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VARIANCE AND THE INEQUALITY OF ARITHMETIC AND GEOMETRIC MEANS

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ABSTRACT. A number of recent papers have been devoted to generalizations of the classical AM-GM inequality. Those generalizations which incorporate *variance* have been the most useful in applications to economics and finance. In this paper, we prove an inequality which yields the best possible upper and lower bounds for the geometric mean of a sequence solely in terms of its arithmetic mean and its variance. A particular consequence is the following: among all positive sequences having given length, arithmetic mean and nonzero variance, the geometric mean is maximal when all terms in the sequence except one are equal to each other and are less than the arithmetic mean.

Introduction. Roughly speaking, the discrepancy between the arithmetic and geometric means of a finite sequence tends to increase as the sequence deviates more and more from being constant. The literature contains several generalizations of the classical arithmetic-geometric mean inequality; they differ, in part, by using different measures for the deviation of the sequences from constancy. *Variance*, or *standard deviation*, is a mathematically natural measure of the deviation of a sequence from constancy. In addition, as noted by Aldaz [**1, 2**], Becker [**4**], Estrada [**6**] and Markowitz [**8**], variance is the most useful such measure from the point of view of economics and finance. (Markowitz [**8**] points out that investors are made aware of the arithmetic mean and variance of a portfolio, but there is a need for them to estimate the geometric mean since that is the portfolio's likely long term return; cf. Remark 2.2 below). In this paper, Theorem 1.2 gives bounds for the geometric mean depending solely on the arithmetic mean and variance; these mean-variance bounds are the best possible.

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A discussion of related previous results is given in Section 2. Corollary 2.1 yields an upper bound, depending only on variance, for the numerical difference between the arithmetic and geometric means; cf. [**2**].

1. Let $x_1, x_2, \ldots, x_n$ be a sequence of n real numbers. The (arithmetic) mean μ and variance σ^2 are defined as:

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2. \tag{1.1}$$

This notation always implies that σ is the standard deviation, i.e., the nonnegative square root of the variance. Thus, *mean* will always refer to the arithmetic mean; the quantity $(x_1 x_2 \ldots x_n)^{1/n}$ will be referred to by its complete name, *geometric mean.*

It is clear that, for a fixed mean $\mu > 0$, if σ^2 is sufficiently small, then each x_i will necessarily be positive. Also, for a fixed mean $\mu > 0$, if all x_i are positive, then the variance of the sequence cannot be too large. The precise conditions for the mean and variance with these two properties are given in the next lemma.

Lemma 1.1. *Let $x_1, x_2, \ldots, x_n$ be a sequence of $n \geqq 2$ real numbers with mean $\mu > 0$ and variance σ^2.*

(a) *If $\sigma/\mu < 1/\sqrt{n-1}$, then all terms of the sequence are necessarily positive.*
(b) *If all terms of the sequence are positive, then $\sigma/\mu < \sqrt{n-1}$.*

Proof. Let $\mu > 0$ be fixed. Let S be the $(n-1)$-simplex

$$S = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R^n} \colon x_1 + x_2 + \cdots + x_n = n\mu \text{ and } x_i \geqq 0 \text{ for } i = 1, \ldots, n\}. \tag{1.2}$$

The variance σ^2 of the coordinates of a point $(x_1, x_2, \ldots, x_n)$ of S is related to the distance r from that point to the centroid $C_0 = (\mu, \mu, \ldots, \mu)$ of S by $r^2 = n\sigma^2$.

Let r_1 be the distance from C_0 to a nearest boundary point of S, and let r_2 be the distance from C_0 to a furthest boundary point of S. If $(x_1, x_2, \ldots, x_n)$ has mean μ and, if the distance from $(x_1, x_2, \ldots, x_n)$ to C_0 is $\leqq r_1$, then each coordinate x_i must be nonnegative. Similarly, if $x_1, x_2, \ldots, x_n$ has mean μ and if each x_i is nonnegative, then the distance from $(x_1, x_2, \ldots, x_n)$ to C_0 must be $\leqq r_2$.

The boundary points of S nearest to C_0 are the centroids of each $(n-2)$-face of S, for example, the point $(0, n\mu/(n-1), \ldots, n\mu/(n-1))$. The distance r_1 from C_0 to such a nearest boundary point satisfies $r_1^2 = \mu^2 n/(n-1)$. Therefore, if a sequence $x_1, x_2, \ldots, x_n$ with mean μ has variance $\sigma^2 < r_1^2/n = \mu^2/(n-1)$, then all terms of that sequence are necessarily positive. This proves Lemma 1.1 (a).

