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THE EXISTENCE OF THREE SOLUTIONS FOR p-LAPLACIAN PROBLEMS WITH CRITICAL AND SUPERCRITICAL GROWTH

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ABSTRACT. In this paper we deal with the existence and multiplicity of solutions for the p-Laplacian problems involving critical and supercritical Sobolev exponent via variational arguments. By means of the truncation combining with the Moser iteration, we extend the result obtained by Ricceri [**14**] to the critical and supercritical case.

1. Introduction and main results. We study the existence and multiplicity of solutions for the quasilinear elliptic problem

$$\begin{cases} -\triangle_p u = f_\lambda(x,u) + \mu|u|^{r-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < N$, $N \geq 3$, $r \geq p^* = (Np)/(N-p)$, μ is a nonnegative constant and f_λ: $\Omega \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function with λ a nonnegative parameter.

If $f_\lambda + \mu|u|^{r-2}u$ with $r < p^*$ satisfies the A-R condition, problem (1.1) can be handled by the well-known critical point theory as the energy functional which satisfies the (PS) conditions. One can refer to the papers [**13, 14**] for the three critical points theorem with the subcritical case. We focus on the case $r \geq p^*$, i.e., the critical and supercritical case. Firstly, let us recall the critical case $r = p^*$. Guedda and Véron [**9**] study problem (1.1) for $f_\lambda(x,u) = \lambda u^{p-1}$ and prove that there exists a solution for $\lambda \in (0, \lambda_1)$, and no solution for

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$\lambda > \lambda_1$, where λ_1 is the first eigenvalue of the p-Laplacian with Dirichlet data. By means of the sub-supersolutions and variational arguments, García Azorero, Peral Alonso and Manfredi [**8**] deal with the problem (1.1) with $f_\lambda(x,u) = \lambda|u|^{q-2}$, $1 < q < p$, $\mu = 1$, and obtain two positive solutions. By the mountain pass lemma and the concentration-compactness principle [**10**], García Azorero and Peral Alonso [**7**] obtain the following results: if $1 < q < p$, there exist infinitely many solutions for $\lambda > 0$ small enough; if $p < q < p^*$, then there exists $\lambda_0 > 0$ such that there exists a nontrivial solution for $\lambda > \lambda_0$. Wei and Wu [**18**] deal with the existence and multiplicity of solutions for the problem (1.1) with $f_\lambda(x,u) = a(x)|u|^{p-2}u + g(x,u)$, $a(x) \in L^\infty(\Omega)$ and $g(x,u)$ a lower-order perturbation of $|u|^{p^*-2}u$, odd with respect to u; from the symmetric mountain pass lemma and the concentration-compactness principle [**10**], they obtain that problem (1.1) has at least k pairs of solutions for given $k \in \mathbf{N}$, when $\mu > 0$ small. Silva and Xavier [**15**] also use the same methods to deal with problem (1.1) when $f_\lambda = f(x,u)$ is a lower-order perturbation of $|u|^{p^*-2}u$ with the symmetry condition, and obtain a result similar to Wei and Wu [**18**], but the assumptions on f are different (see [**15**] for details). If $p = 2$, we can refer to [**1, 2, 16**] for elliptic equations with critical Sobolev exponent.

For the case $r > p^*$, we cannot directly use variational techniques because the corresponding functional is not well-defined on the Sobolev space $W_0^{1,p}(\Omega)$. However, the symmetry domain can raise the Sobolev exponents, and some supercritical cases of problem (1.1) on such a domain can be handled by the variational method (see [**6, 17**]). For the general domain, there are some meaningful results about positive solutions when $p = 2$. For the problem

$$\begin{cases} -\triangle u = \lambda u^q + u^m, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

with $m > (N+2)/(N-2)$, Ambrosetti, Brezis and Cerami [**1**] consider problem (1.2) with $0 < q < 1$, obtain some beautiful results, and pose a question: Are there solutions other than the minimal solution for $\lambda > 0$ small enough? Zhao and Zhong [**20**] give a negative answer to the question; in [**19**], they obtain the existence of infinitely many solutions of problem (1.2) for some suitable λ with $\Omega = B$ the unit ball of $\mathbf{R}^N$ when $q = 1$. By constructing a suitable truncation, the Moser iteration **[11]** and the Ljusternik-Schnirelman category theory, Rabinowitz **[12]** studies the supercritical Dirichlet problem $-\triangle u = \lambda|u|^{q-1}u + \lambda|u|^{m-1}u$ with $1 < q < (N+2)/(N-2)$, $\Omega \subset \mathbf{R}^N$ a bounded smooth domain, and obtains at least k pairs of solutions for given $k \in \mathbf{N}$ when $\lambda > 0$ small. Corrêa and Figueiredo **[4]** also use the same methods to study the nonlocal p-Laplacian problem with the supercritical case. For $\Omega = \mathbf{R}^N$, one can refer to **[3, 5]**.

