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THE LOCAL AND GLOBAL ZETA FUNCTIONS OF GAUSS'S CURVE

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ABSTRACT. The singular curve $C \subset \mathbb{P}^2$ defined over $\mathbb{F}_p$ for a prime p by the equation $x^2t^2 + y^2t^2 + x^2y^2 - t^4 = 0$ is known as Gauss's curve. For $p \equiv 3 \bmod 4$, we give a proof that the zeta function of C is

$$Z_C(u) = \frac{(1+pu^2)(1+u)^2}{(1-pu)(1-u)}.$$

We define the (Hasse-Weil) global zeta function for any geometric genus 1 singular curve and, in particular, find that the global zeta function of C is

$$\zeta_C(s) = \frac{\zeta(s)\zeta(s-1)}{L_E(s)L(s,\chi')^2},$$

where E is a projective nonsingular model for C, $L_E(s)$ is its L-function, and $L(s,\chi')$ is a Dirichlet L-series for a character χ' that we specify. We then consider more generally the ratio $R_X(s)$ of the Hasse-Weil global zeta function of a singular curve X and that of its normalization $\widetilde{X}$. We finish with questions about the analytic properties of $R_X(s)$.

1. Introduction. On July 9, 1814, Gauss made the last entry in his mathematical diary. He recorded the following discovery **[3, 8]**.

Theorem 1.1. *Suppose* $p = a^2+b^2 \equiv 1 \bmod 4$ *is prime, where* $a+bi \equiv 1 \bmod (2+2i)$. *Then the number of solutions to* $x^2 + y^2 + x^2y^2 = 1$ *over* $\mathbb{F}_p$ *is* $p+1-2a$.

Here, Gauss counted the two double points at infinity as four points total. Counting the points at infinity without multiplicity yields leads to the following theorem.

Theorem 1.2. **[7,** Chapter 11.5]. *Consider the curve* $C : x^2t^2 + y^2t^2 + x^2y^2 - t^4 = 0$ *in* $\mathbb{P}^2$ *defined over* $\mathbb{F}_p$ *where* $p \equiv 1 \bmod 4$. *Write*

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$p = a^2 + b^2$ *with* b *even and with* $a \equiv (-1)^{b/2} \bmod 4$. *Then the number of points on* $C(\mathbb{F}_p)$ *is* $N_1 = p - 1 - 2a$. *Furthermore, the zeta function of* C *is*

$$Z_C(u) = \frac{(1 - 2au + pu^2)(1 - u)}{1 - pu}.$$

Notice that setting $t = 1$ yields Gauss's original equation. Gauss did not provide a proof of his statement, and the first known proof uses the complex multiplication of elliptic functions associated to C which is due to Herglotz in 1921 [**6**] (see also [**2**]). Several other proofs have been published over the years, see Lemmermeyer's notes in [**8,** Chapter 10] for a survey. Lemmermeyer also shows that there are $p + 1$ points on C over $\mathbb{F}_p$ for $p \equiv 3 \bmod 4$ by showing a bijection between the $\mathbb{F}_p$-solutions to $x^2 + y^2 + x^2y^2 = 1$ and the $\mathbb{F}_p$-solutions to the equation $w^2 = 1 - v^4$.

In this paper, we first expand Lemmermeyer's work to count points of C over $\mathbb{F}_{p^n}$ for $p \equiv 3 \bmod 4$ and $n \geq 1$ (see also [**7,** Chapter 11, Exercises 10–13]). This allows us to calculate the zeta function of C over $\mathbb{F}_p$ for all primes p. We then extend the definition of the global zeta function to singular curves and compute the (Hasse-Weil) global zeta function of C. We then introduce $R_X(s)$, the ratio of the global zeta function of X and the global zeta function of $\widetilde{X}$, the normalization of X. For Gauss's curve C, the ratio is a product of Dirichlet L-functions. Finally, we ask whether this is a phenomenon that in some sense extends to all geometric genus 1 singular curves over $\mathbb{Q}$.

