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ON ALGEBRAS OF BANACH ALGEBRA-VALUED BOUNDED CONTINUOUS FUNCTIONS

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ABSTRACT. Let X be a completely regular Hausdorff space. We denote by $C(X,A)$ the algebra of all continuous functions on X with values in a complex commutative unital Banach algebra A. Let $C_b(X,A)$ be its subalgebra consisting of all bounded continuous functions and endowed with the uniform norm. In this paper, we give conditions equivalent to the density of a natural continuous image of $X \times \mathcal{M}(A)$ in the maximal ideal space of $C_b(X,A)$.

1. Introduction. Throughout this paper, X will denote a completely regular Hausdorff space, A a complex commutative unital Banach algebra with norm $\|\cdot\|$ and unit element e and $G(A)$ the set of invertible elements of A. We may assume that $\|e\| = 1$. We shall use the following notation for various function spaces:

$C(X,A)$ is the unital algebra of all continuous functions on X with values in A, with pointwise operations and unit element the function on X identically equal to e and which will be denoted simply by e.

$C_b(X,A)$ is the subalgebra of $C(X,A)$ of all bounded continuous functions, provided with the uniform norm $\|\cdot\|_\infty$ given by $\|f\|_\infty = \sup_{x\in X}\|f(x)\|$.

When A is the complex field $\mathbb{C}$, then we shall write $C(X)$ and $C_b(X)$ instead of $C(X,\mathbb{C})$ and $C_b(X,\mathbb{C})$, respectively.

$C_p(X,A)$ is the subalgebra of $C_b(X,A)$ of all continuous functions f such that the closure of its range in A, namely $\mathrm{cl}\,(f(X))$, is compact in A.

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It is easy to see that $C_b(X,A)$ and $C_p(X,A)$ are Banach algebras. In general, $C_p(X,A)$ is a proper subalgebra of $C_b(X,A)$ as the next example shows. Take $X=\mathbb{N}$ endowed with the discrete topology and $A=C([0,1])$ with the uniform norm. Let $f:X\to A$ be the function given by $f(n)(0)=f(n)(1)=1$, $f(n)(1-1/n)=1/n$ and $f(n)$ is linear elsewhere in $[0,1]$. Then $f\in C_b(X,A)\diagdown C_p(X,A)$, since the sequence $(f(n))$ has no uniformly convergent subsequence in $C([0,1])$.

Necessary and sufficient conditions for the equality of the latter algebras are given in the next easily proven result.

Proposition 1.1. *The following assertions are equivalent*:

(i) $C_b(X,A)=C_p(X,A)$.
(ii) *If* $f\in C_b(X,A)$ *and* $f(X)\subset G(A)$, *then* $cl(f(X))$ *is compact.*
(iii) *For every* $f\in C_b(X,A)$, *there exist* $\lambda_1,\lambda_2\in\mathbb{C}$, *with* $\lambda_1\neq 0$, *such that* $\lambda_1 f+\lambda_2 e\in C_p(X,A)$.

For every $f\in C_p(X,A)$, there exists a unique extension f_β of f to the Stone-Čech compactification βX of X.

Let B be any complex commutative unital algebra. We denote by $\mathfrak{M}^{\#}(B)$ the set of all non-zero multiplicative linear functionals on B, provided with the weak star topology w^*. When B is a topological algebra, $\mathfrak{M}(B)$ denotes the topological subspace of $\mathfrak{M}^{\#}(B)$ consisting of all non-zero multiplicative continuous linear functionals on B. For $b\in B$, its Gelfand transform $\widehat{b}$ is given by $\widehat{b}(\varphi)=\varphi(b)$, for $\varphi\in\mathfrak{M}^{\#}(B)$. The set $\mathfrak{M}(B)$ is called the maximal ideal space of B and it coincides with $\mathfrak{M}^{\#}(B)$ if B is a Banach algebra.

There are several papers in which $\mathfrak{M}^{\#}(B)$ or $\mathfrak{M}(B)$ is characterized when B is a function algebra. Well-known results are: $\mathfrak{M}^{\#}(C(X))=X$ if X is a realcompact space ([**5,** page 3609, Theorem 1]) and $\mathfrak{M}(C_b(X))=\beta(X)$ if X is a completely regular Hausdorff space ([**11,** page 123, Theorem (3.2.11)]).