Similarly, the boundary points of S furthest from C_0 are the vertices of S. The distance r_2 from C_0 to a vertex of S satisfies $r_2^2 = \mu^2 n(n-1)$. Therefore, if a sequence $x_1, x_2, \ldots, x_n$ with mean μ and variance σ^2 has all positive terms, then $\sigma^2 < r_2^2/n = \mu^2(n-1)$. This proves Lemma 1.1 (b). $\square$

Theorem 1.2. *Let $n \geqq 2$. Let $x_1, x_2, \ldots, x_n$ be real numbers with mean $\mu > 0$ and variance σ^2.*

(a) *If $0 \leq \sigma/\mu < 1/\sqrt{n-1}$, then each x_i is positive and*

(1.3)
$$(\mu - \sigma\sqrt{n-1})\left(\mu + \frac{\sigma}{\sqrt{n-1}}\right)^{n-1} \leq x_1 x_2 \cdots x_n \leq (\mu + \sigma\sqrt{n-1})\left(\mu - \frac{\sigma}{\sqrt{n-1}}\right)^{n-1}.$$

The upper and lower bounds in (1.3) *are sharp.*

(b) *If every term of the sequence $x_1, x_2, \ldots, x_n$ is positive, then $0 \leq \sigma/\mu < \sqrt{n-1}$ and the inequalities* (1.3) *continue to hold. The upper bound is again sharp. In the subrange $1/\sqrt{n-1} \leq \sigma/\mu \leq \sqrt{n-1}$, the lower bound expression in* (1.3) *becomes negative and should be replaced by* 0; *with that understanding the lower inequality will then be best possible for the entire range $0 \leq \sigma/\mu < \sqrt{n-1}$.*

Remark 1.3. Up to a change in order of terms, the sequences which make (1.3) an equality are the following. For $0 \leq \sigma/\mu < \sqrt{n-1}$, the upper bound is attained when

(1.4) $$x_1 = x_2 = \cdots = x_{n-1} = \mu - \frac{\sigma}{\sqrt{n-1}} \quad \text{and} \quad x_n = \mu + \sigma\sqrt{n-1}.$$

For $0 \leq \sigma/\mu < 1/\sqrt{n-1}$, the lower bound is attained when

$$x_1 = x_2 = \cdots = x_{n-1} = \mu + \frac{\sigma}{\sqrt{n-1}} \quad \text{and} \quad x_n = \mu - \sigma\sqrt{n-1}. \tag{1.5}$$

For $1/\sqrt{n-1} \leq \sigma/\mu < \sqrt{n-1}$, there is no minimum among positive sequences with the given μ and σ, but the infimum is 0.

Proof of Theorem 1.2. Let $n \geqq 2$, $\mu > 0$, and σ^2 be fixed. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a real n-vector, and let

$$\begin{aligned} \mathrm{G}(\mathbf{x}) &= x_1 x_2 \cdots x_n, \\ \mathrm{A}(\mathbf{x}) &= x_1 + x_2 + \cdots + x_n, \\ \mathrm{V}(\mathbf{x}) &= (x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_n - \mu)^2. \end{aligned} \tag{1.6}$$

Consider the problem: maximize or minimize $\mathrm{G}(\mathbf{x})$ subject to the constraints

$$\mathrm{A}(\mathbf{x}) = n\mu \quad \text{and} \quad \mathrm{V}(\mathbf{x}) = n\sigma^2. \tag{1.7}$$

We shall refer to this as the *max-min problem.* By a *critical point* for this problem we mean a point $\mathbf{x}$ which satisfies constraints (1.7) and where **grad** $\mathrm{G}(\mathbf{x})$ is in the space spanned by **grad** $\mathrm{A}(\mathbf{x})$ and **grad** $\mathrm{V}(\mathbf{x})$. The method of Lagrange multipliers asserts that the solutions to the max-min problem, which exist by compactness, will be found among the values of G at the critical points. First, we will find all critical points. Then we will consider the restrictions in the theorem regarding positivity and bounds on σ/μ.