2. Zeta functions over finite fields.

Definition 2.1. [**7,** Chapter 11.1]. Consider a projective curve X defined over $\mathbb{F}_p$. The zeta function of X is given by

$$Z_X(u) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(X)u^n}{n}\right),$$

where $N_n(X)$ denotes the cardinality of $X(\mathbb{F}_{p^n})$.

Fact 2.2. [**8,** Chapter 10.5]. $Z_X(u)$ is a rational function of the form

$$Z_X(u) = \prod_i (1 - \alpha_i u) \prod_j (1 - \beta_j u)^{-1}$$

for some $\alpha_i, \beta_j \in \mathbb{C}$. Furthermore,

$$N_n(X) = \sum_i \alpha_i^n - \sum_j \beta_j^n.$$

In this section, let p be a prime with $p \equiv 3 \bmod 4$, and let ζ_8 be a primitive eighth root of unity, so $\zeta_8 \in \mathbb{F}_{p^n}$ if and only if n is even.

2.1. The zeta function of C. Here we define some plane curves birational to C, for which it will be easier to count $\mathbb{F}_p$-points. Specifically, we will consider the following projective curve E as well as an affine slice E_0:

$$E : y^2t - x^3 + 4xt^2 = 0, \qquad E_0 : y^2 - x^3 + 4x = 0.$$

We also use G_0 and C_0:

$$G_0 : z^2 + w^4 - 1 = 0, \qquad C_0 : x^2 + y^2 + x^2y^2 - 1 = 0.$$

Proposition 2.3. *We have*

$$N_n(C_0) = \begin{cases} N_n(G_0) & \text{if } 2 \nmid n; \\ N_n(G_0) - 2 & \text{if } 2 | n. \end{cases}$$

Proof. Consider the birational map

$$\mu : G_0 \longrightarrow C_0 \quad \text{given by} \quad (w, z) \longmapsto (x, y) = \left(w, \frac{z}{1 + w^2} \right).$$

The map μ is defined for all $(w, z) \in G_0(\mathbb{F}_{p^n})$ such that $w^2 \not\equiv -1 \bmod p$. Note that the inverse map is $\widetilde{\mu}(x, y) = (x, (1 + x^2)y)$. Therefore, μ induces a bijection from $G_0(\mathbb{F}_{p^n})$ to $C_0(\mathbb{F}_{p^n})$ for n odd and a bijection away from the points $(0, \pm\sqrt{-1}) \in G_0(\mathbb{F}_{p^n})$ for n even. Hence, $N_n(C_0) = N_n(G_0)$ for n odd and $N_n(C_0) = N_n(G_0) - 2$ for n even. $\square$

Proposition 2.4. *For even n, we have $N_n(C_0) = N_n(E_0) - 3$.*

Proof. Consider the following birational map defined over $\mathbb{F}_{p^2}$:

$$\eta : E_0 \longrightarrow G_0 \quad \text{given by} \quad (x, y) \longmapsto (w, z) = \left(\frac{\zeta_8 y}{2x}, \frac{y^2 + 8x}{4x^2} \right).$$

This induces a map from $E_0(\mathbb{F}_{p^n})$ to $G_0(\mathbb{F}_{p^n})$ for n even, which is defined away from $(0, 0) \in E_0(\mathbb{F}_{p^n})$. The inverse map is

$$\widetilde{\eta} : G_0 \longrightarrow E_0 \quad \text{given by} \quad (w, z) \longmapsto \left(\frac{2}{z + \zeta_8^2 w^2}, \frac{4\zeta_8^7 w}{z + \zeta_8^2 w^2} \right).$$

The map $\widetilde{\eta}$ is defined for all points of $G_0(\mathbb{F}_{p^n})$ since there is no point (w, z) in $G_0(\mathbb{F}_{p^n})$ such that $z + \zeta_8^2 w^2 = 0$. Therefore, the induced map $\eta^* : E_0(\mathbb{F}_{p^n}) - \{(0, 0)\} \to G_0(\mathbb{F}_{p^n})$ is a bijection and $N_n(G_0) = N_n(E_0) - 1$ for n even. Proposition 2.3 then gives us Proposition 2.4. □

Note that Proposition 2.4 shows that E is the normalization of C over $\mathbb{F}_{p^2}$, so C has geometric genus 1. However, since C is singular, its zeta function is not determined just by the value $N_1(C)$ (as it is for E).