Recently, Ricceri **[14]** dealt with the problem

$$\text{(1.3)} \qquad \begin{cases} -\triangle_p u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain and $g(x,u)$ satisfies subcritical growth, i.e.,

$$|g(x,u)| \le C(1 + |u|^{q-1}), p < q < p^* = \frac{Np}{N-p}, \quad p < N.$$

He established a three critical theorem (the following Theorem A) and obtained three solutions. In the following Theorem 1.1 and Theorem 3.2, we extend the result in **[14]** to the critical and supercritical Sobolev exponent case. For completeness, we list some results of **[14]**.

If X is a real Banach space, we can denote by $\mathfrak{W}_X$ the class of all functionals $\phi : X \to \mathbf{R}$ possessing the following property: if $\{u_n\} \subset X$ is a sequence converging weakly to $u \in X$ and $\liminf_{n\to\infty} \phi(u_n) \le \phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u. For instance, if X is an uniformly convex Banach space and $g : [0, +\infty) \to \mathbf{R}$ is a continuous, strictly increasing function, then the functional $u \to g(\|u\|)$ belongs to the class $\mathfrak{W}_X$.

Theorem A (**[14]**). *Let X be a separable and reflexive real Banach space; $I \subseteq \mathbf{R}$ an interval; $\Phi : X \to \mathbf{R}$ a sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathfrak{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbf{R}$ a C^1 functional with compact derivative. Assume that, for each $\lambda \in I$, the functional $\Phi - \lambda J$ is coercive and has a strict local, not global minimizer, say $\widehat{u}_\lambda$. Then, for each compact interval $[a,b] \subseteq I$ for which $\sup_{\lambda\in[a,b]}(\Phi(\widehat{u}_\lambda) - \lambda J(\widehat{u}_\lambda)) < +\infty$, there exists $\gamma > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1 functional $\Psi : X \to \mathbf{R}$ with compact derivative, there exists $\delta_0 > 0$ such* *that, for each $\mu \in [0, \delta_0]$ the equation*

$$\Phi'(u) = \lambda J'(u) + \mu \Psi'(u)$$

has at least three solutions in X whose norms are less than γ.

Remark. If $|g(x,u)| \le C(1+|u|^{q-1}), p < q < p^*, p < N$, then the functional $\Psi(u) = \int_\Omega \int_0^u g(x,s)\,ds\,dx$ has a compact derivative in $W_0^{1,p}(\Omega)$. Theorem A can be applied to problem (1.3) so as to obtain the existence of three solutions. In this paper, we are interested in existence and multiplicity of the solutions of problem (1.1) with $r \ge p^*$. The main difficulty is obviously the lack of the compactness. Under suitable assumptions on the nonlinearity f, by means of the Moser iteration, a suitable truncation and Theorem A, we obtain the existence of three solutions of problem (1.1) for $\mu > 0$ small enough.

We suppose that the nonlinearity $f_\lambda(x,u) = \lambda f(x,u)$ satisfies the following conditions:

- (f_1) $\lim_{|s|\to+\infty} f(x,s)/|s|^{p-1} = 0$, uniformly in $x \in \Omega$.
- (f_2) $\lim_{|s|\to 0} f(x,s)/|s|^{p-1} = 0$, uniformly in $x \in \Omega$.
- (f_3) $\sup_{u \in W_0^{1,p}(\Omega)} \int_\Omega F(x,u)\,dx > 0$, where $F(x,u) = \int_0^u f(x,s)\,ds$.
- (f_4) For each $M > 0$, the function $\sup_{|u|\le M} |f(\cdot,u)| \in L^\infty(\Omega)$.