Thus, $\mathbf{x}$ will be a critical point if equations (1.7) are satisfied and if there exist numbers λ_1, λ_2 such that

$$\mathbf{grad}\ \mathrm{G}(\mathbf{x}) = \lambda_1\, \mathbf{grad}\ \mathrm{A}(\mathbf{x}) + \lambda_2\, \mathbf{grad}\ \mathrm{V}(\mathbf{x}). \tag{1.8}$$

We have

$$\begin{aligned} \mathbf{grad}\ \mathrm{G}(\mathbf{x}) &= (x_2 x_3 \cdots x_n, \ldots, x_1 x_2 \cdots x_{n-1}), \\ \mathbf{grad}\ \mathrm{A}(\mathbf{x}) &= (1, 1, \ldots, 1), \\ \mathbf{grad}\ \mathrm{V}(\mathbf{x}) &= 2(x_1 - \mu, x_2 - \mu, \ldots, x_n - \mu). \end{aligned} \tag{1.9}$$

If we multiply each side of (1.8) by x_i and equate the ith components on each side, then, via the three equations (1.9), we obtain the n scalar equations

$$x_1 x_2 \cdots x_n = (\lambda_1 - 2\mu\lambda_2)x_i + 2\lambda_2 x_i^2, \quad i = 1, 2, \ldots, n. \tag{1.10}$$

If **grad** $\mathrm{G}(\mathbf{x}) = 0$, then at least two of the coordinates of $\mathbf{x}$ are 0; the converse is also true. Suppose that **grad** $\mathrm{G}(\mathbf{x}) \neq 0$. Then λ_1 and λ_2 are not both 0. Equations (1.10) show that all of the n ordered pairs (x_i, x_i^2) lie on a line

$$(\lambda_1 - 2\mu\lambda_2)x + 2\lambda_2 y = \text{constant}; \tag{1.11}$$

of course, they also lie on the parabola $y = x^2$. Therefore, there are at most two distinct values in the set $\{x_1, x_2, \ldots, x_n\}$.

We have seen that a point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a critical point for the max-min problem only if $\mathbf{x}$ has at most two distinct coordinates, or else **grad** $\mathrm{G}(\mathbf{x}) = 0$. The case of one distinct coordinate $x_1 = x_2 = \cdots = x_n$ occurs if and only if $\sigma = 0$. In this case, the inequalities (1.3) become trivial equalities.

Consider a critical point $\mathbf{x}$ such that **grad** $\mathrm{G}(\mathbf{x}) \neq \mathbf{0}$ and such that the coordinates of $\mathbf{x}$ have exactly two distinct values; denote these two values by a and b, with $b < \mu < a$.

Suppose that the value a occurs i times and b occurs j times, where $0 < i$, $j < n$ and $i + j = n$. The constraints (1.7) require that

$$ia + jb = n\mu \quad \text{and} \quad i(a-\mu)^2 + j(b-\mu)^2 = n\sigma^2. \tag{1.12}$$

To express a and b in terms of i, j, n, μ and σ, solve the first equation in (1.12) for b, substitute the solution into the second equation and obtain

$$i(a-\mu)^2 = j\sigma^2. \tag{1.13}$$

Since $a > \mu$,

$$a = \mu + \sigma\sqrt{\frac{j}{i}}. \tag{1.14}$$

Now, the first equation in (1.12) yields

$$b = \mu - \sigma\sqrt{\frac{i}{j}}. \tag{1.15}$$

Given i, $1 \leqq i \leqq n-1$, there are $\binom{n}{i}$ points which have i coordinates a given by (1.14), and $j = n - i$ coordinates b given by (1.15); these will be called *critical points of type* i. If $\mathbf{x_i}$ is a critical point of type i, then the corresponding critical value of G is

$$G(\mathbf{x}_i) = \left(\mu + \sigma\sqrt{\frac{j}{i}}\right)^i \cdot \left(\mu - \sigma\sqrt{\frac{i}{j}}\right)^j, \qquad i = 1, \ldots, n-1; \qquad j = n - i. \tag{1.16}$$

We now want to order the $n - 1$ critical values in (1.16) according to magnitude.

Let $t = \sigma/\mu$. Define $P_i(t)$ by

$$P_i(t) = \frac{G(\mathbf{x}_i)}{\mu^n}. \tag{1.17}$$

Each $P_i(t), 1 \leqq i \leqq n-1$, can be considered a polynomial in t of degree n:

$$P_i(t) = \left(1 + t\sqrt{\frac{j}{i}}\right)^i \cdot \left(1 - t\sqrt{\frac{i}{j}}\right)^j, \qquad i = 1, \ldots, n-1; \qquad j = n - i. \tag{1.18}$$