Theorem 2.5. *Consider the curve* $C : x^2t^2 + y^2t^2 + x^2y^2 - t^4$ *over* $\mathbb{F}_p$ *where* $p \equiv 3 \bmod 4$*. Then*

$$N_n(C) = \begin{cases} p^n - 2(\sqrt{-p})^n - 1 & \textit{if } n \textit{ even,} \\ p^n + 3 & \textit{if } n \textit{ odd.} \end{cases}$$

Furthermore,

$$Z_C(u) = \frac{(1 + u)^2(1 + pu^2)}{(1 - u)(1 - pu)}.$$

Proof. It is elementary to calculate that $N_n(G_0) = p^n + 1$ when n is odd [**8,** page 318]. Then, since E is a smooth curve of genus 1 with $p + 1$ points over $\mathbb{F}_p$, the Weil conjectures allow us to easily calculate that

$$N_n(E) = (1^n + p^n) - ((\sqrt{-p})^n + (-\sqrt{-p})^n).$$

Therefore, when n is even, $N_n(E) = p^n - 2(\sqrt{-p})^n + 1$ and $N_n(E_0) = p^n - 2(\sqrt{-p})^n$.

Since

$$N_n(C_0) = \begin{cases} N_n(E_0) - 3 & \text{if } n \text{ even,} \\ N_n(G_0) & \text{if } n \text{ odd,} \end{cases}$$

and the curve C has two points at infinity regardless of n, we have deduced the values of $N_n(C)$. In order to calculate the zeta function of C, notice that $N_n(C)$ can be rewritten for any value of n as

$$N_n(C) = p^n + 1 - (\sqrt{-p})^n - (-\sqrt{-p})^n - 2(-1)^n.$$

Therefore,

$$Z_C(u) = \exp\left(\sum_{n=1}^{\infty} \frac{(p^n + 1 - (\sqrt{-p})^n - (-\sqrt{-p})^n - 2(-1)^n)u^n}{n}\right).$$

Using the identity $\sum_{n=1}^{\infty} w^n n^{-1} = -\ln(1-w)$, we get the desired result

$$Z_C(u) = \frac{(1+u)^2(1+pu^2)}{(1-u)(1-pu)}. \qquad \square$$

2.2. Comparison to zeta functions of normalizations. The relationship between the zeta function of a singular curve over a finite field and its normalization has been studied in [**12, 14**]. Gauss's curve C is an example of a projective plane curve with singularities. By [**5,** Chapter 17], for every such singular curve X, there exists a nonsingular projective curve $\widetilde{X}$ along with a normalization map $\nu : \widetilde{X} \to X$. For every nonsingular point P of X, the preimage $\nu^{-1}(P)$ consists of only one point.

Another approach to determining $Z_X(u)$ is to consider $\widetilde{X}$ and its zeta function $Z_{\tilde{X}}(u)$. Then $N_n(X)$ can be calculated by comparing it to $N_n(\widetilde{X})$ while considering the size and field of definition of the preimages of the singular points of X. More precisely, let X_{sing} represent the set of singular points of X. Let $Q \mid P$ denote the set of points $Q \in \widetilde{X}$ such that $\nu(Q) = P$. Then we let $\deg(P)$ be the degree of the field extension of the residue field of P over $\mathbb{F}_p$. The following lemma explains how the zeta function of a singular curve is related to the zeta function of its normalization. It is a consequence of the Euler product representation of the zeta function [**9,** Chapter 8.4].

Lemma 2.6. (*see, e.g.,* [**1,** Section 2]). *Let X be a complete irreducible algebraic projective curve with normalization $\widetilde{X}$. Then*

$$\frac{Z_X(u)}{Z_{\tilde{X}}(u)} = \prod_{P \in X_{\text{sing}}} \frac{\prod_{Q|P}(1 - u^{\deg(Q)})}{1 - u^{\deg(P)}}.$$

We now apply this lemma, using that $\widetilde{C} = E$. For p any odd prime, C has two degree-one singular points $P_1 = [1, 0, 0]$ and $P_2 = [0, 1, 0]$. If $p \equiv 3 \bmod 4$, there is one point of degree 2 on E for each of these, hence $Z_C(u)/Z_E(u) = (1 + u)^2$. When $p \equiv 1 \bmod 4$, there are two points of degree 1 on E for each of these, yielding $Z_C(u)/Z_E(u) = (1 - u)^2$.

