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TENSOR PRODUCTS OF UNBOUNDED OPERATOR ALGEBRAS

M. FRAGOULOPOULOU, A. INOUE AND M. WEIGT

ABSTRACT. The term GW^*-algebra means a generalized W^*-algebra and corresponds to an unbounded generalization of a standard von Neumann algebra. It was introduced by the second named author in 1978 for developing the Tomita-Takesaki theory in algebras of unbounded operators. In this note we consider tensor products of unbounded operator algebras resulting in a GW^*-algebra. Existence and uniqueness of the GW^*-tensor product is encountered, while "properly W^*-infinite" GW^*-algebras are introduced and their structure is investigated.

1. Introduction. It is known that the Tomita-Takesaki theory plays a significant role in the study of the structure of von Neumann algebras and physical applications. The extension of the Tomita-Takesaki theory to algebras of unbounded operators is a contribution of a long period of systematic studies by the second named author and some of his collaborators. These studies resulted in a monograph (see **[13]**) that builds on the structure of the so-called O^*-algebras on which the unbounded Tomita-Takesaki theory is based. O^*-algebras were introduced by Lassner **[15]**, in 1972, aiming for the solution of questions that appear in quantum statistics and quantum dynamics, that the algebraic formulation of quantum theories presented by Haag and Kastler, in 1964, could not face; in this regard, see also **[2, 18]** and the corresponding bibliography therein.

Among O^*-algebras, one finds the GW^*-algebras (Inoue) (see **[12]**), which are unbounded generalizations of the standard von Neumann algebras. Other unbounded generalizations of von Neumann algebras have been given by Dixon **[6]**, in 1971, Araki and Jurzak **[3]**, in 1982, and Schmüdgen **[17]**, in 1988. From all these, we consider only the

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corresponding concept due to Dixon (see Definition 2.1) and we connect it with that of a GW^*-algebra. We want to emphasize that there is a physical justification for using tensor products. For instance, tensor products are used to describe two quantum systems as one joint system (see [**1**]). Another fact is that the physical significance of tensor products always depends on the applications, which may involve wave functions, spin states, oscillators etc.; see e.g., [**4, 9**]. To our knowledge, there is almost no literature on tensor products of unbounded operator algebras. So, motivated by all the above, our first attempt was to investigate the tensor product of Allan's GB^*-algebras, which generalize the classical C^*-algebras and, as Dixon has proved in [**5**], they are algebras of unbounded operators. For details on this study, see [**8**]. In the present paper, we define tensor products of O^*-algebras resulting in GW^*-algebras. Thus, Section 2 is devoted to prerequisites for our study. Section 3 deals with unbounded commutants that play an important role in the definition and main results of GW^*-algebras, as it happens in the case of standard von Neumann algebras. In Section 4, three types of GW^*-tensor products are constructed, by using closed O^*-algebras and weak commutants. We prove that all three coincide when our initial closed O^*-algebras belong to a class of unbounded operator algebras that contains Dixon's "extended von Neumann algebras," called EW^*-algebras (Proposition 4.5). This gives the existence and uniqueness of our GW^*-tensor product in the aforementioned class of unbounded operator algebras. In the final Section 5, we introduce the notion of "properly W^*-infinite" GW^*-algebras and investigate their structure. In this regard we prove a generalization of the known von Neumann algebra result that reads as follows: If $\mathcal{M}_0$ is a properly infinite von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathcal{B}(\mathcal{K})$ is the von Neumann algebra of all bounded linear operators on a Hilbert space $\mathcal{K}$, then $\mathcal{M}_0$ is realized (with respect to a $*$-isomorphism) by the von Neumann algebra tensor product of $\mathcal{M}_0$ and $\mathcal{B}(\mathcal{K})$, for every separable Hilbert space $\mathcal{K}$ (see Theorem 5.3 and Corollary 5.4).

