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QUALITATIVE PROPERTIES AND STANDARD ESTIMATES OF SOLUTIONS FOR SOME FOURTH ORDER ELLIPTIC EQUATIONS

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ABSTRACT. In this paper, first, we make the estimates for a class of fourth order elliptic equations in different domains and boundary conditions. Consequently, we study the qualitative properties of solutions with prescribed Q-curvature. Finally, we also will obtain some radially symmetric results by using moving plane methods.

1. Introduction. In this paper, we make estimates to the following fourth order elliptic equation:

$$(*)\qquad \begin{cases} \Delta^2 u(x) = Q(x)e^{4u} & \text{in } \Omega \subset R^4; \\ u = \triangle u = 0 & \text{in } \partial\Omega, \end{cases}$$

and investigate properties of the solutions to the following fourth order elliptic equation:

$$(**)\qquad \Delta^2 u(x) = Q(x)e^{4u}, \quad x \in R^4,$$

where Ω is a bounded smooth domain and $Q(x)$ is the given function in $L^p(\Omega)$ for some $1 < p \leq \infty$. We assume that $u \in L^1(\Omega)$, $e^{4u} \in L^{p'}(\Omega)$ (where p' is the conjugate exponent of p) so that $(*)$ has a meaning in the sense of distributions. A first question is whether one can conclude that $u \in L^\infty(\Omega)$ for $(*)$. As we will see in Section 2, the answer is positive.

Recently, a series of works has been done to understand the existence and qualitative properties of the solutions of $(**)$. When $Q(x) = 6$, Lin **[5]** had given a complete classification of u in terms of its growth,

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or of the behavior of $\triangle u$ at ∞. Xu [**10**] had done similar work by using moving sphere methods. Wei and Xu [**8**] and Martinazzi [**6**] also gave a complete classification of solutions for higher order conformally invariant equations compared to $(**)$. In Section 3, we consider more general functions $Q(x)$. This is considered as a generalization of [**5**]. First, we obtain the asymptotic behavior of solutions near infinity. Consequently, we prove that all solutions satisfy an identity, which is similar to the well-known Kazdan-Warner condition. Finally, using the harmonic asymptotic expansion at ∞ in [**5**], we show that all the solutions are radially symmetric provided Q is radially symmetric and non-increasing. This part can be viewed as the completion of [**5**].

2. L^∞-boundedness for a single solution of $\triangle^2 u = Q(x)e^{4u}$. Assume $\Omega \subset R^4$ is a bounded domain, and let h be a solution of

$$\begin{cases} \Delta^2 h(x) = f(x) & \text{in } \Omega \subset R^4; \\ h = \triangle h = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.1}$$

Following the argument of Brezis and Merle [**1**], Lin obtained the following lemma:

Lemma 2.1. [**5**] *Suppose $f \in L^1(\overline{\Omega})$. For any $\delta \in (0, 32\pi^2)$, there exists a constant $C_\delta > 0$ such that the inequality*

$$\int_\Omega \exp\left(\frac{\delta|h|}{||f||_{L^1}}\right) dx \leq C_\delta(\operatorname{diam}\Omega)^4,$$

where $\operatorname{diam}\Omega$ *denotes the diameter of* Ω.

By use of the above lemma, we obtain the following consequent results:

Theorem 2.1. *Let u be a solution of equation* (2.1) *with $f \in L^1(\Omega)$. Then, for every constant $k > 0$,*

$$e^{ku} \in L^1(\Omega).$$

Proof. Letting $0 < \varepsilon < 1/k$, we may split f as $f = f_1 + f_2$ with $||f_1||_1 < \varepsilon$ and $f_2 \in L^\infty(\Omega)$. Write u_i as the solution of

$$\begin{cases} \Delta^2 u_i = f_i & \text{in } \Omega; \\ u_i = \triangle u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1, we find $\int_\Omega \exp\left[|u_1(x)|/||f_1||_1\right] < \infty$, and thus

$$\int_\Omega \exp\left[k|u_1|\right] < \infty.$$

The conclusion follows since $|u| \leq |u_1| + |u_2|$ and $u_2 \in L^\infty(\Omega)$. □

