., [**8,** Proposition 4.5]). Note that ${E_n}^* \bigotimes_{\mathrm{alg}} E_n$ is a dense subset of $M_n(\mathcal{B})$. Set $\sigma_1, \dots, \sigma_n \in S$, and let $\mathbf{K}$ be defined as above. For each $k \in \mathbb{N}$, we define $\mathbf{K}^k : (E_n)^k \to (F_n)^k$ by

$$\mathbf{K}^k(\mathbf{x}^k) := (\mathbf{K}(\mathbf{x}_1), \mathbf{K}(\mathbf{x}_2), \dots, \mathbf{K}(\mathbf{x}_k))^t,$$

where $\mathbf{x}^k = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k)^t \in (E_n)^k$. Define a linear map $[\mathfrak{K}^{\sigma_i,\sigma_j}]_{i,j=1}^n : {E_n}^* \bigotimes_{\mathrm{alg}} E_n \to M_n(\mathcal{C})$ by

$$[\mathfrak{K}^{\sigma_i,\sigma_j}] \left( \sum_{l=1}^k \langle \mathbf{x}_l, \mathbf{y}_l\rangle \right) := \langle \mathbf{K}^k(\mathbf{x}^k), \mathbf{K}^k(\mathbf{y}^k)\rangle,$$

where $\mathbf{x}^k = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k)^t$, $\mathbf{y}^k = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k)^t \in (E_n)^k$, i.e., $\langle \mathbf{x}^k, \mathbf{y}^k\rangle = \sum_{i=1}^k \langle \mathbf{x}_i, \mathbf{y}_i\rangle$.

First, we prove that $[\mathfrak{K}^{\sigma_i,\sigma_j}]$ is bounded. We have

$$\left\| [\mathfrak{K}^{\sigma_i,\sigma_j}] \left( \sum_{l=1}^k \langle \mathbf{x}_l, \mathbf{y}_l\rangle \right) \right\| = \|\langle \mathbf{K}^k(\mathbf{x}^k), \mathbf{K}^k(\mathbf{y}^k)\rangle\| \le \|\mathbf{K}\|_{cb}^2 \|\mathbf{x}^k\| \|\mathbf{y}^k\|.$$

For $0 < \alpha < 1$, we decompose $\mathbf{x}^{k*}$ as $\mathbf{w}_\alpha^k |\mathbf{x}^{k*}|^\alpha$ (cf. [**8,** Lemma 4.4]; [**15,** Lemma 2.9]), where $\mathbf{w}_\alpha^k := |\mathbf{x}^{k*}|^{1-\alpha}$. So, as $\alpha \to 1$, we have

$$\begin{aligned} \left\| \sum_{l=1}^k \langle \mathbf{x}_l, \mathbf{y}_l\rangle \right\| &= \|\langle \mathbf{x}^k, \mathbf{y}^k\rangle\| = \|\mathbf{x}^{k*} \otimes \mathbf{y}^k\| \\ &= \|\mathbf{w}_\alpha^k |\mathbf{x}^{k*}|^\alpha \otimes \mathbf{y}^k\| = \|\mathbf{w}_\alpha^k \otimes |\mathbf{x}^{k*}|^\alpha \mathbf{y}^k\| \\ &\le \|\mathbf{w}_\alpha^k\| \| |\mathbf{x}^{k*}|^\alpha \mathbf{y}^k\| \longrightarrow \| |\mathbf{x}^{k*}| \mathbf{y}^k\| = \|\langle \mathbf{x}^k, \mathbf{y}^k\rangle\|. \end{aligned}$$

In the above equation array, we have used the facts that $\|\mathbf{w}_\alpha^k\| = \sup_{\lambda \in \sigma(|\mathbf{x}^{k*}|)} \lambda^{1-\alpha} = \|\mathbf{x}^{k*}\|^{1-\alpha} \to 1$, and $|\mathbf{x}^{k*}|^\alpha$ converges in norm to $|\mathbf{x}^{k*}|$. We deduce that, for each $\epsilon > 0$, there exists an $\alpha$ such that

$$\|\mathbf{w}_\alpha^k\| \| |\mathbf{x}^{k*}|^\alpha \mathbf{y}^k\| \le \left\| \sum_{l=1}^k \langle \mathbf{x}_l, \mathbf{y}_l\rangle \right\| + \epsilon.$$