2. Preliminaries. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. The algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\mathcal{H}_1 \otimes \mathcal{H}_2 := \left\{ \sum_{i=1}^{n} \xi_{1,i} \otimes \xi_{2,i},\ \xi_{1,i} \in \mathcal{H}_1,\ \xi_{2,i} \in \mathcal{H}_2 \right\}$$

is a pre-Hilbert space under the inner product

$$(\xi \mid \eta) := \sum_{i=1}^{n} \sum_{j=1}^{m} (\xi_{1,i} \mid \eta_{1,j})(\xi_{2,i} \mid \eta_{2,j}),$$

with

$$\xi = \sum_{i=1}^{n} \xi_{1,i} \otimes \xi_{2,i} \quad \text{and} \quad \eta = \sum_{j=1}^{m} \eta_{1,j} \otimes \eta_{2,j}.$$

The completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$, under the norm induced by the previous inner product, is a Hilbert space denoted by $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$, and it is called the *Hilbert space tensor product* of the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be dense subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then the algebraic tensor product $\mathcal{D}_1 \otimes \mathcal{D}_2$ of $\mathcal{D}_1$ and $\mathcal{D}_2$ is a dense subspace of $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$.

Let x_1 and x_2 be linear operators on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. The linear operator $x_1 \otimes x_2$ is defined by

$$(x_1 \otimes x_2)(\xi_1 \otimes \xi_2) := x_1\xi_1 \otimes x_2\xi_2, \ \xi_1 \in \mathcal{D}_1, \ \xi_2 \in \mathcal{D}_2.$$

In particular, if $x_1 \in \mathcal{B}(\mathcal{H}_1)$ and $x_2 \in \mathcal{B}(\mathcal{H}_2)$, where $\mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, is the algebra of all bounded linear operators on $\mathcal{H}_i$, $i = 1, 2$, then $x_1 \otimes x_2$ has a continuous extension $\overline{x_1 \otimes x_2}$ to the Hilbert space $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$, called the tensor product of x_1 and x_2, and denoted by $x_1 \bar{\otimes} x_2$.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then the von Neumann algebra on $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$ generated by the operators $\{x_1 \bar{\otimes} x_2, \ x_1 \in \mathcal{M}_1, \ x_2 \in \mathcal{M}_2\}$ is called the *W^*-tensor product* of $\mathcal{M}_1$ and $\mathcal{M}_2$, denoted by $\mathcal{M}_1 \overset{\mathrm{W}^*}{\otimes} \mathcal{M}_2$. We shall use this W^*-tensor product in order to define unbounded tensor products generated by $\mathcal{M}_1$ and $\mathcal{M}_2$, in Section 4.

To define tensor products of O^*-algebras, we review the basic definitions and properties of O^*- and GW^*-algebras. For more details, we refer to **[13, 18]**. Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. Denote by $\mathcal{L}^\dagger(\mathcal{D})$ the set of all linear operators x from $\mathcal{D}$ to $\mathcal{D}$ such that the domain $\mathcal{D}(x^*)$ of the adjoint x^* of x contains $\mathcal{D}$ and $x^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra under the usual algebraic operations, and the involution $x \mapsto x^\dagger := x^* \restriction_{\mathcal{D}}$. A $*$-subalgebra $\mathcal{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$ is called an *O^*-algebra* on $\mathcal{D}$ in $\mathcal{H}$. The locally convex topology on $\mathcal{D}$ induced by the seminorms $\{\|\cdot\|_x : x \in \mathcal{M}\}$, where $\|\xi\|_x := \|\xi\| + \|x\xi\|$, $\xi \in \mathcal{D}$, is called the *graph topology* on $\mathcal{D}$ and is denoted by $\tau_{\mathcal{M}}$. If the locally convex space $\mathcal{D}[\tau_{\mathcal{M}}]$ is complete, then $\mathcal{M}$ is called *closed.* Denote by $\widetilde{\mathcal{D}}(\mathcal{M})$ the completion of $\mathcal{D}[\tau_{\mathcal{M}}]$. Then $\widetilde{\mathcal{D}}(\mathcal{M}) = \bigcap_{x\in\mathcal{M}} \mathcal{D}(\overline{x})$, where $\overline{x}$ denotes the closure of x. Now put $\widetilde{x} = \overline{x} \restriction_{\widetilde{\mathcal{D}}(\mathcal{M})}$, $x \in \mathcal{M}$ and $\widetilde{\mathcal{M}} = \{\widetilde{x} : x \in \mathcal{M}\}$. Then $\widetilde{\mathcal{M}}$ is the smallest closed extension of $\mathcal{M}$, which is called the *closure* of $\mathcal{M}$. It is easily shown that $\mathcal{M}$ is closed if and only if $\mathcal{M} = \widetilde{\mathcal{M}}$ if and only if $\mathcal{D} = \widetilde{\mathcal{D}}(\mathcal{M}) = \bigcap_{x\in\mathcal{M}} \mathcal{D}(\overline{x})$.