We introduce the functionals

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx, \qquad J(u) = \int_\Omega F(x,u)\,dx,$$

and denote $\|u\| = \|u\|_{W_0^{1,p}} = (\int_\Omega |\nabla u|^p dx)^{1/p}$. Obviously, condition (f_3) implies

$$\theta := \sup_{\Phi(u)\neq 0} \frac{J(u)}{\Phi(u)} > 0. \tag{1.4}$$

One of the main results in this paper is stated as follows:

Theorem 1.1. *Let f satisfy (f_1), (f_2), (f_3) and (f_4). Then, for each compact interval $[a,b] \subset (1/\theta, +\infty)$ there exists $\gamma > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such* *that, for each $\mu \in [0, \delta]$, problem* (1.1) *has at least three solutions in $W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$, whose $W_0^{1,p}(\Omega)$-norms are less than γ.*

2. Preliminaries.

Lemma 2.1. *Let f satisfy (f_1) and (f_4). Then, for every $\lambda \in (0, +\infty)$, the functional $\Phi - \lambda J$ is sequentially weakly lower continuous and coercive on $W_0^{1,p}(\Omega)$, and has a global minimizer v_λ.*

Proof. Let us fix $\lambda \in (0, +\infty)$. By (f_1), for all $\varepsilon > 0$, there exists $M_0 > 0$, such that

$$|f(x, s)| \le \varepsilon |s|^{p-1} \quad \text{as } |s| \ge M_0.$$

Considering this inequality and (f_4), we may find a constant $c_0 > 0$, such that

$$|f(x, s)| \le c_0 + \varepsilon |s|^{p-1} \quad \text{as } s \in \mathbf{R}, \tag{2.1}$$

which implies

$$|F(x, u)| \le c_0 |u| + \frac{\varepsilon}{p} |u|^p. \tag{2.2}$$

Thus, for $u \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned}
\Phi(u) - \lambda J(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega F(x, u)\, dx \\
&\ge \frac{1}{p} \int_\Omega |\nabla u|^p\, dx - \lambda \int_\Omega \left(\frac{\varepsilon}{p} |u|^p + c_0 |u| \right) dx \\
&= \frac{1}{p} \|u\|^p - \frac{\lambda \varepsilon}{p} \int_\Omega |u|^p\, dx - \lambda c_0 \int_\Omega |u|\, dx \\
&\ge \frac{1}{p} \|u\|^p - \frac{\lambda \varepsilon C_p^p}{p} \|u\|^p - \lambda c_0 C_1 \|u\|,
\end{aligned}$$

where $\|u\|_{L^p} \le C_p \|u\|$ and $\|u\|_{L^1} \le C_1 \|u\|$, with constants $C_1, C_p > 0$. Since $p > 1$, $\varepsilon > 0$ is small enough, so we have $\Phi(u) - \lambda J(u) \to +\infty$ as $\|u\| \to \infty$. Hence, $\Phi - \lambda J$ is coercive.

Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact and (2.2), J is weakly continuous. Obviously, $\Phi(u) = 1/p \int_\Omega |\nabla u|^p\, dx \in \mathfrak{W}_X$ is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. We can deduce that $\Phi - \lambda J$

is sequentially weakly lower semicontinuous. So $\Phi - \lambda J$ has a global minimizer v_λ. The proof is complete. $\square$

Next, we will show that $\Phi - \lambda J$ has a strictly local, not global minimizer for some λ, when f satisfies (f_1)–(f_4).

Lemma 2.2. *Let f satisfy (f_1), (f_2), (f_3) and (f_4). Then*

(i) 0 *is a strict local minimizer of the functional $\Phi - \lambda J$ for $\lambda \in (0, +\infty)$.*
(ii) $0 \neq v_\lambda$, *i.e., 0 is not the global minimizer v_λ for $\lambda \in (1/\theta, +\infty)$, where v_λ is given by Lemma 2.1.*

Proof. Firstly, we prove

$$\lim_{\|u\|\to 0} \frac{J(u)}{\Phi(u)} = 0. \tag{2.3}$$

In fact, by (f_2), for all $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x,u)| \leq \varepsilon |u|^{p-1}, \quad \text{as } |u| < \delta. \tag{2.4}$$

Considering inequality (2.4), (f_1) and (f_4), we have

$$|f(x,u)| \leq \varepsilon |u|^p + |u|^q, \quad \text{for } x \in \Omega,\ u \in \mathbf{R}, \tag{2.5}$$

with fixed $q \in (p, p^*)$. By continuous embedding, we have

$$|J(u)| \leq \varepsilon C_p \|u\|^p + C_q \|u\|^q. \tag{2.6}$$

This implies

$$\lim_{\|u\|\to 0} \frac{J(u)}{\Phi(u)} = 0. \tag{2.7}$$

Next, we will prove (i) and (ii).