Theorem 2.2. *Assume $u \in L^1_{\mathrm{Loc}}(\Omega)$, $\triangle^2 u \in L^1_{\mathrm{Loc}}(\Omega)$. Then for every constant $k > 0$*

$$e^{ku} \in L^1_{\mathrm{loc}}(\Omega).$$

Proof. Without loss of generality, we may assume that Ω as B_R defines the ball of radius R centered at θ. For ε small enough, we split $\triangle^2 u = f_1 + f_2$ with $||f_1||_1 < \varepsilon$ and $f_2 \in L^\infty(\Omega)$. Write $u = u_1 + u_2 + u_3$, where u_i $(i = 1, 3)$ are, respectively, the solutions of

$$\begin{cases} \Delta^2 u_1 = f_1 & \text{in } B_{R/2}; \\ u_1 = \triangle u_1 = 0 & \text{on } \partial B_{R/2} \end{cases}$$

and

$$\begin{cases} \Delta^2 u_3 = f_2 & \text{in } B_{R/2}; \\ u_3 = \triangle u_3 = 0 & \text{on } \partial B_{R/2}. \end{cases}$$

It follows from Lemma 2.1 that $e^{k|u_1|} \in L^1_{\mathrm{Loc}}(B_R)$. Since standard elliptic estimates apply (see **[4]**), we have $|u_3|_{L^\infty(B_{R/2})} \leq c$. Hence, we have $e^{k|u_3|} \in L^1_{\mathrm{Loc}}(B_R)$. Since $\triangle u_2$ is harmonic and $\triangle u \in L^1_{\mathrm{Loc}}(B_R)$, which is obtained from the Ehrling-Nirenberg-Gagliardo inequality, by the mean value theorem for harmonic functions, we have

$$|\triangle u_2|_{L^\infty(B_{R/4})} \leq c.$$

Thus, by $u \in L^1_{\mathrm{Loc}}(B_R)$ and the above inequality, we have

$$|u_2|_{L^\infty(B_{R/8})} \leq c.$$

So $e^{ku_2} \in L^1_{\text{Loc}}(B_R)$. At last, the conclusion follows since $|u| \leq |u_1| + |u_2| + |u_3|$. □

Remark 2.1. Theorem 2.2 is a local form of Theorem 2.1.

Theorem 2.3. *Suppose u is a solution of equation $(*)$ with $Q \in L^p(\Omega)$ and $e^{4u} \in L^{p'}(\Omega)$ for some $1 < p \leq \infty$. Then $u \in L^\infty(\Omega)$.*

Proof. By Theorem 2.1, we know that $e^{ku} \in L^1(\Omega)$ for all k, i.e., $e^u \in L^r(\Omega)$ for all $r < \infty$. It follows that $Qe^{4u} \in L^{p-\delta}$ for all $\delta > 0$ if $p < \infty$, and $Qe^{4u} \in L^r(\Omega)$ for all $r < \infty$ if $p = \infty$. Standard elliptic estimates imply that $\triangle u \in L^\infty(\Omega)$. Hence, combining $u = 0$ with $\partial\Omega$, we have $u \in L^\infty(\Omega)$. □

Corollary 2.1. *Suppose u is a solution of*

$$\begin{cases} \Delta^2 u = Qe^{4u} + f(x) & \textit{in } \Omega; \\ u = g_1, \ \triangle u = g_2 & \textit{on } \partial\Omega \end{cases}$$

with $Q \in L^p(\Omega)$ and $e^{4u} \in L^{p'}(\Omega)$ for some $1 < p \leq \infty$, where $g_1, \ g_2 \in L^\infty(\partial\Omega)$ and $f \in L^q(\Omega)$ for some $q > 1$. Then $u \in L^\infty(\Omega)$.