Lemma 1.4. *Let* i, j *and* $P_i(t)$ *be given by* (1.18). *If* $1 \leqq i \leqq n-2$, *then*

$$P_i(t) > P_{i+1}(t) \quad \textit{for } 0 < t < \sqrt{\frac{j-1}{i+1}}. \tag{1.19}$$

Consequently,

$$P_{n-1}(t) < \cdots < P_2(t) < P_1(t) \quad \textit{for } 0 < t < \frac{1}{\sqrt{n-1}}. \tag{1.20}$$

Proof of Lemma 1.4. From (1.18), we find that, for $1 \leqq i \leqq n-1$,

$$\frac{d}{dt}\log P_i(t) = \frac{-nt}{(1 + t\sqrt{j/i})(1 - t\sqrt{i/j})}. \tag{1.21}$$

Thus, $P_i(t)$ decreases from 1 to 0 as t goes from 0 to $\sqrt{j/i}$. We have

$$\log P_i(t) = \int_0^t \frac{-n\tau d\tau}{(1+\tau\sqrt{j/i})(1-\tau\sqrt{i/j})}, \quad 0 < t < \sqrt{j/i}. \tag{1.22}$$

According to representation (1.22), in order to prove that, for $1 \leqq i \leqq n-2$,

$$\log P_i(t) > \log P_{i+1}(t) \quad \text{for } 0 < t < \sqrt{\frac{j-1}{i+1}}, \tag{1.23}$$

it suffices to show that, for $0 < \tau < \sqrt{(j-1)/(i+1)}$,

$$\frac{-n\tau}{(1+\tau\sqrt{j/i})(1-\tau\sqrt{i/j})} > \frac{-n\tau}{(1+\tau\sqrt{(j-1)/(i+1)})(1-\tau\sqrt{(i+1)/(j-1)})}. \tag{1.24}$$

For τ in this range, the factors in the denominators of (1.24) are positive and inequality (1.24) can be algebraically simplified to become

$$\sqrt{\frac{j-1}{i+1}} - \sqrt{\frac{i+1}{j-1}} < \sqrt{\frac{j}{i}} - \sqrt{\frac{i}{j}}. \tag{1.25}$$

Replace j by $n-i$ in (1.25); the result may be written as

$$\frac{n-2i}{\sqrt{i(n-i)}} > \frac{n-2(i+1)}{\sqrt{(i+1)(n-(i+1))}}. \tag{1.26}$$

It is evident that the left hand side of (1.26) is a strictly decreasing function of a real variable i in the interval $0 < i < n$ since its derivative with respect to i is negative. Therefore, (1.26) is valid for integers i in the range $1 \leqq i \leqq n-2$, and (1.23) follows. Inequality (1.19) follows from (1.23). Inequality (1.19) implies (1.20) since, for $1 \leqq i \leqq n-2$, $\sqrt{(j-1)/(i+1)}$ takes its minimum when $i = n-2$. This completes the proof. □

We have found all critical points for the max-min problem, namely, for each i, $1 \leqq i \leqq n-1$, there are $\binom{n}{i}$ critical points of type i; the corresponding critical value is given by (1.16). (A critical point of type i can be described geometrically as follows. Consider a ray from the centroid C_0 of the $(n-1)$-simplex S given in (1.2) to the centroid of a k-dimensional face of S, $0 \leqq k \leqq n-2$. The intersection of this ray with the sphere of radius $\sigma\sqrt{n}$ centered at C_0 is a critical point of type

$k+1$.) In addition, there are the critical points $\mathbf{x}$ where $\mathbf{grad}\, G(\mathbf{x}) = 0$, i.e., points $\mathbf{x}$ which have two or more of their coordinates equal to 0, and there is the critical point $(\mu, \ldots, \mu)$ when $\sigma = 0$.

Now, consider Theorem 1.2 (a) where $0 \leq \sigma/\mu < 1/\sqrt{n-1}$. We may assume that $0 < \sigma/\mu < 1/\sqrt{n-1}$ since, as remarked earlier, if $\sigma = 0$, then (1.3) is trivial. Consider the set of points $\mathbf{x}$ in n-space whose coordinates have the given mean μ and variance σ^2. By Lemma 1.1 (a), these $\mathbf{x}$ have positive coordinates. By compactness, the function $G(\mathbf{x})$ restricted to this set attains a maximum and a minimum. Therefore, the maximum and minimum must occur among the critical values $G(\mathbf{x_i})$ given by (1.16). Since $0 < t = \sigma/\mu < 1/\sqrt{n-1}$, we see from (1.17) and (1.20) that, for all $\mathbf{x}$ with the given mean and variance,

$$P_{n-1}(t) = \frac{G(\mathbf{x_{n-1}})}{\mu^n} \leq \frac{G(\mathbf{x})}{\mu^n} \leq \frac{G(\mathbf{x_1})}{\mu^n} = P_1(t), \tag{1.27}$$

which proves (1.3).