Along these lines, Dierolf, Schröder and Wengenroth proved in [**3,** page 54, Theorem 1], the formula $\mathfrak{M}^{\#}(C(X,E))=X\times\mathfrak{M}^{\#}(E)$ for a (completely regular Hausdorff) realcompact space X and a metrizable topological algebra E. Under the same assumption on X this formula was previously proved in [**8,** page 371, Theorem 5 (a)] by Hery supposing that E is a unital commutative topological Q-algebra with continuous inversion and either $\mathfrak{M}(E)$ is locally equicontinuous or X is discrete.

Concerning the maximal ideal spaces of functions algebras, Hausner in [**7,** page 248, Theorem], Dietrich in [**4,** page 207, Theorem 4] and Kahn in ([**9,** page 89, Theorem 5.2.4]) proved that $\mathfrak{M}(C(X, E)) = X \times \mathfrak{M}(E)$. In the first of these works X is a compact Hausdorff space and E is a unital complex commutative Banach algebra. In the second one, X is any completely regular k-space and E is a unital complete locally convex algebra such that $\mathfrak{M}(E)$ is locally equicontinuous. In Kahn's, X is a completely regular space of finite covering dimension and E is a unital topological algebra with non-trivial dual and such that $\mathfrak{M}(E)$ is locally equicontinuous. In all these papers $C(X, E)$ carries the compact-open topology. Using any of these results or [**1,** page 314, Corollary 6], the equality $\mathfrak{M}(C_p(X, A)) = \beta X \times \mathfrak{M}(A)$, which is a particular case of [**8,** page 369, Corollary 2 (a)], is easily obtained in Proposition 2.1 under our general hypothesis on X and A.

In contrast, little is known in general about the maximal ideal space of $C_b(X, A)$. Govaerts showed in [**6,** page 156, Counterexample 1] that $\mathfrak{M}(C_b(X, A)) = \beta X \times \mathfrak{M}(A)$ is false in general, and Kahn proved in [**9,** page 89, Corollary 5.2.3] that $\mathfrak{M}(C_b(X, E)) = X \times \mathfrak{M}(E)$, where $C_b(X, E)$ is endowed with the strict topology for any completely regular space X of finite covering dimension and a unital topological algebra E with non-trivial dual for which $\mathfrak{M}(E)$ is locally equicontinuous. The notion of the strict topology on $C_b(X, E)$ was first introduced by Buck in [**2,** page 97, Definition] in the case of X locally compact and E locally convex.

Here we study $\mathfrak{M}(C_b(X, A))$. We define a natural continuous transformation T from $X \times \mathfrak{M}(A)$, with the product topology, into $\mathfrak{M}(C_b(X, A))$. Therefore, each function $f \in C_b(X, A)$ has its proper Gelfand transform $\widehat{f} \in C(\mathfrak{M}(C_b(X, A)))$ and also another Gelfand transform $\widetilde{f} = \widehat{f} \circ T$ belonging to $C_b(X \times \mathfrak{M}(A))$. We prove that the transformation $f \to \widetilde{f}$ is a continuous homomorphism.

Let A be a complex *completely symmetric* algebra, i.e., a complex commutative unital Banach algebra with involution * satisfying $\|a\| = \|a^*\|$ and $F(a^*) = \overline{F(a)}$ (the complex conjugate of $F(a)$) for all $a \in A$

and $F \in \mathfrak{M}(A)$. We show that property "$f \in C_b(X, A)$ is invertible if $\widetilde{f}$ is invertible" is equivalent to "$T(X \times \mathfrak{M}(A))$ is dense in $\mathfrak{M}(C_b(X, A))$."

We do not know if these two properties are still equivalent if A is not assumed as above, but we exhibit an example, orally proposed by V. Müller, in which A is a complex completely symmetric algebra and nevertheless there exists $f \in C_b(X, A)$ such that $\widetilde{f}$ is invertible and f is not. Therefore even for completely symmetric algebras, the set $\mathfrak{M}(C_b(X, A))$ is in general larger than the w^*-closure $cl_{w^*}(T(X \times \mathfrak{M}(A)))$ of $T(X \times \mathfrak{M}(A))$.

Any C^*-algebra is an example of a completely symmetric algebra ([**10,** page 233, Corollary 4]), but here we are not going to assume that the involution on A satisfies $\|aa^*\| = \|a\|^2$, not even the weaker condition $\|aa^*\| = \|a\|\|a^*\|$, for $a \in A$.