(i) For $\lambda \in (0, +\infty)$, since $\lim_{\|u\|\to 0}(J(u)/\Phi(u)) = 0 < 1/\lambda$ and $\Phi(u) > 0$ for each $u \neq 0$ in some neighborhood U of 0, there exists a neighborhood $V \subseteq U$ of 0 such that $\Phi(u) - \lambda J(u) > 0$ for all $u \in V \setminus \{0\}$. Hence, 0 is a strict local minimum of $\Phi - \lambda J$.
(ii) For $\lambda \in (1/\theta, +\infty)$, from the definition of θ, there exists $u^* \in W_0^{1,p}(\Omega)$, with $\min\{\Phi(u^*),\ J(u^*)\} > 0$, such that $J(u^*)/\Phi(u^*) > 1/\lambda$, i.e., $\Phi(u^*) - \lambda J(u^*) < 0 = \Phi(0) - \lambda J(0)$.

So $0 \neq v_\lambda$ is not a global minimum of $\Phi - \lambda J$. The proof is complete. $\square$

3. Proof of Theorem 1.1. We follow the ideas from **[3, 12]**. Let $K > 0$ be a real number, whose value will be fixed later. Let the truncation of $|u|^{r-2}u$ with $r \geq p^*$, be given by

$$g_K(u) = \begin{cases} |u|^{r-2}u & \text{if } 0 \leq |u| \leq K, \\ K^{r-q}|u|^{q-2}u & \text{if } |u| \geq K, \end{cases}$$

where $q \in (p, p^*)$, almost everywhere $x \in \Omega$. Then $g_K(u)$ satisfies

$$|g_K(u)| \leq K^{r-q}|u|^{q-1}.$$

Set

$$h_K(x,u) = \lambda f(x,u) + \mu g_K(u).$$

We study the truncated problem associated to h_K

$$(T_{\mu,K}) \qquad \begin{cases} -\triangle_p u = h_K(x,u) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

Definition 3.1. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the truncated problem $(T_{\mu,K})$ if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega h_K(x,u)v \, dx$$

for every $v \in W_0^{1,p}(\Omega)$. We introduce

$$\Psi(u) = \int_\Omega G_K(u) \, dx,$$

where $G_K(u) = \int_0^u g_K(s) \, ds$. It is easy to see that the functional $E = \Phi - \lambda J - \mu\Psi$ is of class C^1, and its derivative is given by

$$\begin{aligned} &\langle \Phi'(u) - \lambda J'(u) - \mu\Psi'(u), v\rangle \\ &\quad = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx - \lambda \int_\Omega f(x,u)v \, dx - \mu \int_\Omega g_K(u)v \, dx, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairs between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, $1/p + 1/p' = 1$. Moreover, the critical points of the functional E are the weak solutions of problem $(T_{\mu,K})$.

From (f_2), for small $\varepsilon > 0$, there exists $\delta > 0$ such that the function f satisfies

$$|f(x,s)| \leq \varepsilon |s|^{p-1}, \quad \text{for } |s| < \delta. \tag{3.1}$$

For each compact interval $[a,b] \subset (1/\theta, +\infty)$, $\lambda \in [a,b]$, considering (f_1) and $|g_K| \leq K^{r-q}|u|^{q-1}$, $q \in (p, p^*)$, we can choose constants $C > 0$ such that

$$|h_K(x,s)| \leq C|s|^{p-1} + \mu K^{r-q}|s|^{q-1}, \tag{3.2}$$

for all $s \in \mathbf{R}$. Hence, $h_K(x,u)$ is a superlinear function with subcritical growth. $\Psi(u)$ has a compact derivative in $W_0^{1,p}(\Omega)$. By Lemma 2.1 and Lemma 2.2, all the hypotheses of Theorem A are satisfied. Then there exists $\gamma > 0$, with the following the property: for every $\lambda \in [a,b] \subset (1/\theta, +\infty)$, there exists $\delta_0 > 0$ such that, for $\mu \in [0, \delta_0]$, problem $(T_{\mu,K})$ has at least three solutions u_0, u_1 and u_2 in $W_0^{1,p}(\Omega)$ whose $W_0^{1,p}(\Omega)$-norms are less than γ, i.e., $\|u_i\|_{W_0^{1,p}} \leq \gamma$, $i = 0,1,2$, where γ depends on λ, but does not depend on μ or K (see **[14,** (5)] for details). If the three solutions u_i, $i = 0,1,2$, satisfy