Let $\mathbf{x}'^k := \mathbf{w}_\alpha^{k*} \in (E_n)^k$ and $\mathbf{y}'^k = |\mathbf{x}^{k*}|^\alpha \mathbf{y}^k \in (E_n)^k$. Then

$$\|\langle \mathbf{x}'^k, \mathbf{y}'^k \rangle\| \le \|\mathbf{x}'^k\| \|\mathbf{y}'^k\| \le \left\| \sum_{l=1}^k \langle \mathbf{x}_l, \mathbf{y}_l \rangle \right\| + \epsilon,$$

and

$$\begin{aligned}\langle \mathbf{x}'^k, \mathbf{y}'^k \rangle &= \mathbf{x}'^{k*} \otimes \mathbf{y}'^k = \mathbf{x}'^{k*} \otimes \mathbf{y}'^k \\ &= \mathbf{w}_\alpha^k \otimes |\mathbf{x}^{k*}|^\alpha \mathbf{y}^k = \mathbf{w}_\alpha^k |\mathbf{x}^{k*}|^\alpha \otimes \mathbf{y}^k = \langle \mathbf{x}^k, \mathbf{y}^k \rangle.\end{aligned}$$

Therefore, $[\mathfrak{K}^{\sigma_i,\sigma_j}]$ is bounded.

Because $E_n$ is full, as in the case $(c) \Leftrightarrow (d)$, we can obtain the approximate unit $e_\lambda = \langle \mathbf{X}_\lambda, \mathbf{Y}_\lambda \rangle$ for $M_n(\mathcal{B})$, where

$$\mathbf{X}_\lambda = (\mathbf{x}_1^\lambda, \mathbf{x}_2^\lambda, \ldots, \mathbf{x}_{k_\lambda}^\lambda)^t, \mathbf{Y}_\lambda = (\mathbf{y}_1^\lambda, \mathbf{y}_2^\lambda, \ldots, \mathbf{y}_{k_\lambda}^\lambda)^t \in (E_n)^{k_\lambda}.$$

Let $B$ be a positive element in $M_n(\mathcal{B})$, and let $t_\lambda$ be the positive square root of the rank 1 operator $\mathbf{X}_\lambda B \mathbf{X}_\lambda^*$ in $\mathcal{K}((E_n)^{k_\lambda})$. Finally, using $e_\lambda^* B e_\lambda \xrightarrow{\lambda} B$ in norm and

$$\begin{aligned}[\mathfrak{K}^{\sigma_i,\sigma_j}](e_\lambda^* B e_\lambda) &= [\mathfrak{K}^{\sigma_i,\sigma_j}](\mathbf{Y}_\lambda^* \mathbf{X}_\lambda B \mathbf{X}_\lambda^* \mathbf{Y}_\lambda) \\ &= [\mathfrak{K}^{\sigma_i,\sigma_j}](\langle t_\lambda \mathbf{Y}_\lambda, t_\lambda \mathbf{Y}_\lambda \rangle) \\ &= \langle \mathbf{K}^{k_\lambda}(t_\lambda \mathbf{Y}_\lambda), \mathbf{K}^{k_\lambda}(t_\lambda \mathbf{X}_\lambda) \rangle \ge 0,\end{aligned}$$

we infer that $[\mathfrak{K}^{\sigma_i,\sigma_j}](B) \ge 0$. □

Let $G$ be a locally compact group. Suppose that $E$ is a full Hilbert $C^*$-module over a unital $C^*$-algebra $\mathcal{B}$ and that $(G, \eta, E)$ is a dynamical system on $E$. We define a $C^*$-dynamical system on the linking algebra $\mathcal{L}_E$ as follows. For each $s \in G$, let us define $\mathrm{Ad}\,\eta_s(a) := \eta_s a \eta_{s^{-1}}$ for $a \in \mathcal{B}^a(E)$, and define $\eta_s^*(x^*) := \eta_s(x)^*$ for $x \in E$. Denote by $\theta$ the action of $G$ on $\mathcal{L}_E$, which is given by

$$\theta_s \begin{pmatrix} b & x^* \\ y & a \end{pmatrix} := \begin{pmatrix} \alpha_s^\eta(b) & \eta_s^*(x^*) \\ \eta_s(y) & \mathrm{Ad}\,\eta_s a \end{pmatrix}$$

for all $s \in G$, $a \in \mathcal{B}^a(E)$, $b \in \mathcal{B}$ and $x, y \in E$. It is easy to check that we obtain a $C^*$-dynamical system $(G, \theta, \mathcal{L}_E)$.