Proof. Let w be the solution of

$$\begin{cases} \Delta^2 w = f(x) & \text{in } \Omega; \\ w = g_1, \ \triangle w = g_2 & \text{on } \partial\Omega \end{cases}$$

so that $w \in L^\infty(\Omega)$. The function $\widetilde{u} = u - w$ satisfies

$$\begin{cases} \Delta^2 \widetilde{u} = Qe^{4w}e^{4\tilde{u}} & \text{in } \Omega; \\ \widetilde{u} = 0, \ \triangle \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

and we are reduced to the assumption of Theorem 2.3. □

Theorem 2.4. *Suppose $u \in L^1_{\text{Loc}}(R^4)$ is solution of equation $(**)$ with $Q \in L^p_{\text{Loc}}(R^4)$ and $e^{4u} \in L^{p'}_{\text{Loc}}(R^4)$ for some $1 < p \leq \infty$. Then $u \in L^\infty_{\text{Loc}}(R^4)$.*

Proof. Without loss of generality, let $B_R(\theta) \subset R^4$. Fix $\varepsilon > 0$ small enough and split Qe^{4u} as $Qe^{4u} = f_1 + f_2$ with $||f_1||_1 < \varepsilon$ and

$f_2 \in L^\infty(B_R)$, u_1, u_2, respectively, solutions of

$$\begin{cases} \Delta^2 u_1 = f_1 & \text{in } B_R; \\ u_1 = \triangle u_1 = 0 & \text{on } \partial B_R, \end{cases}$$

and

$$\begin{cases} \Delta^2 u_2 = f_2 & \text{in } B_R; \\ u_2 = \triangle u_2 = 0 & \text{on } \partial B_R. \end{cases}$$

It follows from Lemma 2.1 that $e^{k|u_1|} \in L^1(B_R)$. Since standard elliptic estimates apply (see **[4]**), we have $|u_2|_{L^\infty(B_R)} \leq c$. Hence, we have $e^{k|u_2|} \in L^1(B_R)$. Let $u_3 = u - u_1 - u_2$. Since $\triangle u_3$ is harmonic, by the mean value theorem for harmonic functions, we have

$$|\triangle u_3|_{L^\infty(B_{R/2})} \leq c.$$

Thus,

$$|u_3|_{L^\infty(B_{R/4})} \leq c.$$

It follows from

$$\triangle^2 u = (Qe^{4u_1})e^{4u_2+4u_3}$$

and standard elliptic estimates that

$$||\triangle u||_{L^\infty(B_{R/8})} \leq c.$$

So

$$||u||_{L^\infty(B_{R/16})} \leq c. \qquad \square$$

From **[1]**, Brezis and Merle imply that u is bounded from above when u satisfies $-\triangle u = V(x)e^u$ and other conditions. This result is used to study the qualitative properties and classification of solutions for some second order elliptic equation (see **[2, 3, 7]**). Now, one naturally asks: is any solution u to equation $(**)$ with $\int_{R^4} Qe^{4u} < +\infty$ bounded from above? We will partially answer this problem and obtain the following result:

Theorem 2.5. *Assume $Q(x)$ is a positive bounded away from 0 and bounded from the above function and u is a C^2 solution of $(**)$ with $\int_{R^4} e^{4u} < +\infty$, $u(x) = \circ(|x|^2)$. Then $u^+ \in L^\infty(R^4)$.*

Before we begin our proof, we need the following lemmas.

Lemma 2.2. **[9, 11]**. *Suppose u is a C^2 function on R^4 such that*

(a) *Qe^{4u} is in $L^1(R^4)$ with $0 < m \le Q \le M$ for some constants m, M;*
(b) *in the sense of weak derivatives, u satisfies the following equation:*

$$\triangle u + \frac{2}{\beta_0}\int_{R^4} \frac{Q(y)e^{4u(y)}}{|x-y|^2}\,dy = 0.$$

Then there is a constant $c > 0$, depending on u, such that $|\triangle u|(x) \le c$ on R^4, where β_0 is given by $(-\triangle_x)^2(\ln(1/|x-y|)) = \beta_0\delta_y(x)$.