$$|u_i(x)| \leq K, \quad \text{a.e. } x \in \Omega, \ i = 0,1,2, \tag{3.3}$$

then, in view of the definition g_K, we have $h_K(x,u) = \lambda f(x,u) + \mu|u|^{r-2}u$ and therefore u_i, $i = 0,1,2$, are also solutions of the original problem (1.1). Thus, in order to prove Theorem 1.1, it suffices to show that there exists $\delta > 0$ such that, for $\mu \in [0,\delta]$, the solutions obtained by Theorem A satisfy inequality (3.3).

Proof of Theorem 1.1. Our aim is to show that there exists $\delta > 0$ such that, for $\mu \in [0,\delta]$, the solutions u_i, $i = 0,1,2$ verify inequality (3.3). To save notation, we will denote $u := u_i$, $i = 0,1,2$.

Set $u_+ = \max\{u, 0\}$, $u_- = -\min\{u,0\}$; then $|u| = u_+ + u_-$. We can argue with the positive and negation part of u separately.

Firstly, we deal with u_+. For each $L > 0$, we define the following functions

$$u_L = \begin{cases} u_+ & \text{if } 0 \leq u_+ \leq L, \\ L & \text{if } u_+ \geq L, \end{cases}$$

$z_L = u_L^{p(\beta-1)}u_+$ and $w_L = u_L^{\beta-1}u_+$, where $\beta > 1$ will be fixed later.

Taking z_L as a test function in problem $(T_{\mu,K})$, we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla z_L\,dx = \int_\Omega h_K(x,u) z_L\,dx. \tag{3.4}$$

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u\nabla z_L\,dx\\
&= \int_\Omega |\nabla u_+ - \nabla u_-|^{p-2}(\nabla u_+ - \nabla u_-)\nabla(u_L^{p(\beta-1)}u_+)\,dx\\
&= \int_\Omega |\nabla u_+ - \nabla u_-|^{p-2}(\nabla u_+ - \nabla u_-)\\
&\quad \times (u_L^{p(\beta-1)}\nabla u_+ + p(\beta-1)u_L^{p(\beta-1)-1}u_+\nabla u_L)\,dx\\
&= \int_\Omega |\nabla u_+|^p u_L^{p(\beta-1)}dx + p(\beta-1)\\
&\quad \times \int_\Omega |\nabla u_+|^{p-2}u_L^{p(\beta-1)-1}u_+\nabla u_L\nabla u_+ dx\\
&= \int_\Omega u_L^{p(\beta-1)}|\nabla u_+|^p dx + p(\beta-1)\\
&\quad \times \int_{\{u_+\le L\}} u_L^{p(\beta-1)-1}u_+\nabla u_L|\nabla u_+|^{p-2}\nabla u_+ dx.
\end{aligned} \tag{3.5}$$

From the definition of u_L, we have

$$\begin{aligned}
p(\beta-1)&\int_{\{u_+\le L\}} u_L^{p(\beta-1)-1}u_+\nabla u_L|\nabla u_+|^{p-2}\nabla u_+ dx\\
&= p(\beta-1)\int_{\{u_+\le L\}} u_+^{p(\beta-1)}|\nabla u_+|^p dx \ge 0.
\end{aligned} \tag{3.6}$$

Due to (3.2) and (3.4)–(3.6), we have

$$\begin{aligned}
&\int_\Omega u_L^{p(\beta-1)}|\nabla u_+|^p dx\\
&\le \int_\Omega h_K(x,u)z_L\,dx\\
&\le \int_\Omega |h_K(x,u)z_L|\,dx\\
&\le \int_\Omega (C|u|^{p-1} + \mu K^{r-q}|u|^{q-1})u_L^{p(\beta-1)}u_+ dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega (C(u_+ + u_-)^{p-1} + \mu K^{r-q}(u_+ + u_-)^{q-1}) u_L^{p(\beta-1)} u_+ dx \\
&= \int_\Omega (C u_+^p u_L^{p(\beta-1)} + \mu K^{r-q} u_+^q u_L^{p(\beta-1)})\, dx. \qquad (3.7)
\end{aligned}
$$