**Theorem 3.3.** *Let $E$ be a full Hilbert $C^*$-module over a unital $C^*$-algebra $\mathcal{B}$, and let $F$ be a $C^*$-correspondence from $\mathcal{D}$ to $\mathcal{C}$ where $\mathcal{C}$ and*

*$\mathcal{D}$ are unital $C^*$-algebras. Let $u : G \to \mathcal{UC}$ and $u' : G \to \mathcal{UD}$ be unitary representations of a locally compact group $G$, and let $(G, \eta, E)$ be a dynamical system on $E$. Assume $S$ to be a set and $\mathcal{K}^\sigma$ to be a linear map from $E$ to $F$ for each $\sigma \in S$. Let $F_\mathcal{K} := \overline{\operatorname{span}}\{\mathcal{K}^\sigma(x)c : x \in E,\ c \in \mathcal{C},\ \sigma \in S\}$. Then the following statements are equivalent:*

(a) *there exists a unique* CPD-*kernel* $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ *such that* $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is a* $(u', u)$-*covariant* $\mathfrak{K}$-*family with respect to the dynamical system* $(G, \eta, E)$.

(b) $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *extends to block-wise bounded linear maps* $\begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix}$ *from* $\mathcal{L}_E$ *to* $\mathcal{L}_{F_\mathcal{K}}$ *forming a* CPD-*kernel over* $S$ *from* $\mathcal{L}_E$ *to* $\mathcal{L}_{F_\mathcal{K}}$, *where* $\vartheta$ *is a* $*$-*homomorphism, i.e.,* $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is a* CPD-H-*extendable family. The kernel* $\begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix}$ *is* $\omega$-*covariant with respect to* $(G, \theta, \mathcal{L}_E)$ *where* $\omega : G \to \mathcal{UL}_{F_\mathcal{K}}$ *is a unitary representation.*

(c) *For each finite choice* $\sigma_1, \ldots, \sigma_n \in S$ *the map from* $E_n$ *to* $F_n$ *defined by*
$$\mathbf{x} \longmapsto (\mathcal{K}^{\sigma_1}(x_1), \mathcal{K}^{\sigma_2}(x_2), \ldots, \mathcal{K}^{\sigma_n}(x_n))$$
*for* $\mathbf{x} = (x_1, x_2 \ldots, x_n) \in E_n$, *is completely bounded. Moreover,* $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is* $(u', u)$-*covariant with respect to* $(G, \eta, E)$, $F_\mathcal{K}$ *is a correspondence from* $\mathcal{B}^a(E)$ *to* $\mathcal{C}$ *such that the action of* $\mathcal{B}^a(E)$ *on* $F_\mathcal{K}$ *is non-degenerate and, for each* $\sigma \in S$, $\mathcal{K}^\sigma$ *is a left* $\mathcal{B}^a(E)$-*linear map.*

(d) *For each finite choice* $\sigma_1, \ldots, \sigma_n \in S$ *the map from* $E_n$ *to* $F_n$ *defined by*
$$\mathbf{x} \longmapsto (\mathcal{K}^{\sigma_1}(x_1), \mathcal{K}^{\sigma_2}(x_2), \ldots, \mathcal{K}^{\sigma_n}(x_n))$$
*for* $\mathbf{x} = (x_1, x_2 \ldots, x_n) \in E_n$, *is completely bounded, and* $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is* $(u', u)$-*covariant with respect to* $(G, \eta, E)$ *satisfying, for* $x, y, x', y' \in E$,
$$\langle \mathcal{K}^\sigma(y), \mathcal{K}^{\sigma'}(x\langle x', y'\rangle)\rangle = \langle \mathcal{K}^\sigma(x'\langle x, y\rangle), \mathcal{K}^{\sigma'}(y')\rangle.$$

*Proof.* We use the same notation as in the proof of part (a) $\Rightarrow$ (b) of Theorem 3.2. For each $s \in G$, define a map $\omega_s : \mathcal{L}_F \to \mathcal{L}_F$ by
$$\omega_s \begin{pmatrix} c & x^* \\ y & a \end{pmatrix} := \begin{pmatrix} u_s c & u_s x^* \\ u'_s y & u'_s a \end{pmatrix}$$

for all $c \in \mathcal{C}$, $x, y \in F$ and $a \in \mathcal{B}^a(F)$. The mapping $\omega : G \to \mathcal{UL}_F$ is a unitary representation. Using Theorem 2.4, we obtain a unitary representation $w' : G \to \mathcal{U}\mathcal{B}^a(E \bigotimes_{\mathcal{B}} \mathcal{F})$ defined by