2. Results. In this section, we define a natural continuous transformation T from $X \times \mathfrak{M}(A)$, with the product topology, into $\mathfrak{M}(C_b(X, A))$ and through it and the classical Gelfand transform $\widehat{f}$ for $f \in C_b(X, A)$, we introduce the Gelfand transform $\widetilde{f}$ with respect to $X \times \mathfrak{M}(A)$. Using T and $\widetilde{f}$, we shall state and prove almost all the results. In order to avoid confusion on the scope of these, we recall that we are assuming that X is a completely regular Hausdorff space and A is a complex commutative unital Banach algebra. From Lemma 2.5 on, A is a complex completely symmetric algebra with continuous involution.

Proposition 2.1. *The function $H : C_p(X, A) \to C(\beta X, A)$, with $H(f) = f_\beta$ is an isometric isomorphism of algebra and $\mathfrak{M}(C_p(X, A)) = \beta X \times \mathfrak{M}(A)$.*

Proof. It is readily seen that H is a bijective homomorphism of algebras. We also have that $\|f\|_\infty = \|f_\beta\|_\infty$, since X is dense in βX and then H is an isometry. Thus, $\mathfrak{M}(C_p(X, A)) = \mathfrak{M}(C(\beta X, A))$. Since $\mathfrak{M}(C(\beta X, A)) = \beta X \times \mathfrak{M}(A)$, the result follows. $\square$

Proposition 2.2. *There exists a continuous mapping T from $X \times \mathfrak{M}(A)$ into $\mathfrak{M}(C_b(X, A))$, given by $T(x, F) = T_{(x,F)}$, where*

$$T_{(x,F)}(f) = F(f(x)) = \widehat{f(x)}(F),$$

for every $f \in C_b(X, A)$ *and* $\widehat{f(x)}$ *is the Gelfand transform of* $f(x)$. *This mapping* T *has a continuous extension* T_β *to* $\beta(X \times \mathfrak{M}(A))$.

Proof. It is clear that $T_{(x,F)} \in \mathfrak{M}(C_b(X, A))$. Given the w^*-neighborhood $U = V(T_{(x,F)}, f_1, \ldots, f_n, \epsilon)$ of $T_{(x,F)}$ take the w^*- neighborhood $W = V(F,\ f_1(x), \ldots, f_n(x), \epsilon/2)$ of F and a neighborhood $V(x)$ of x satisfying $\|f_i(x) - f_i(y)\| < \epsilon/2$ if $y \in V(x)$ and $1 \leq i \leq n$. Then, for $(y, G) \in V(x) \times W$, we have that $T_{(y,G)} \in U$.

Since $\mathfrak{M}(C_b(X, A))$ is compact, then T has a continuous extension T_β to $\beta(X \times \mathfrak{M}(A))$. □

Corollary 2.3. $T_\beta(\beta(X \times \mathfrak{M}(A))) = \mathrm{cl}_{w^*}(T(X \times \mathfrak{M}(A)))$.

Proof. Since T_β is continuous and $X \times \mathfrak{M}(A)$ is dense in $\beta(X \times \mathfrak{M}(A))$, we get that $T_\beta(\beta(X \times \mathfrak{M}(A))) \subset \mathrm{cl}_{w^*}(T(X \times \mathfrak{M}(A)))$. But $T_\beta(\beta(X \times \mathfrak{M}(A)))$, being weak*- compact, contains the weak*-closure of $T(X \times \mathfrak{M}(A))$. □

Taking $f \in C_b(X, A)$, we define its Gelfand's transform $\widetilde{f}$ with respect to $X \times \mathfrak{M}(A)$ as $\widetilde{f} = \widehat{f} \circ T$, i.e.,

$$\widetilde{f}\,(x, F) = F\,(f\,(x))\,,$$

for $(x, F) \in X \times \mathfrak{M}(A)$. Therefore, $\widetilde{f} \in C_b(X \times \mathfrak{M}(A))$ and $\|\widetilde{f}\|_\infty \leq \|f\|_\infty$.

The mapping $f \to \widetilde{f}$ is a continuous homomorphism from $C_b(X, A)$ into $C_b(X \times \mathfrak{M}(A))$. Thus, if f is invertible in $C_b(X, A)$, then $\widetilde{f}$ is invertible in $C_b(X \times \mathfrak{M}(A))$.