By the Sobolev embedding, we get

$$
\begin{aligned}
&\left(\int_\Omega |w_L|^{p^*} dx \right)^{p/p^*} \\
&\leq S^{-1} \int_\Omega |\nabla w_L|^p dx \\
&= S^{-1} \int_\Omega |\nabla (u_+ u_L^{\beta-1})|^p dx \\
&= S^{-1} \int_\Omega |(\beta-1) u_+ u_L^{\beta-2} \nabla u_L + u_L^{\beta-1} \nabla u_+|^p dx \\
&\leq 2^{p-1} S^{-1} \Bigg(\int_\Omega |(\beta-1) u_+ u_L^{\beta-2} \nabla u_L|^p dx \\
&\qquad + \int_\Omega |u_L^{\beta-1} \nabla u_+|^p dx \Bigg) \\
&= 2^{p-1} S^{-1} \Bigg(\int_{\{u_+ \leq L\}} (\beta-1)^p |u_L|^{p(\beta-1)} |\nabla u_+|^p dx \\
&\qquad + \int_\Omega |u_L^{\beta-1} \nabla u_+|^p dx \Bigg) \\
&\leq 2^{p-1} S^{-1} \Bigg(\int_\Omega (\beta-1)^p |u_L|^{p(\beta-1)} |\nabla u_+|^p dx \\
&\qquad + \int_\Omega |u_L^{\beta-1} \nabla u_+|^p dx \Bigg) \\
&= 2^{p-1} S^{-1} \Bigg((\beta-1)^p \int_\Omega |u_L|^{p(\beta-1)} |\nabla u_+|^p dx \\
&\qquad + \int_\Omega |u_L^{\beta-1} \nabla u_+|^p dx \Bigg) \\
&= 2^{p-1} S^{-1} ((\beta-1)^p + 1) \int_\Omega |u_L|^{p(\beta-1)} |\nabla u_+|^p dx \\
&= 2^{p-1} S^{-1} \beta^p \left(\left(\frac{\beta-1}{\beta} \right)^p + \frac{1}{\beta^p} \right) \int_\Omega |u_L|^{p(\beta-1)} |\nabla u_+|^p dx, \qquad (3.8)
\end{aligned}
$$

where S is given by

$$S = \inf_{u\neq 0, u\in W_0^{1,p}} \frac{\int_\Omega |\nabla u|^p dx}{(\int_\Omega |u|^{p^*} dx)^{p/p^*}}.$$

Since $\beta > 1$, we have $1/\beta^p < 1$ and $((\beta-1)/\beta)^p < 1$. From (3.8), we get

$$\begin{aligned}
2^{p-1}S^{-1}\beta^p\left(\left(\frac{\beta-1}{\beta}\right)^p + \frac{1}{\beta^p}\right)&\int_\Omega |u_L|^{p(\beta-1)}|\nabla u_+|^p dx \\
&\leq 2^p S^{-1}\beta^p \int_\Omega |u_L|^{p(\beta-1)}|\nabla u_+|^p dx \\
&= C'\beta^p \int_\Omega |u_L|^{p(\beta-1)}|\nabla u_+|^p dx \\
&\leq C'\beta^p \int_\Omega (Cu_+^p u_L^{p(\beta-1)} + \mu K^{r-q} u_+^q u_L^{p(\beta-1)}) dx, \qquad (3.9)
\end{aligned}$$

where $C' = 2^p S^{-1}$ and the last inequality is given by (3.7). Considering the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $\|u_+\|_{W_0^{1,p}} \leq \gamma$, we have

$$\|u_+\|_{L^{p^*}} \leq S^{-1/p}\|u_+\|_{W_0^{1,p}} \leq \gamma S^{-1/p}. \qquad (3.10)$$