$$w'_t(x \otimes b\mathfrak{i}(\sigma)c) := \eta_t(x) \otimes v_t(b\mathfrak{i}(\sigma)c)$$

for all $b \in \mathcal{B}$, $c \in \mathcal{C}$, $x \in E$, $\sigma \in S$ and $t \in G$. Further, it satisfies $\nu w'_t = u'_t \nu$ for all $t \in G$. Thus, we have

$$\vartheta(\eta_s a \eta_{s^{-1}}) = \nu((\eta_s a \eta_{s^{-1}}) \otimes \mathrm{id}_{\mathcal{F}})\nu^* = \nu w'_s(a \otimes \mathrm{id}_{\mathcal{F}}) w'_{s^{-1}} \nu^* = u'_s \vartheta(a) u'_{s^{-1}}$$

for all $s \in G$ and $a \in \mathcal{B}^a(E)$. Therefore,

$$\begin{aligned}\begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix} \theta_s \begin{pmatrix} b & x^* \\ y & a \end{pmatrix} &= \begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'}(\alpha^\eta_s(b)) & \mathcal{K}^{\sigma^*}(\eta^*_s(x^*)) \\ \mathcal{K}^{\sigma'}(\eta_s(y)) & \vartheta(Ad\eta_s a) \end{pmatrix} \\ &= \omega_s \begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix} \begin{pmatrix} b & x^* \\ y & a \end{pmatrix} \omega^*_s\end{aligned}$$

for all $s \in G$, $a \in \mathcal{B}^a(E)$, $b \in \mathcal{B}$, $\sigma, \sigma' \in S$ and $x, y \in E$. $\square$

**4. Application to the dilation theory of CPD-kernels.** Suppose $E$ and $F$ are Hilbert $C^*$-modules over $C^*$-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Let $S$ be a set, and let $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ be a CPD-kernel. Let $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ be a $\mathfrak{K}$-family where $\mathcal{K}^\sigma$ is a map from $E$ to $F$ for each $\sigma \in S$. Recall that there exists a Kolmogorov decomposition $(\mathcal{F}, \mathfrak{i})$ of $\mathfrak{K}$. From Theorem 2.2, it follows that there is an isometry $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F} \to F$ such that

$$\nu(x \otimes \mathfrak{i}(\sigma)) = \mathcal{K}^\sigma(x) \quad \text{for all } x \in E,\ \sigma \in S.$$

If $F_{\mathcal{K}}$ is complemented in $F$, then we obtain a $*$-homomorphism $\vartheta$ from $\mathcal{B}^a(E)$ to $\mathcal{B}^a(F)$ defined by $\nu(\bullet \otimes \mathrm{id}_{\mathcal{F}})\nu^*$. Also, if $\xi$ is a unit vector in $E$, i.e., $\langle \xi, \xi \rangle = 1$, then the next diagram commutes.

$$\text{(4.1)} \qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{\mathfrak{K}^{\sigma,\sigma'}} & \mathcal{C} \\ {\scriptstyle \xi\bullet\xi^*}\downarrow & & \uparrow{\scriptstyle \langle \nu(\xi\otimes\mathfrak{i}(\sigma)), \bullet\nu(\xi\otimes\mathfrak{i}(\sigma'))\rangle} \\ \mathcal{B}^a(E) & \xrightarrow[\vartheta]{} & \mathcal{B}^a(F) \end{array}$$

Here, $b \mapsto \xi b \xi^*$ is a representation of $\mathcal{B}$ on $E$. In fact, to obtain the above commuting diagram, it is sufficient to assume that there exist a $C^*$-correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$, a map $\mathfrak{i} : S \to \mathcal{F}$, a Hilbert

$\mathcal{B}$-module $E$, an adjointable isometry $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F} \to F$ and a unit vector $\xi \in E$. For this, we set $\mathfrak{K}^{\sigma,\sigma'} := \langle \mathfrak{i}(\sigma), \bullet \mathfrak{i}(\sigma') \rangle$ for $\sigma, \sigma' \in S$ and $\vartheta := \nu(\bullet \otimes \mathrm{id}_{\mathcal{F}})\nu^*$.