3. Global zeta functions. Let X be an elliptic curve defined over $\mathbb{Q}$ by a global minimal model (so defined by a generalized Weierstrass equation) over $\mathbb{Z}$ with discriminant Δ, let X_p be the reduction of X mod p, and let $\mathcal{S} = \{p \text{ prime: } p \mid \Delta\}$ be the set of primes of bad reduction for X (see [**7,** Chapter 18.2] for reference). Then the above function $Z_{X_p}(u)$ is defined for primes $p \notin \mathcal{S}$. Via the change in variables $u = p^{-s}$, we can define

$$\zeta_{X_p}(s) = Z_{X_p}(p^{-s})$$

to be the (local) zeta function of X at p.

The global zeta function of X is a function which incorporates the local zeta functions of X for all primes $p \notin \mathcal{S}$, as well as zeta factors which we will define below for $p \in \mathcal{S}$. Global zeta functions have been studied extensively and are the subject of the well-known Birch and Swinnerton-Dyer conjecture (see [**4,** Lecture 2] for a more complete discussion of BSD).

Let $\mathcal{N}_p(X) = |X_p(\mathbb{F}_p)|$, and let $\alpha_p = p + 1 - \mathcal{N}_p$. We then have that, for $p \notin \mathcal{S}$,

$$\zeta_{X_p}(s) = \frac{1 - \alpha_p p^{-s} + p^{1-2s}}{(1 - p^{-s})(1 - p^{1-s})}.$$

The (incomplete) global zeta function of X is defined to be the product of the local zeta functions:

$$\zeta_X^*(s) = \prod_{p \nmid \Delta} \zeta_{X_p}(s).$$

Let $L_X^*(s) = \prod_{p \nmid \Delta}(1 - \alpha_p p^{-s} + p^{1-2s})^{-1}$, which is called the (incomplete) L-function of X.

To complete these functions, we include local ζ-factors corresponding to $p \in \mathcal{S}$. For elliptic curves, we use the following definitions (see [**4, 11**])

$$\zeta_{X_p}(s) = \begin{cases} \frac{1}{(1-p^{-s})(1-p^{1-s})} & \text{if } X \text{ has additive reduction at } p, \\ \frac{1}{(1-p^{1-s})} & \text{if } X \text{ has split multiplicative reduction at } p, \\ \frac{1+p^{-s}}{(1-p^{-s})(1-p^{1-s})} & \text{if } X \text{ has non-split multiplicative reduction at } p. \end{cases}$$

Taking the product over all p, we have a formula for the global ζ function of X

$$\zeta_X(s) = \prod_p \zeta_{X_p}(s) = \zeta(s)\zeta(s-1)L_X(s)^{-1},$$

where $L_X(s)$ is the corresponding completed L-function of X. Determining the global zeta function of X is equivalent to determining its L-function.

We note that, for an elliptic curve E defined over $\mathbb{Q}$, one can add a factor corresponding to infinity to obtain the function

$$\Lambda(E, s) = (2\pi)^{-s}\Gamma(s)L_E(s),$$

where $\Gamma(s)$ is the usual Gamma function. Wiles, Taylor and others proved that $\Lambda(E, s)$ has an analytic continuation to the entire complex plane and satisfies a functional equation [**4**]. While it is necessary to carefully specify ζ_{X_p} for primes of bad reduction to get the functional equation of $\Lambda(E, s)$, it is not necessary to do so just for determining whether $L_E(s)$ has a meromorphic continuation to the whole s-plane. So, while the formal definition for L_{X_p} comes from the characteristic polynomial of Frobenius acting on the dual of the inertial invariants of the Tate module of E (see [**4**]), one could naively define $L_{X_p}^{-1}$ as $(1-p^{-s})(1-p^{1-s})$ times $Z_{X_p}(p^{-s}z)$. When X is a global minimal model over $\mathbb{Q}$, these two definitions for ζ_{X_p} agree, but this motivates our following ad hoc definition. We call the following elementary global zeta function of a singular curve the (Hasse-Weil) global zeta function (to contrast with [**1, 13**], where the "global" zeta function has a different meaning; see also [**10**]).