In fact, $\beta_0 = 8\pi^2$.

Lemma 2.3. **[9]**. *Suppose u is a C^2 function on R^4 such that $0 \le (-\triangle)u(x) \le A$ on R^4 for some constant A and $\int_{R^4} Q(y)e^{4u(y)}dy = \alpha < \infty$ with $0 < m \le Q \le M$. Then there exists a constant B, depending only on A, m, M and α such that $u(x) \le B$ on R^4.*

Lemma 2.4. *Suppose u is a solution of $(**)$. Let*

$$w(x) = \frac{1}{8\pi^2}\left(\int_{R^4} \frac{\ln|x-y|}{|y|+1} Q(y)e^{4u(y)}dy\right).$$

Then there exists a constant c such that

$$w(x) \le \beta\ln(|x|+1) + c,$$

where $\beta = (\int_{R^4} Q(y)e^{4u(y)}dy)/8\pi^2$.

Proof. For $|x| \ge 4$, we decompose $R^4 = A_1 \cup A_2$, where $A_1 = \{y||y-x| \le |x|/2\}$ and $A_2 = \{y||y-x| \ge |x|/2\}$. For $y \in A_1$, we have $|y| \ge |x| - |x-y| \ge |x|/2 \ge |x-y|$, which implies

$$\ln\frac{|x-y|}{|y|+1} \le 0.$$

Since $|x-y| \le |x|+|y| \le |x|(|y|+1)$ for $|x|, |y| \ge 2$ and $\ln|x-y| \le \ln|x| + c$ for $|x| \ge 4$ and $|y| \le 2$, we have

$$\begin{aligned} w(x) &\le \frac{1}{8\pi^2} \int_{A_2} \ln \frac{|x-y|}{|y|+1} Q(y) e^{4u(y)} dy \\ &\le \frac{1}{8\pi^2} \left(\int_{R^4} Q(y) e^{4u(y)} dy \right) \ln|x| + c \\ &= \beta \ln(|x|+1) + c. \end{aligned}$$

□

Lemma 2.5. *Suppose u is a solution of $(**)$ with $u(x) = \circ(|x|^2)$. Then $\triangle u(x)$ can be represented by*

$$\triangle u(x) = -\frac{1}{4\pi^2} \int_{R^4} \frac{Q(y) e^{4u(y)}}{|x-y|^2} dy. \tag{2.2}$$

Proof. Let $v = u + w$. It is obvious that $\triangle^2 v \equiv 0$ in R^4. Similar to the proof of Lin **[5]**, we have for any $x_0 \in R^4$ and $r > 0$,

$$2\pi^2 r^3 \exp\left(\frac{r^2}{2} \triangle v(x_0) \right) \le e^{-4v(x_0)} \int_{|x-x_0|=r} e^{4v} d\sigma.$$

Since $v = u + w \le u(x) + \beta \ln|x| + c$ follows from Lemma 2.4, we have

$$r^{3-4\beta} \exp\left(\frac{\triangle v(x_0)}{2} r^2 \right) \in L^1[1, +\infty].$$

Thus, $\triangle v(x_0) \le 0$ for all $x_0 \in R^4$. By Liouville's theorem, $\triangle v(x) \equiv -c_1$ in R^4 for some constant $c_1 \ge 0$. Hence, we have

$$\triangle u(x) = -\frac{1}{4\pi^2} \int_{R^4} \frac{Q(y) e^{4u(y)}}{|x-y|^2} dy - c_1. \tag{2.3}$$