If $\mathfrak{i}(\sigma)$s are also unit vectors, then $\mathfrak{K}^{\sigma,\sigma'}$ is a unital map for each $\sigma, \sigma' \in S$, and, in this case, we say that kernel $\mathfrak{K}$ is *Markov* and the dilation $\vartheta$ of $\mathfrak{K}$ is a *weak dilation.* Change the map $\xi \bullet \xi^*$ by the map $\langle \xi, \bullet \xi \rangle$ and reverse the arrow of this map. Now substitute $\mathcal{K}^\sigma(\xi) = \nu(\xi \otimes \mathfrak{i}(\sigma))$ in the above diagram to obtain the commuting diagram:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mathfrak{K}^{\sigma,\sigma'}"] & \mathcal{C} \\ \mathcal{B}^a(E) \arrow[u, "{\langle \xi, \bullet \xi \rangle}"] \arrow[r, "\vartheta"'] & \mathcal{B}^a(F) \arrow[u, "{\langle \mathcal{K}^\sigma(\xi), \bullet \mathcal{K}^{\sigma'}(\xi) \rangle}"'] \end{tikzcd} \tag{4.2}$$

This motivates us to introduce a notion of dilation of a CPD-kernel $\mathfrak{K}$ over $S$ whenever there is a family of maps $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ between some Hilbert $C^*$-modules and a commuting diagram similar to (4.2).

**Definition 4.1.** Let $E$ and $F$ be Hilbert $C^*$-modules over $C^*$-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Let $S$ be a set, and let $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ be a CPD-kernel. A $*$-homomorphism $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is a CPDH-*quasi-dilation* of $\mathfrak{K}$ if there is a linear map $\mathcal{K}^\sigma$ from $E$ to $F$ for each $\sigma \in S$ such that

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mathfrak{K}^{\sigma,\sigma'}"] & \mathcal{C} \\ \mathcal{B}^a(E) \arrow[u, "{\langle x, \bullet x' \rangle}"] \arrow[r, "\vartheta"'] & \mathcal{B}^a(F) \arrow[u, "{\langle \mathcal{K}^\sigma(x), \bullet \mathcal{K}^{\sigma'}(x') \rangle}"'] \end{tikzcd} \tag{4.3}$$

commutes for all $x, x' \in E$. A CPDH-quasi-dilation $\vartheta$ is called

(a) a CPDH-*dilation* if $E$ is full.
(b) *strict* if the $*$-homomorphism $\vartheta$ is strict.

A CPDH-(quasi-)dilation $\vartheta$ is called a CPDH$_0$-(*quasi-*)*dilation* if $\vartheta$ is a unital $*$-homomorphism.

**Proposition 4.2.** *Let $\vartheta$ be a* $\mathrm{CPDH}_0$*-quasi-dilation of a* CPD*-kernel* $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$. *If* $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is a family of maps from $E$ to $F$ such that the diagram* (4.3) *commutes, then* $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is a $\mathfrak{K}$-family where*

$$\mathcal{K}^\sigma(ax) = \vartheta(a)\mathcal{K}^\sigma(x) \quad \textit{for } x \in E,\ a \in \mathcal{B}^a(E),\ \sigma \in S.$$

*Proof.* Since diagram (4.3) commutes, for $x \in E$, $a \in \mathcal{B}^a(E)$ and $\sigma, \sigma' \in S$, we get

$$\langle \mathcal{K}^\sigma(x), \vartheta(a)\mathcal{K}^{\sigma'}(x') \rangle = \langle \mathcal{K}^\sigma(x), \mathcal{K}^{\sigma'}(ax') \rangle. \tag{4.4}$$

As $\vartheta$ is unital, $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is a $\mathfrak{K}$-family. Thus, by setting $F_{\mathcal{K}} := \overline{\mathrm{span}}\{\mathcal{K}^\sigma(e)c : e \in E,\ c \in \mathcal{C},\ \sigma \in S\}$ and using equation 4.4 we get a $*$-homomorphism $\vartheta_{\mathcal{K}} : \mathcal{B}^a(E) \to \mathcal{B}^a(F_{\mathcal{K}})$ which is defined by $\vartheta_{\mathcal{K}}(a)\mathcal{K}^\sigma(x) = \mathcal{K}^\sigma(ax)$ for $x \in E$, $a \in \mathcal{B}^a(E)$, $\sigma, \sigma' \in S$. We obtain

$$\langle y, \vartheta_{\mathcal{K}}(a)y' \rangle = \langle y, \vartheta(a)y' \rangle \quad \text{for all } a \in \mathcal{B}^a(E) \text{ and } y, y' \in F_{\mathcal{K}}.$$

Thus, $\vartheta(a)y = \vartheta_{\mathcal{K}}(a)y$ for all $y \in F_{\mathcal{K}}$ and $a \in \mathcal{B}^a(E)$. □

**Definition 4.3.** A family of maps $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ from $E$ to $F$ is called (*strict*) $\mathrm{CPDH}_0$-*family*, if it extends to block-wise bounded linear maps from $\mathcal{L}_E$ to $\mathcal{L}_F$ forming a CPD-kernel over $S$ whose $(2,2)$-corner is a unital (strict) $*$-homomorphism.