Now consider Theorem 1.2 (b). Here, we are given an n-vector $(x_1, x_2, \ldots, x_n)$ where the x_i are positive with mean μ and variance σ^2. By Lemma 1.1 (b), $0 \leq \sigma/\mu < \sqrt{n-1}$. As before, we can dispense with the trivial case $\sigma = 0$. We want to find the maximum of $G(\mathbf{y})$ among all n-vectors $\mathbf{y}$ whose coordinates are positive and have the given μ and σ^2. By compactness, G attains a maximum on the intersection

$$\{\mathrm{A}(\mathbf{y}) = n\mu\} \cap \{\mathrm{V}(\mathbf{y}) = n\sigma^2\} \cap \{y_1 \geqq 0, y_2 \geqq 0, \ldots, y_n \geqq 0\}. \tag{1.28}$$

This maximum must occur at a point $\mathbf{y_0}$ with positive coordinates. Therefore, this maximum is a local maximum for the max-min problem (1.7) and hence occurs at a critical point. That is, $\mathbf{y_0} = \mathbf{x_i}$ for some i, where $\mathbf{x_i}$ is a critical point of type i. Although it is possible in case (b) for $\mathrm{G}(\mathbf{x_1}) < \mathrm{G}(\mathbf{x_i})$ for some $\mathbf{i}$, we can make use of the observation

$$\mathrm{G}(\mathbf{x_1}) = \max_{\mathbf{i}}\{\mathrm{G}(\mathbf{x_i}) : \\ \mathbf{x_i} \text{ is a critical point having all coordinates positive}\}, \tag{1.29}$$

which follows from (1.15), (1.16) and (1.19). Therefore, $\mathbf{y_0} = \mathbf{x_1}$ for some critical point $\mathbf{x_1}$ of type 1. Hence, $x_1 x_2 \cdots x_n \leqq \mathrm{G}(\mathbf{x_1}) = \mu^n P_1(\sigma/\mu)$, which establishes the upper bound in (1.3) for case (b) of Theorem 1.2. This completes the proof of Theorem 1.2. $\square$

2. As explained in [**1, 2, 4, 6, 8**], generalizations of the arithmetic and geometric means inequality which involve only the variance of the sequence are the most useful in applications to economics and finance. Becker [**4**] provides a discussion, with historical references, of the heuristics behind the approximation $R_{\mathrm{A}} - R_{\mathrm{G}} \approx \sigma^2/2$, where R_{A} and R_{G} denote the arithmetic and geometric mean. Markowitz [**8**] considers five different mean-variance approximations for the geometric mean and compares their accuracy for sequences of historical economic data. A general inequality involving weighted means and generalized variances which is optimal within its class is given in [**2,** Theorem 2.4]. When this general inequality is specialized by setting the weights $\alpha = (1/n, 1/n, \ldots, 1/n)$ and $s = 2$, the result for nonnegative sequences becomes:

$$R_{\mathrm{A}} - R_{\mathrm{G}} \leq n\sigma. \tag{2.1}$$

Theorem 1.2 can be applied to obtain a similar type of upper bound for $R_{\mathrm{A}} - R_{\mathrm{G}}$. Indeed, the lower bound in (1.3) implies that $(\mu - \sigma\sqrt{n-1})^n \leq x_1 x_2 \cdots x_n$, and hence, $R_{\mathrm{A}} - R_{\mathrm{G}} \leq \sqrt{n-1}\sigma$. We record this result as a corollary to Theorem 1.2.

Corollary 2.1. *Fix $n \geqq 2$. If $x_1, x_2, \ldots, x_n$ is a positive sequence with mean μ and variance σ^2, then*

$$\mu - (x_1 x_2 \cdots x_n)^{1/n} \leq \sqrt{n-1}\,\sigma. \tag{2.2}$$

Aldaz has shown that there can be no similar lower bound, i.e., there does not exist a constant $k > 0$ such that $k\sigma \leq \mu - (x_1 x_2 \cdots x_n)^{1/n}$ is valid for all positive sequences $x_1, x_2, \ldots, x_n$ with mean μ and variance σ^2, see [**1,** Example 2.1].