The function $\widetilde{f}$ is invertible in the algebra $C_b(X \times \mathfrak{M}(A))$ if and only if $\widetilde{f}$ is bounded away from zero, i.e., $|F(f(x))| > \epsilon$ for some $\epsilon > 0$ and all $(x, F) \in X \times \mathfrak{M}(A)$. In particular, f is invertible in $C(X, A)$ if $\widetilde{f}$ is invertible.

Theorem 2.4. *For the following four assertions we have that*: (i) *implies* (ii); (ii) *implies* (iv); *and* (ii) *and* (iii) *are equivalent to each other.*

(i) *If* $f_1, \ldots, f_n \in C_b(X,A)$ *and* $\epsilon > 0$ *are such that, for every* $(x,F) \in X \times \mathfrak{M}(A)$, *there exist* $1 \le i \le n$ *for which* $|\widetilde{f}_i(x,F)| > \epsilon$, *then there exist* $g_1, \ldots, g_n \in C_b(X,A)$ *satisfying* $f_1 g_1 + \cdots + f_n g_n = e$.

(ii) *If* $f \in C_b(X,A)$ *and* $\widetilde{f}$ *is invertible, then* f *is invertible.*

(iii) *If* $f \in C_b(X,A)$ *and there exists* $\epsilon > 0$ *such that* $\|f(x) - y\| > \epsilon$ *for all* $x \in X$ *and* $y \in A \diagdown G(A)$, *then* f *is invertible.*

(iv) *If* $f \in C_b(X,A)$ *and*

$$\sup \left\{ \left| \widetilde{f}\,(x,F) \right| : (x,F) \in X \times \mathfrak{M}\,(A) \right\} < 1,$$

then $e - f$ *is invertible.*

Proof. Obviously, (i) implies (ii) and (ii) implies (iv).

(ii) $\Rightarrow$ (iii). Assume that there exists $\epsilon > 0$ such that $\|f(x) - y\| > \epsilon$ for all $x \in X$ and $y \in A \diagdown G(A)$. Put $y = f(x) - F(f(x))e$ for $x \in X$ and $F \in \mathfrak{M}(A)$. We have that $y \notin G(A)$ and $|\widetilde{f}(x,F)| = |F(f(x))| = \|f(x) - y\| > \epsilon$, then $\widetilde{f}$ is invertible and, by (ii), f is invertible.

(iii) $\Rightarrow$ (ii). Take $f \in C_b(X,A)$, and suppose $\widetilde{f}$ is invertible. There exists an $\epsilon > 0$ such that $|\widetilde{f}(x,F)| > \epsilon$ for all $(x,F) \in X \times \mathfrak{M}(A)$. Given $x \in X$ and $y \in A \diagdown G(A)$, choose $F \in \mathfrak{M}(A)$ such that $F(y) = 0$ and put $y = f(x) - F(f(x))e$. Then, $\|f(x) - y\| = |\widetilde{f}(x,F)| > \epsilon$; hence by (iii), f is invertible. $\square$

In the rest of this section we shall assume that A is a complex completely symmetric algebra with continuous involution *.

Lemma 2.5. *For every* $f \in C_b(X,A)$, *there exists a* $g \in C_b(X,A)$ *such that* $\widetilde{g}(x,F)$ *is the complex conjugate* $\overline{\widetilde{f}(x,F)}$ *of* $\widetilde{f}(x,F)$ *for each* $(x,F) \in X \times \mathfrak{M}(A)$. *Furthermore, we have* $|\widetilde{f}|^2 = \widetilde{fg}$.

Proof. If $f \in C_b(X,A)$, then the function g defined by $g(x) = f(x)^*$ belongs to $C_b(X,A)$ because the involution is a continuous function. Then, we have

$$\widetilde{g}\,(x,F) = F\left(f\,(x)^*\right) = \overline{\widetilde{f}\,(x,F)}$$

and

$$\widetilde{fg}\,(x,F)=F\left(f\,(x)\,f\,(x)^{*}\right)=\left|\widetilde{f}\,(x,F)\right|^{2},$$

for all $(x,F)\in X\times\mathfrak{M}(A)$. $\square$

Theorem 2.6. *Assertions* (i)–(iv) *in Theorem* 2.4 *are all equivalent.*

Proof.