We remark that the acronym CPDH is used instead of CPD-H *extendable* if we have the Hilbert $C^*$-module $F$ instead of $F_{\mathcal{K}}$ in the statement of Theorem 3.2 (b).

**Proposition 4.4.** *Let $\mathcal{B}$ be unital. If $\vartheta$ is a strict* $\mathrm{CPDH}_0$*-dilation of a* CPD*-kernel* $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ *and* $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is a family of maps from $E$ to $F$ such that diagram* (4.3) *commutes, then* $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ *is a strict* $\mathrm{CPDH}_0$*-family.*

*Proof.* Let $(\mathcal{F}_{\mathfrak{K}}, \mathfrak{i})$ be the Kolmogorov decomposition of the CPD-kernel $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$. Because $\vartheta$ is a strict unital homomorphism from $\mathcal{B}^a(E)$ to $\mathcal{B}^a(F)$, using the representation theorem [**9,** Theorem 1.4], we obtain a $C^*$-correspondence $\mathcal{F}_\vartheta := E^* \bigotimes_\vartheta F$ (another notation for $E^* \bigotimes_{\mathcal{B}^a(E)} F$) from $\mathcal{B}$ to $\mathcal{C}$ and a unitary $\nu : E \bigotimes_{\mathcal{B}} \mathcal{F}_\vartheta \to F$, defined by

$$\nu(x' \otimes (x^* \otimes y)) := \vartheta(x'x^*)y \quad \text{for all } x, x' \in E \text{ and } y \in F$$

such that we obtain $\vartheta = \nu(\bullet \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^*$. It is immediate from Proposition 4.2 that the map from $\mathcal{F}_\mathfrak{K}$ onto $E^* \bigotimes_\vartheta F_\mathcal{K} \subset \mathcal{F}_\vartheta$ defined by $\langle x, x'\rangle\mathfrak{i}(\sigma) \mapsto x^* \otimes \mathcal{K}^\sigma(x')$ for all $x, x' \in E$ and $\sigma \in S$, is a bilinear unitary. Now we identify $\mathcal{F}_\mathfrak{K} \subset \mathcal{F}_\vartheta$, and we have $\mathfrak{i}(\sigma) \in \mathcal{F}_\vartheta$ for all $\sigma \in S$. Further, we obtain

$$\nu(x \otimes \langle x', x''\rangle\mathfrak{i}(\sigma)) = \nu(x \otimes (x'^* \otimes \mathcal{K}^\sigma(x''))) = \vartheta(xx'^*)\mathcal{K}^\sigma(x'') = \mathcal{K}^\sigma(x\langle x', x''\rangle)$$

for all $x, x', x'' \in E$, where the last equality follows from Proposition 4.2. Since $E$ is full and $\mathcal{B}$ is unital, we get $\mathcal{K}^\sigma(x) = \nu(x \otimes \mathfrak{i}(\sigma))$ for $x \in E$.

For each $\sigma \in S$, we have

$$\begin{pmatrix} \mathfrak{i}(\sigma) & 0 \\ 0 & \nu^* \end{pmatrix} \in \mathcal{B}^r\left(\begin{pmatrix} \mathcal{C} \\ F \end{pmatrix}, \begin{pmatrix} \mathcal{B} \\ E \end{pmatrix} \bigotimes_\mathcal{B} \mathcal{F}_\vartheta\right).$$