Let $\alpha^* = (pp^*)/(p^*-q+p)$. Since $u_+^q u_L^{p(\beta-1)} = u_+^{q-p} w_L^p$ and $u_+^p u_L^{p(\beta-1)} = w_L^p$, we can use Hölder's inequality and (3.8)–(3.10) to conclude that, whenever $w_L \in L^{\alpha^*}(\Omega)$, the following holds:

$$\begin{aligned}
\|w_L\|_{L^{p^*}}^p &\leq C'\beta^p \int_\Omega (Cu_+^p u_L^{p(\beta-1)} + \mu K^{r-q} u_+^q u_L^{p(\beta-1)}) dx, \\
&= C'\beta^p (C\int_\Omega w_L^p dx + \mu K^{r-q}\int_\Omega u_+^{q-p} w_L^p dx), \\
&\leq C'\beta^p \bigg[C(m(\Omega))^{(q-p)/p^*}\left(\int_\Omega w_L^{\alpha^*} dx\right)^{p/\alpha^*} \\
&\quad + \mu K^{r-q}\left(\int_\Omega |u_+|^{p^*} dx\right)^{(q-p)/p^*}\left(\int_\Omega w_L^{\alpha^*} dx\right)^{p/\alpha^*}\bigg] \\
&\leq C'\beta^p (C(m(\Omega))^{(q-p)/p^*} + \mu K^{r-q} S^{-(q-p)/p}\gamma^{q-p})\|w_L\|_{L^{\alpha^*}}^p. \qquad (3.11)
\end{aligned}$$

Set $\beta := p^*/\alpha^*$. Since $u_L \leq u_+$, we conclude that $w_L \in L^{\alpha^*}(\Omega)$, whenever $u_+^\beta \in L^{\alpha^*}(\Omega)$. If this is the case, it follows from the above inequality that

$$\left(\int_\Omega u_L^{p^*(\beta-1)} u_+^{p^*}\,dx\right)^{p/p^*} \le C'\beta^p [C(m(\Omega))^{(q-p)/p^*}$$

$$+\mu K^{r-q} S^{-(q-p)/p}\gamma^{q-p}]\left(\int_\Omega |u_L^{\beta-1}u_+|^{\alpha^*}\,dx\right)^{p/\alpha^*}$$

$$\le C'\beta^p C_{\mu,K}\|u_+\|^{\beta p}_{L^{\beta\alpha^*}}, \tag{3.12}$$

where $C_{\mu,K} = C(m(\Omega))^{(q-p)/p^*} + \mu K^{r-q} S^{-(q-p)/p}\gamma^{q-p}$. By Fatou's lemma in the variable L, we get

$$\|u_+\|_{L^{\beta p^*}} \le (C'C_{\mu,K})^{1/(p\beta)}\beta^{1/\beta}\|u_+\|_{L^{\beta\alpha^*}}, \tag{3.13}$$

where $u_+^{\beta\alpha^*} \in L^1(\Omega)$. Since $\beta = p^*/\alpha^* > 1$ and $u_+ \in L^{p^*}(\Omega)$, the inequality (3.13) holds for this choice of β. Thus, since $\beta^2\alpha^* = \beta p^*$, it follows that (3.13) also holds with β replaced by β^2. Hence,

$$\begin{aligned}\|u_+\|_{L^{\beta^2 p^*}} &\le (C'C_{\mu,K})^{1/\beta^2 p}\beta^{2/\beta^2}\|u_+\|_{L^{\beta^2\alpha^*}}\\ &\le (C'C_{\mu,K})^{(1/p)((1/\beta)+(1/\beta^2))}\beta^{(1/\beta)+(2/\beta^2)}\|u_+\|_{L^{\beta\alpha^*}}.\end{aligned}$$

By iterating this process and $\beta\alpha^* = p^*$, we obtain

$$\|u_+\|_{L^{\beta^m p^*}} \le (C'C_{\mu,K})^{p^{-1}\sum_{i=1}^m \beta^{-i}}\beta^{\sum_{i=1}^m i\beta^{-i}}\|u_+\|_{L^{p^*}}. \tag{3.14}$$

Taking the limit as $m \to \infty$ in (3.14), we have

$$\|u_+\|_{L^\infty} \le (C'C_{\mu,K})^{\sigma_1}\beta^{\sigma_2}\|u_+\|_{L^{p^*}} \le (C'C_{\mu,K})^{\sigma_1}\beta^{\sigma_2}\gamma S^{-1/p}, \tag{3.15}$$

with $\sigma_1 = p^{-1}\sum_{i=1}^\infty \beta^{-i}$, $\sigma_2 = \sum_{i=1}^\infty i\beta^{-i}$, $\beta > 1$. Next, we will find some suitable value of K and μ, such that the inequality

$$(C'C_{\mu,K})^{\sigma_1}\beta^{\sigma_2}\gamma S^{-1/p} \le \frac{K}{2} \tag{3.16}$$

holds, where $C' = 2^p S^{-1}$, $C_{\mu,K} = C(m(\Omega))^{(q-p)/p^*} + \mu K^{r-q} S^{-(q-p)/p}\gamma^{q-p}$. This implies

$$C(m(\Omega))^{(q-p)/p^*} + \mu K^{r-q} S^{-(q-p)/p}\gamma^{q-p} \le \frac{1}{C'}\left(\frac{K}{2\gamma S^{-1/p}\beta^{\sigma_2}}\right)^{1/\sigma_1}. \tag{3.17}$$