Definition 3.1. Let Y be a singular curve over $\mathbb{Q}$ with normalization $\widetilde{Y}$ over $\mathbb{Q}$, where $\widetilde{Y}$ is an elliptic curve. Let $\mathcal{S}$ be the set of primes of bad reduction for $\widetilde{Y}$. Define $\zeta_{Y_p}(s) = \zeta_{\tilde{Y}_p}(s)$ for $p \in \mathcal{S}$. Define the

Hasse-Weil global zeta function of Y to be

$$\zeta_Y(s) = \prod_p \zeta_{Y_p}(s).$$

Our Hasse-Weil global zeta function of a singular curve X uses the local zeta functions of X at all primes of good reduction for $\widetilde{X}$ and the same zeta factors as $\widetilde{X}$ for primes of bad reduction for $\widetilde{X}$. Since X is singular, it does not have a minimal model in the traditional sense, so we consider the global minimal model of $\widetilde{X}$ as a proxy for studying the primes of bad reduction.

3.1. The global zeta functions of E and C. As above, E is the normalization of Gauss's curve C over $\mathbb{Q}$. The function $L_E(s)$ is well known. To fix notation, let P be a prime of $\mathbb{Z}[i]$, $P \nmid 2$. Let $N(P)$ be the norm of P. For $A \in \mathbb{Z}[i]$, let $(A/P)_4 \in \{0, \pm 1, \pm i\}$ be the quartic residue symbol of A modulo P. That is,

$$\left(\frac{A}{P}\right)_4 = 0 \quad \text{if } P \mid A$$

and

$$\left(\frac{A}{P}\right)_4 = A^{[N(P)-1]/4} \bmod p, \text{ otherwise.}$$

Define a Hecke character χ on primes P of $\mathbb{Z}[i]$. If P divides 2, define $\chi(P) = 0$. If $N(P) = p^2$ for some rational prime p, then $p \equiv 3 \bmod 4$ and $(P) = (p)$, where p is inert in $\mathbb{Z}[i]$. In this case define $\chi(P) = -p$. If $N(P) = p$, i.e., (p) splits in $\mathbb{Z}[i]$ and $p \equiv 1 \bmod 4$, then $P = (\pi)$ for some $\pi \in \mathbb{Z}[i]$ with $\pi \equiv 1 \bmod (2+2i)$. Define $\chi(P) = \overline{(4/(\pi))_4 \pi}$, where a bar denotes complex conjugation.

The Hecke L-function associated to χ is defined as

$$L(s, \chi) = \prod_{P \text{ prime of } \mathbb{Z}[i]} (1 - \chi(P) N(P)^{-s})^{-1}.$$

For the case of the elliptic curve $E = \widetilde{C} : y^2 t - x^3 + 4xt^2 = 0$, it is shown in [**7,** Chapter 18.6] that $L_E(s) = L(s, \chi)$. This reflects that fact that the curve E has complex multiplication by $\mathbb{Z}[i]$ (see [**11,** subsection 11.10]).

To express the Hasse-Weil global zeta function of the singular curve C, we use the following character.

Definition 3.2. Let χ' be the Dirichlet character $\chi' : \mathbb{Z} \to \{0, \pm 1\}$, where $\chi'(n) = 0$ if n is even, $\chi'(n) = 1$ if $n \equiv 1 \bmod 4$, and $\chi'(n) = -1$ if $n \equiv 3 \bmod 4$, i.e., the non-trivial character associated to the extension $\mathbb{Q}(i)/\mathbb{Q}$.