Now, we claim that $c_1 = 0$. Otherwise, we have $\triangle u(x) \le -c_1 < 0$ for $|x| \ge R_0$ where R_0 is sufficiently large. Let

$$h(y) = u(y) + \varepsilon |y|^2 + A(|y|^{-2} - R_0^{-2}), \tag{2.4}$$

where ε is small enough such that

$$\triangle h(y) = \triangle u + 8\varepsilon < -\frac{c_1}{2} < 0 \tag{2.5}$$

for $|y| > R_0$, and A is sufficiently large so that $\inf\limits_{|y|\geq R_0} h(y)$ is achieved by some $y_0 \in R^4$ with $|y_0| > R_0$. Applying the maximum principle to (2.5) at y_0, we have a contradiction. Hence, our claim is proved. □

Proof of Theorem 2.5. By Lemma 2.5 and Lemma 2.3, our conclusion holds. □

3. Qualitative properties of solutions of $\triangle^2 u = Q(x)e^{4u}$. In this section, we study the qualitative properties of solutions of equation $(**)$. Following our Theorem 2.5 and Chen **[3]**, we obtain the following results:

Theorem 3.1. *Assume that $Q(x)$ is a positive C^1 function bounded away from and above 0 and u is a C^2 solution of equation $(**)$ with $\int_{R^4} e^{4u}dx < \infty$, $u(x) = \circ(|x|^2)$. Then*

$$-\beta \ln(|x|+1) - c \leq u(x) \leq -\beta \ln(|x|+1) + c \tag{3.1}$$

with $\beta > 1$. Furthermore, we have the following identity

$$\int_{R^4} (x, \nabla Q)e^{4u}dx = \pi^2\beta(16\beta - 32). \tag{3.2}$$

Theorem 3.2. *Suppose u satisfies the assumptions of Theorem* 3.1 *and Q is radially symmetric and monotone decreasing. Then u is radially symmetric and monotone decreasing.*

Lemma 3.1. *Assume u satisfies the assumptions of Theorem* 3.1. *Then*

$$\frac{w(x)}{\ln|x|} \longrightarrow \beta, \quad \textit{uniformly as } |x| \to \infty,$$

where $w(x)$ and β have been given in Section 2.

Proof. We need only to verify that

$$I = \int_{R^4} \frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|} Q(y)e^{4u(y)}dy \longrightarrow 0$$
$$\text{as } |x| \to \infty.$$

Write $I = I_1 + I_2 + I_3$ as the integrals on the regions $D_1 = \{y : |x-y| \leq 1\}$, $D_2 = \{y : |x-y| > 1 \quad \text{and} \quad |y| \leq k\}$ and $D_3 = \{y : |x-y| > 1 \quad \text{and} \quad |y| > k\}$, respectively. We may assume that $|x| \geq 3$.

(a) To estimate I_1, we simply notice that

$$|I_1| \leq C \int_{|x-y|\leq 1} Q(y)e^{4u(y)}dy - \frac{1}{\ln|x|} \int_{|x-y|\leq 1} \ln|x-y| Q(y)e^{4u(y)}dy.$$

Then, by the boundedness of Qe^{4u} (see Theorem 2.5 in Section 2) and $\int_{R^4} Q(y)e^{4u(y)}dy$, we see that $I_1 \to 0$ as $|x| \to \infty$.

(b) For each fixed k, in region D_2, we have, as $|x| \to \infty$,

$$\frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|} \longrightarrow 0.$$

Hence, $I_2 \to 0$.

(c) To see $I_3 \to 0$, we use the fact that, for $|x-y| > 1$,

$$\left|\frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|}\right| \leq c.$$

Then let $k \to \infty$. □

Lemma 3.2. *Assume u satisfies the assumptions of Theorem* 3.1. *Then*

$$u(x) = \frac{1}{8\pi^2} \int_{R^4} \ln \frac{|y|+1}{|x-y|} Q(y)e^{4u(y)}dy + c_0,$$

where c_0 is a constant.