Choose K to satisfy the inequality

$$\frac{1}{C'}\left(\frac{K}{2\gamma S^{-1/p}\beta^{\sigma_2}}\right)^{1/\sigma_1} - C(m(\Omega))^{(q-p)/p^*} > 0 \tag{3.18}$$

and fix μ' such that
(3.19)

$$\mu \le \mu' = \frac{S^{(q-p)/p}}{K^{r-q}\gamma^{q-p}}\left[\frac{1}{C'}\left(\frac{K}{2\gamma S^{-1/p}\beta^{\sigma_2}}\right)^{1/\sigma_1} - C(m(\Omega))^{(q-p)/p^*}\right].$$

Thus, we obtain (3.16) for $\mu \in [0, \mu']$, i.e.,

$$\|u_+\|_{L^\infty} \le \frac{K}{2}, \quad \text{for } \mu \in [0, \mu']. \tag{3.20}$$

Similarly, we can also have the estimate for the u_-, i.e.,

$$\|u_-\|_{L^\infty} \le \frac{K}{2}, \quad \text{for } \mu \in [0, \mu'].$$

Set $\delta = \min\{\delta_0, \mu'\}$, where $\delta_0 > 0$ is given by Theorem A. Since $|u| = u_+ + u_-$, we have

$$\|u\|_{L^\infty} \le K \quad \text{for } \mu \in [0, \delta].$$

Considering this fact and $u := u_i$, $i = 1, 2, 3$, we get

$$\|u_i\|_{L^\infty} \le K, \quad i = 0, 1, 2, \text{for } \mu \in [0, \delta].$$

We obtain inequality (3.3). The proof is complete. □

Remark. By means of the truncation and the Moser iteration **[11]**, we can deal with problem (1.3) with critical and supercritical growth, i.e.,

$$\begin{cases} -\triangle_p u = \lambda f(x,u) + \mu g(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $|g(x,s)| \le C_0(1+|s|^{r-1})$, $r \ge p^* = (Np)/(N-p)$, $C_0 > 0$ a constant, and $f\colon \Omega \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function, with μ and λ nonnegative parameters. For problem (1.3), if f satisfies the conditions (f_1)–(f_4), we also have the following result similar to Theorem 1.1.

The second of the main results in this paper is stated as follows:

Theorem 3.2. *Let f satisfy (f_1)–(f_4). Then, for each compact interval $[a,b] \subset (1/\theta, +\infty)$, there exists $\gamma > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, problem* (1.3) *has at least three solutions u_i, $i = 0,1,2$, in $W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$, whose $W_0^{1,p}(\Omega)$-norms are less than γ.*

In fact, the truncation of $g_K(x,s)$ can be given by

$$g_K(x,s) = \begin{cases} g(x,s) & \text{as } |s| \le K, \\ \min\{g(x,s), C_0(1+K^{r-q}|s|^{q-2}s)\} & \text{as } |s| > K, \end{cases} \tag{3.21}$$

where $q \in (p, p^*)$. Then g_K satisfies

$$|g_K(x,s)| \le C_0(1+K^{r-q}|s|^{q-1}) \quad \text{as } s \in \mathbf{R}. \tag{3.22}$$

Set $h_K(x,s) = \lambda f(x,s) + \mu g_K(x,s)$. The truncated problems associated to h_K

$$\begin{cases} -\triangle_p u = h_K(x,u) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{3.23}$$

By Theorem A and the technique of Theorem 1.1 or (3.1)-(3.20), we can prove that there exists $\delta > 0$ such that the solutions for the truncated problems (3.23) satisfy $\|u\|_{L^\infty} \le K$ for $\mu \in [0,\delta]$; and, in view of the definition g_K, we have $h_K(x,u) = \lambda f(x,u) + \mu g(x,u)$. Therefore, $u := u_i$, $i = 0,1,2$, obtained by Theorem A, are also solutions of the original problem (1.3).

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