Since

$$\left(\begin{pmatrix} b & x^* \\ x' & a \end{pmatrix} \otimes \mathrm{id}_{\mathcal{F}_\vartheta}\right)\begin{pmatrix} \mathfrak{i}(\sigma) & 0 \\ 0 & \nu^* \end{pmatrix}\begin{pmatrix} c \\ y \end{pmatrix} = \begin{pmatrix} b\mathfrak{i}(\sigma)c + (x^* \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y \\ x' \otimes \mathfrak{i}(\sigma)c + (a \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y \end{pmatrix},$$

we have

$$\begin{aligned}
&\Big\langle \left(\begin{pmatrix} b_1 & x_1^* \\ x_1' & a_1 \end{pmatrix} \otimes \mathrm{id}_{\mathcal{F}_\vartheta}\right)\begin{pmatrix} \mathfrak{i}(\sigma) & 0 \\ 0 & \nu^* \end{pmatrix}\begin{pmatrix} c_1 \\ y_1 \end{pmatrix}, \\
&\quad \left(\begin{pmatrix} b_2 & x_2^* \\ x_2' & a_2 \end{pmatrix} \otimes \mathrm{id}_{\mathcal{F}_\vartheta}\right)\begin{pmatrix} \mathfrak{i}(\sigma') & 0 \\ 0 & \nu^* \end{pmatrix}\begin{pmatrix} c_2 \\ y_2 \end{pmatrix}\Big\rangle \\
&\quad = c_1^*\langle \mathfrak{i}(\sigma), b_1^* b_2 \zeta_j\rangle c_2 + c_1^*\langle \mathfrak{i}(\sigma), b_1^*(x_2^* \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y_2\rangle \\
&\qquad + \langle (x_1^* \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y_1, b_2\mathfrak{i}(\sigma')\rangle c_2 \\
&\qquad + \langle (x_1^* \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y_1, (x_2^* \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y_2\rangle \\
&\qquad + c_1^*\langle x_1' \otimes \mathfrak{i}(\sigma), x_2' \otimes \mathfrak{i}(\sigma')\rangle c_2 + c_1^*\langle x_1' \otimes \mathfrak{i}(\sigma), (a_2 \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y_2\rangle \\
&\qquad + \langle (a_1 \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y_1, x_2' \otimes \mathfrak{i}(\sigma')\rangle c_2 \\
&\qquad + \langle (a_1 \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y_1, (a_2 \otimes \mathrm{id}_{\mathcal{F}_\vartheta})\nu^* y_2\rangle \\
&\quad = c_1^*\mathfrak{K}^{\sigma,\sigma'}(b_1^* b_2)c_2 + c_1^*\langle \mathcal{K}^\sigma(x_2 b_1), y_2\rangle + \langle y_1, \mathcal{K}^{\sigma'}(x_1 b_2)\rangle c_2 \\
&\qquad + \langle y_1, \vartheta(x_1 x_2^*)y_2\rangle + c_1^*\mathfrak{K}^{\sigma,\sigma'}(\langle x_1', x_2'\rangle)c_2 \\
&\qquad + c_1^*\langle \mathcal{K}^\sigma(a_2^* x_1'), y_2\rangle + \langle y_1, \mathcal{K}^{\sigma'}(a_1^* x_2')\rangle c_2 + \langle y_1, \vartheta(a_1^* a_2)y_2\rangle
\end{aligned}$$

$$= \left\langle \begin{pmatrix} c_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix} \left( \begin{pmatrix} b_1 & x_1^* \\ x_1' & a_1 \end{pmatrix}^* \begin{pmatrix} b_2 & x_2^* \\ x_2' & a_2 \end{pmatrix} \right) \begin{pmatrix} c_2 \\ y_2 \end{pmatrix} \right\rangle$$

for all $x_1, x_2, x_1', x_2' \in E$; $b_1, b_2 \in \mathcal{B}$; $c_1, c_2 \in \mathcal{C}$; $y_1, y_2 \in F$ and $a_1, a_2 \in \mathcal{B}^a(E)$. Therefore, $\begin{pmatrix} \mathfrak{K}^{\sigma,\sigma'} & \mathcal{K}^{\sigma^*} \\ \mathcal{K}^{\sigma'} & \vartheta \end{pmatrix}$ forms a CPD-kernel, and hence, $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is a strictly CPDH$_0$-family. □

We further generalize the notion of CPDH-dilation as follows:

**Definition 4.5.** Suppose $E$ and $F$ are Hilbert $C^*$-modules over $C^*$-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Let $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ be a CPD-kernel. Let $\mathfrak{P}$ be a CPD-kernel over the set $E$ from $\mathcal{B}^a(E)$ to $\mathcal{B}$, and let $\mathfrak{L}$ be a CPD-kernel over the set $\{\mathcal{K}^\sigma(x) : \sigma \in S, x \in E\}$ from $\mathcal{B}^a(F)$ to $\mathcal{C}$. A homomorphism $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is called a *generalized* CPDH-*quasi-dilation* of $\mathfrak{K}$ if $\{\mathcal{K}^\sigma\}_{\sigma \in S}$ is a collection of linear maps from $E$ to $F$ such that the next diagram commutes for all $x, x' \in E$ and $\sigma, \sigma' \in S$:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mathfrak{K}^{\sigma,\sigma'}"] & \mathcal{C} \\ \mathcal{B}^a(E) \arrow[u, "\mathfrak{P}^{x,x'}"] \arrow[r, "\vartheta"'] & \mathcal{B}^a(F) \arrow[u, "\mathfrak{L}^{\mathcal{K}^\sigma(x), \mathcal{K}^{\sigma'}(x')}"'] \end{tikzcd} \tag{4.5}$$

A generalized CPDH-quasi-dilation $\theta$ is called a *generalized* CPDH-*dilation* if $E$ is full.

Let $\mathfrak{L}$ be a CPD-kernel over the set $S' = \{\mathcal{K}^\sigma(x) : \sigma \in S, x \in E\}$ from a unital $C^*$-algebra $\mathcal{B}^a(F)$ to a $C^*$-algebra $\mathcal{C}$. We get the Kolmogorov decomposition $(\mathcal{F}, \mathfrak{i})$ such that

$$\langle \mathfrak{i}(y), a\mathfrak{i}(y') \rangle = \mathfrak{L}^{y,y'}(a) \quad \text{for all } y, y' \in S',\ a \in \mathcal{B}^a(F)$$

and

$$\mathcal{F} = \overline{\operatorname{span}}\{a\mathfrak{i}(y)c : a \in \mathcal{B}^a(F),\ y \in S',\ c \in C\}.$$

Hence, we get

$$\mathfrak{K}^{\sigma,\sigma'}(\mathfrak{P}^{x,x'}(a)) = \langle \mathfrak{i}(\mathcal{K}^\sigma(x)), \vartheta(a)\mathfrak{i}(\mathcal{K}^{\sigma'}(x')) \rangle$$

for each $\sigma, \sigma' \in S$, $x, x' \in E$ and $a \in \mathcal{B}^a(F)$. We denote the homomorphism which gives the left action on $\mathcal{F}$ by $\theta : \mathcal{B}^a(F) \to \mathcal{B}^a(\mathcal{F})$.

Observe that the next diagram commutes for all $x, x' \in E$ and $\sigma, \sigma' \in S$:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\mathfrak{K}^{\sigma,\sigma'}"] & \mathcal{C} \\
\mathcal{B}^a(E) \arrow[u, "\mathfrak{P}^{x,x'}"] \arrow[r, "\theta\circ\vartheta"'] & \mathcal{B}^a(\mathcal{F}) \arrow[u, "\langle \mathfrak{i}(\mathcal{K}^{\sigma}(x)), \bullet \mathfrak{i}(\mathcal{K}^{\sigma'}(x'))\rangle"']
\end{tikzcd}$$

**Proposition 4.6.** *Suppose $E$ and $F$ are Hilbert $C^*$-modules over $C^*$-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Let $\mathfrak{K} : S \times S \to \mathcal{B}(\mathcal{B}, \mathcal{C})$ be a* CPD-*kernel. Let $\mathfrak{P}$ be a* CPD-*kernel over the set $E$ from $\mathcal{B}^a(E)$ to $\mathcal{B}$ defined by $\mathfrak{P}^{x,x'} := \langle x, \bullet x' \rangle$, where $x, x' \in E$, and let $\mathfrak{L}$ be a* CPD-*kernel over the set $\{\mathcal{K}^\sigma(x) : \sigma \in S, x \in E\}$ from $\mathcal{B}^a(F)$ to $\mathcal{C}$. If $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is a generalized* CPDH-*quasi-dilation of $\mathfrak{K}$ with respect to* CPD-*kernels $\mathfrak{P}$ and $\mathfrak{L}$, then $\theta \circ \vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(\mathcal{F})$ is a* CPDH-*quasi-dilation of $\mathfrak{K}$ with respect to maps $\{\mathfrak{i} \circ \mathcal{K}^\sigma : E \to \mathcal{F}\}_{\sigma \in S}$ where $(\mathcal{F}, \mathfrak{i})$ is the Kolmogorov decomposition of $\mathfrak{L}$ and $\theta : \mathcal{B}^a(F) \to \mathcal{B}^a(\mathcal{F})$ is a homomorphism which gives the left action on $\mathcal{F}$.*

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