A number of generalizations of the AM-GM inequality in the literature involve properties other than variance. Cartwright and Field [**5**] prove an inequality involving weighted arithmetic means, variance and upper and lower bounds for the sequence. In the special case of equal weights their result reduces to

$$\frac{\sigma^2}{2b} \leq R_{\mathrm{A}} - R_{\mathrm{G}} \leq \frac{\sigma^2}{2a}, \tag{2.3}$$

where a and b denote lower and upper bounds, respectively, for the positive sequences being considered, see also [**12**]. For easier comparison with (1.3), we can rewrite (2.3) in the equivalent form

$$\left(\mu - \frac{\sigma^2}{2a}\right)^n \leq x_1 x_2 \cdots x_n \leq \left(\mu - \frac{\sigma^2}{2b}\right)^n. \tag{2.4}$$

Alzer [**3**] proves a refinement of the inequality of [**5**] which incorporates variance but also retains bounds a and b defined above. Tung [**10**] derives inequalities depending only on bounds a and b; his inequalities do not involve variance. Meyer [**9**] extends those results to the harmonic mean. Aldaz [**1**] makes use of the variance of the square roots of the terms of the sequence, and in [**2**] he extends those results to more general weights and variances. Loewner and Mann [**7**] derive an upper bound which involves the maximum and minimum of x_i/μ and does not incorporate variance.

Remark 2.2. We illustrate the relevance of Theorem 1.2 to finance. Consider an investment in a certain asset, A. Suppose that, for n consecutive time periods, the investment returns are $r_1, r_2, \ldots, r_n$; $-1 < r_i$. For example, if the time period is years and if asset A returned 6 percent in the ith year, then $r_i = 0.06$; if it lost 6 percent that year, then $r_i = -0.06$.

An initial investment of \$1 dollar in asset A will be worth \$$X_n$ at the end of the nth year, where $X_n = (1 + r_1) \cdots (1 + r_n)$. Suppose that the sequence $r_1, r_2, \ldots, r_n$ has mean μ_n and variance σ_n^2. Then the sequence $1 + r_1, \ldots, 1 + r_2$ will have mean $1 + \mu_n$ and variance σ_n^2. The terms of this sequence are positive since $-1 < r_i$. By Theorem 1.2,

$$\begin{aligned} (1 + \mu_n - \sigma_n\sqrt{n-1})\left(1 + \mu_n + \frac{\sigma_n}{\sqrt{n-1}}\right)^{n-1} &\leq X_n \\ &\leq (1 + \mu_n + \sigma_n\sqrt{n-1})\left(1 + \mu_n - \frac{\sigma_n}{\sqrt{n-1}}\right)^{n-1}. \end{aligned} \tag{2.5}$$

Remark 1.3 shows, perhaps unexpectedly, that, for fixed μ_n and σ_n with $\sigma_n > 0$, the best investment outcome X_n occurs when all returns but one are identical and below the mean μ_n; the worst outcome X_n occurs when all returns but one are identical and above the mean μ_n.

Before making the initial investment, an investor can estimate the mean and variance of returns for asset A from its historical performance record. Let μ_0 and σ_0 be the values so obtained. For example, suppose that asset A is the S&P 500 index, and suppose that the unit of time is days. Based on the historical record from January 3, 1950 through July 31, 2012, it has been estimated that the daily returns on this asset will have a mean μ_0 of 1.0003 and a standard deviation σ_0 of 0.0098; cf. [**11**].

Suppose that one expects that μ_n and σ_n will be close to their estimated values μ_0 and σ_0; say $|\mu_n - \mu_0| < \epsilon$ and $|\sigma_n - \sigma_0| < \epsilon$, $\epsilon > 0$. Then (2.5) will provide the following estimate for $X_n/(1+\mu_n)^n$, the ratio of outcomes for an n-term investment in asset A to that of an n-term investment in a risk free asset with the same mean:

$$\frac{X_n}{(1+\mu_n)^n} \le \left(1 + \frac{(\sigma_0+\epsilon)\sqrt{n-1}}{1+\mu_0-\epsilon}\right)\left(1 - \frac{\sigma_0-\epsilon}{(1+\mu_0+\epsilon)\sqrt{n-1}}\right)^{n-1}. \tag{2.6}$$

Note that, if $\sigma_0 - \epsilon > 0$ and $1 + \mu_0 - \epsilon > 0$, then the right hand side of (2.6) will tend to 0 as n tends to ∞.

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