The Dirichlet L-function associated to χ' is

$$L(s, \chi') = \prod_{p \text{ prime of } \mathbb{Z}} (1 - \chi'(p)p^{-s})^{-1}.$$

Theorem 3.3. *The Hasse-Weil global zeta function for C is given by*

$$\zeta_C(s) := \prod_p \zeta_{C_p}(s) = \frac{\zeta(s)\zeta(s-1)}{L_E(s)L(s,\chi')^2} = \frac{\zeta_E(s)}{L(s,\chi')^2}.$$

Proof. Recall from Theorem 1.2 that $\mathcal{N}_p(C) = p - 1 - 2a_p$ for $p \equiv 1 \bmod 4$, where a_p is the value such that $a_p^2 + b^2 = p$ with b even and $a_p \equiv (-1)^{b/2} \bmod 4$. Also note that E has additive reduction at $p = 2$, the only prime of bad reduction for E. We then have

$$\zeta_C(s) = \prod_p \zeta_{C_p}(s) = \frac{1}{(1-2^{-s})(1-2^{1-s})}$$
$$\prod_{p\equiv 1(4)} \frac{(1-2a_p p^{-s}+p^{1-2s})(1-p^{-s})}{1-p^{1-s}} \prod_{p\equiv 3(4)} \frac{(1+p^{-s})^2(1+p^{1-2s})}{(1-p^{-s})(1-p^{1-s})}.$$

A few simplifications yield the form:

$$\zeta_C(s) = \zeta(s)\zeta(s-1) \prod_{p\equiv 1(4)} (1 - 2a_p p^{-s} + p^{1-2s})(1-p^{-s})^2$$
$$\prod_{p\equiv 3(4)} (1+p^{-s})^2(1+p^{1-2s})$$

Now consider the relationship between a_p and α_p, where $\alpha_p = p + 1 - \mathcal{N}_p(E)$. When $p \equiv 1 \bmod 4$, the two singularities on C_p are double points, which in the normalization E_p yield two points each. That means that $\mathcal{N}_p(E) = \mathcal{N}_p(C) + 2$, giving

$$p + 1 - \alpha_p = p - 1 - 2a_p + 2,$$

so $\alpha_p = 2a_p$ for $p \equiv 1 \bmod 4$. When $p \equiv 3 \bmod 4$, we know that $\mathcal{N}_p(E) = p + 1$, so $\alpha_p = 0$.

Therefore,

$$\begin{aligned} \zeta_C(s) &= \zeta(s)\zeta(s-1)\prod_{p\neq 2}(1 - \alpha_p p^{-s} + p^{1-2s})(1 - (-1)^{(p-1)/2}p^{-s})^2 \\ &= \zeta(s)\zeta(s-1)L_E(s)^{-1}L(s,\chi')^{-2}. \end{aligned}$$

□

3.2. Comparison of global zeta functions. For any singular curve X of geometric genus 1 over $\mathbb{Q}$, consider now

$$R_X(s) := \frac{\zeta_{\tilde{X}}(s)}{\zeta_X(s)}.$$

This function captures the information of how $\zeta_{X_p}(s)$ differs from $\zeta_{\tilde{X}_p}(s)$ for all primes p. For Gauss's curve, we have that

$$R_C(s) = L(s,\chi')^2.$$

In particular, as a product of L-series, this ratio has an analytic continuation to the entire complex s-plane and a functional equation relating $R_C(s)$ and $R_C(1-s)$. This ratio is also equal to the product over all primes of the ratios of the local zeta functions:

$$R_X(s) = \prod_p \frac{\zeta_{\tilde{X}_p}(s)}{\zeta_{X_p}(s)} = \prod_{p\notin\mathcal{S}}\prod_{P\in X_{p_{\text{sing}}}} \frac{1 - p^{-s\deg(P)}}{\prod_{Q|P}(1 - p^{-s\deg(Q)})}.$$

This ratio can be calculated by studying the preimages of the singular points of X_p in $\widetilde{X_p}$ for all p of good reduction for the normalized curve.

A few questions naturally arise regarding $R_X(s)$. Is the particularly nice form of $R_C(s)$ due to the fact that $\widetilde{C}$ is an elliptic curve with complex multiplication? Would $R_X(s)$ also be a product of Dirichlet or other nice L-functions for other singular geometric genus 1 curves? Here, $R_C(s)$ has a meromorphic continuation to the entire complex plane. Is this the case for any singular curve of any geometric genus?

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