(iv) $\Rightarrow$ (ii). Take $f\in C_b(X,A)$, and suppose that $\widetilde{f}$ is invertible. Then, $\widetilde{f}$ is bounded away from zero. Take g as in Lemma 2.5, and set $M=\sup|\widetilde{f}(x,F)|^2$ and $N=\sup|e-\widetilde{(1/M)fg}(x,F)|$, where the suprema are taken over all points (x,F) in $X\times\mathfrak{M}(A)$. Since $N=\sup|1-(1/M)|\widetilde{f}(x,F)|^2|<1$, we have by (iv) that $(1/M)fg$ is invertible and then (ii) holds.

(ii) $\Rightarrow$ (i). Suppose $f_1,\ldots,f_n\in C_b(X,A)$ and $\epsilon>0$ are as in (i). Let $g_i\in C_b(X,A)$ be such that $|\widetilde{f_i}|^2=\widetilde{f_ig_i}$ for every $i=1,2,\ldots,n$. For $(x,F)\in X\times\mathfrak{M}(A)$ we have that $\sum_{i=1}^n|\widetilde{f_i}(x,F)|^2=\sum_{i=1}^n\widetilde{f_ig_i}(x,F)=\widetilde{\sum_{i=1}^n f_ig_i}(x,F)>\epsilon$. Thus, $\widetilde{\sum_{i=1}^n f_ig_i}$ is invertible in $C_b(X\times\mathfrak{M}(A))$. By (ii), $\sum_{i=1}^n f_ig_i$ is invertible; therefore, there exists $h\in C_b(X,A)$ such that $\sum_{i=1}^n f_ig_ih=e$, that is, (i) holds. $\square$

Proposition 2.7. *If $T(X\times\mathfrak{M}(A))$ is not dense in $\mathfrak{M}(C_b(X,A))$, then there exists an $f\in C_b(X,A)$ such that $\widetilde{f}$ is invertible and f is not.*

Proof. Let us assume that $T\,(X\times\mathfrak{M}(A))$ is not dense in $\mathfrak{M}(C_b(X,A))$, and take $G\in\mathfrak{M}(C_b(X,A))\backslash\mathrm{cl}_{w^*}(T(X\times\mathfrak{M}(A)))$. Then, there exist $f_1,\ldots,f_n\in C_b(X,A)$ and $\epsilon>0$ such that, for each $(x,F)\in X\times\mathfrak{M}(A)$, there is a $1\le i\le n$ such that $|G(f_i)-F(f_i(x))|>\epsilon$. Put $g_i=f_i-G(f_i)e$, and take $h_i\in C_b(X,A)$ such that $\widetilde{h_i}(x,F)=\overline{\widetilde{g_i}(x,F)}$ for $1\le i\le n$ and $(x,F)\in X\times\mathfrak{M}(A)$. Then, for each $(x,F)\in X\times\mathfrak{M}(A)$, $|\widetilde{g_i}(x,F)|>\epsilon$ for some $1\le i\le n$ and $G(g_i)=0$ for all $1\le i\le n$.

Take

$$f=\sum_{i=1}^n g_ih_i.$$

Then $G(f) = 0$ and

$$\left|\widetilde{f}\,(x,F)\right| = \sum_{i=1}^{n} |\widetilde{g}_i\,(x,F)|^2 > \epsilon$$

for all $(X, F) \in X \times \mathfrak{M}(A)$. Therefore, f is not invertible and $\widetilde{f}$ is invertible. $\square$

Theorem 2.8. *Assertions* (i)–(iv) *of Theorem* 2.4 *are all equivalent to the following*:

(v) $T(X \times \mathfrak{M}(A))$ *is dense in* $\mathfrak{M}(C_b(X, A))$.

Proof. From Proposition 2.7, (ii) implies (v). On the other hand, let us assume that $T\ (X \times \mathfrak{M}(A))$ is dense in $\mathfrak{M}(C_b(X, A))$ and take $f \in C_b(X, A)$ such that $\widetilde{f}$ is invertible. Then there exists an $\epsilon > 0$ such that $|\widetilde{f}(x, F)| > \epsilon$ for every $(x, F) \in X \times \mathfrak{M}(A)$; hence, $\widehat{f}(G) \neq 0$ for all $G \in \mathfrak{M}(C_b(X, A))$. Therefore, f is invertible. $\square$

Corollary 2.9. *If* X *is a pseudocompact space, then* $T(X \times \mathfrak{M}(A))$ *is dense in* $\mathfrak{M}(C_b(X, A))$.