Proof. By Lemma 2.5, we have $\triangle(u+w) = 0$ in R^4. By Theorem 2.5 in Section 2, we have $u^+ \in L^\infty$, So, combining this result with Lemma 2.4, we have $u + w \leq c\ln|x| + c$, since $u + w$ is a harmonic function and, by the gradient estimates of harmonic functions, we have $u(x) + w(x) \equiv c$. □

Lemma 3.3. *Suppose u satisfies the assumptions of Theorem* 3.1. *Then $u(x) \geq -\beta\ln(|x|+1) - c$ and $\beta > 1$.*

Proof. By Lemma 2.4 and Lemma 3.2, we have

$$u(x) > -\beta\ln(|x|+1) - c.$$

From the above inequality and $\int_{R^4} e^{4u}dx < +\infty$, we have $\beta > 1$. $\square$

Lemma 3.4. *Suppose u satisfies the assumptions of Theorem 3.1. Then $u(x) \leq -\beta \ln(|x|+1) + c$.*

Proof. In fact, for $|x-y| \geq 1$, we have

$$|x| \leq |x-y|(|y|+1).$$

Then

$$\ln|x| - 2\ln(|y|+1) \leq \ln|x-y| - \ln(|y|+1).$$

Consequently,

$$\begin{aligned}
w(x) \geq& \frac{1}{8\pi^2}\int_{|x-y|\geq 1}(\ln|x| - 2\ln(|y|+1))Q(y)e^{4u(y)}dy \\
&+ \frac{1}{8\pi^2}\int_{|x-y|\leq 1}(\ln|x-y| - \ln(|y|+1))Q(y)e^{4u(y)}dy \\
\geq& \beta\ln|x| - \frac{\ln|x|}{8\pi^2}\int_{|x-y|\leq 1}Q(y)e^{4u(y)}dy \\
&+ \frac{1}{8\pi^2}\int_{|x-y|\leq 1}\ln|x-y|Q(y)e^{4u(y)}dy \\
&- \frac{1}{8\pi^2}\int_{R^4}\ln(|y|+1)Q(y)e^{4u(y)}dy \\
=& \beta\ln|x| + I_1 + I_2 + I_3.
\end{aligned}$$

Taking into account the fact that

$$\frac{u(x)}{\ln|x|} \longrightarrow -\beta \quad \text{and} \quad \beta > 1,$$

and, by the boundedness of $Q(x)$, we have

$$I_1, I_2 \longrightarrow 0 \quad \text{as} \quad |x| \to \infty$$

and I_3 is finite. Therefore,

$$w(x) \geq \beta\ln(|x|+1) - c.$$

By Lemma 3.2, we have

$$u(x) \leq -\beta\ln(|x|+1) + c. \qquad \square$$

Proof of Theorem 3.1. By Lemma 3.3 and Lemma 3.4, then (3.1) holds. By Lin's Lemma 2.6 and Lemma 2.7 **[5]**, we can similarly infer that (3.2) holds. □

Proof of Theorem 3.2. By Theorem 3.1, we have $u(x) \to -\beta \ln |x|$ as $|x| \to \infty$, where $\beta > 1$. Let $\widetilde{v}(x) = -\triangle u(x)$. By Lin's revised Lemma 2.8 **[5]**, $\widetilde{v}(x)$ has a harmonic asymptotic expansion at ∞:

$$
\begin{cases} \widetilde{v}(x) & = \frac{1}{|x|^2}\left(2\beta + \sum_{j=1}^{4} \frac{a_j}{|x|^2}\right) + \bigcirc\left(\frac{1}{|x|^4}\right), \\ \widetilde{v}_{x_i} & = -\frac{4\beta x_i}{|x|^4} + \bigcirc\left(\frac{1}{|x|^4}\right), \\ \widetilde{v}_{x_i x_j} & = \bigcirc\left(\frac{1}{|x|^4}\right), \end{cases} \tag{3.3}
$$

where a_j $(j = 1, \ldots, 4)$ are constants. The remainder of the proof is essentially equal to Lin's proof. We omit it here. □

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