Proof. Suppose $f \in C_b(X, A)$ and $\widetilde{f}$ is invertible. Then, f is invertible in $C(X, A)$. Since the function $x \to \|f(x)^{-1}\|$ is continuous in X, then it is bounded. Therefore, f is invertible in $C_b(X, A)$. $\square$

3. The example. We thank Vladimir Müller who orally communicated the next example to us that enables us to show that there is a completely symmetric algebra A for which $T(\mathbb{N} \times \mathfrak{M}(A))$ is not dense in $\mathfrak{M}(C_b(\mathbb{N}, A))$.

Let S be the free commutative group with countably many generators $a_1, a_2, \ldots$. Define a function $p : S \to (0, \infty)$ by $p(a_j^k) = 1$ for $k \geq 0$, $p(a_j^k) = j$ for $k < 0$ and $p(a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}) = p(a_1^{k_1}) p(a_2^{k_2}) \cdots p(a_n^{k_n})$. Then, p is a positive multiplicative function.

Let A be the weighted group algebra over S, i.e., A is the set of functions $x : S \to \mathbb{C}$ satisfying that

$$\|x\| = \sum_{s \in S} |x\,(s)|\, p\,(s) < \infty,$$

provided with the usual linear structure and the convolution product

$$(xy)(s) = \sum_{t\in S} x(t)\, y\left(t^{-1}s\right).$$

For each $s \in S$, let χ_s be the characteristic function of the singleton $\{s\}$. Then, $x = \sum_{s\in S} \alpha_s \chi_s$, with $\alpha_s = x(s)$, for $x \in A$. Identifying χ_s with s in this expansion, we have

$$x = \sum_{s\in S} \alpha_s s,$$
$$\|x\| = \sum_{s\in S} |\alpha_s|\, p(s),$$
$$xy = \sum_{s\in S}\sum_{t\in S} \alpha_t \beta_{t^{-1}s} s,$$

if

$$x = \sum_{s\in S} \alpha_s s \quad \text{and} \quad y = \sum_{s\in S} \beta_s s,$$

and

$$F(x) = \sum_{s\in S} \alpha_s F(s) \quad \text{for every } F \in \mathfrak{M}(A).$$

The algebra A under the involution defined by

$$\left(\sum_{s\in S} \alpha_s s\right)^* = \sum_{s\in S} \overline{\alpha_t} s$$

becomes a completely symmetric algebra.

If $B = \{a_1, a_2, \ldots\}$, then clearly $B \subset G(A)$ and B is a bounded set, keeping in mind that $\|a_n\| = 1$ for each n. Since A is a unital commutative Banach algebra, we have that $\sigma(x) = \{F(x) : F \in \mathfrak{M}(A)\}$ for each $x \in A$. From this and applying the spectral radius formula to a_n and a_n^{-1}, we have $|F(a_n)| = 1$ for each $n \in \mathbb{N}$ and $F \in \mathfrak{M}(A)$. Therefore, we have that $\mathfrak{M}(A) = S_1^{\mathbb{N}}$, associating each $F \in \mathfrak{M}(A)$ with the unique sequence $(e^{i\theta_1}, e^{i\theta_2}, \ldots)$ in the complex unit sphere S_1 such that $F(a_j) = e^{i\theta_j}$ for each $j = 1, 2, \ldots$.

Let us consider the algebra $C_b(\mathbb{N}, A)$ and the function $f \in C_b(\mathbb{N}, A)$ defined by $f(n) = a_n$ for all $n \geq 1$. Since $|\widetilde{f}(n, F)| = 1$ for every $(n, F) \in \mathbb{N} \times \mathfrak{M}(A)$, the function $\widetilde{f}$ is invertible. Nevertheless, f is not invertible because $(f(\mathbb{N}))^{-1} = B^{-1}$ is not bounded. Therefore, we have that $T(\mathbb{N} \times \mathfrak{M}(A))$ is not dense in $\mathfrak{M}(C_b(\mathbb{N}, A))$. We point out that it can be shown that $\sigma(f) = \{z : |z| \leq 1\}$.

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