Now let $\mathcal{M}_1$ and $\mathcal{M}_2$ be O^*-algebras on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Then

$$\mathcal{M}_1 \otimes \mathcal{M}_2 := \left\{ \sum_{k=1}^{n} x_{1,k} \otimes x_{2,k}, \text{ with } x_{1,k} \in \mathcal{M}_1 \text{ and } x_{2,k} \in \mathcal{M}_2 \right\}$$

is an O^*-algebra on the algebraic tensor product $\mathcal{D}_1 \otimes \mathcal{D}_2$. The closure of $\mathcal{M}_1 \otimes \mathcal{M}_2$ is called *tensor product of the O^*-algebras* $\mathcal{M}_1$ and $\mathcal{M}_2$ and is denoted by $\mathcal{M}_1 \widetilde{\otimes} \mathcal{M}_2$. The domain of the closed O^*-algebra $\mathcal{M}_1 \widetilde{\otimes} \mathcal{M}_2$ is denoted by $\mathcal{D}_1 \widetilde{\otimes} \mathcal{D}_2$; namely,

$$\mathcal{D}_1 \widetilde{\otimes} \mathcal{D}_2 = \bigcap_{x \in \mathcal{M}_1 \otimes \mathcal{M}_2} \mathcal{D}(\overline{x}).$$

For $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$, $\overline{x_1 \otimes x_2} \restriction_{\mathcal{D}_1 \widetilde{\otimes} \mathcal{D}_2}$ is denoted by $x_1 \widetilde{\otimes} x_2$, and

$$\mathcal{M}_1 \widetilde{\otimes} \mathcal{M}_2 := \left\{ \sum_{k=1}^{n} x_{1,k} \widetilde{\otimes} x_{2,k},\ x_{1,k} \in \mathcal{M}_1,\ x_{2,k} \in \mathcal{M}_2 \right\}.$$

If $\mathcal{M}_1$ and $\mathcal{M}_2$ *are bounded O^*-algebras*, then $\mathcal{D}_1 \widetilde{\otimes} \mathcal{D}_2 = \mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$ and $x_1 \widetilde{\otimes} x_2 = x_1 \bar{\otimes} x_2 = \overline{x_1 \otimes x_2}$, for all $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$. $\mathcal{M}_1 \widetilde{\otimes} \mathcal{M}_2$ is the linear span of $\{x_1 \bar{\otimes} x_2;\ x_1 \in \mathcal{M}_1,\ x_2 \in \mathcal{M}_2\}$ and the double commutant $(\mathcal{M}_1 \widetilde{\otimes} \mathcal{M}_2)''$ of $\mathcal{M}_1 \widetilde{\otimes} \mathcal{M}_2$ equals the W^*-tensor product $\mathcal{M}_1'' \overset{W^*}{\otimes} \mathcal{M}_2''$ of the von Neumann algebras $\mathcal{M}_1''$ and $\mathcal{M}_2''$ (see [**19,** Theorem IV.5.9]).

We next define EW^*- and GW^*-algebras, which are unbounded generalizations of von Neumann algebras. A crucial role, for the study of the structure of these unbounded operator algebras, is played by a von Neumann subalgebra, related to the “bounded part” of the algebras under consideration. The same is true for the Allan's GB^*-algebras (see the Introduction).

Definition 2.1. A closed O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ in $\mathcal{H}$, containing the identity operator $\mathbf{1}$, is said to be an extended W^*-algebra (*abbreviated to* EW^*-algebra), if it is symmetric, that is, $(\mathbf{1}+x^*x)^{-1}$ exists and belongs to $\mathcal{M}_b$, for all $x \in \mathcal{M}$, and $\overline{\mathcal{M}_b} := \{\overline{x} : x \in \mathcal{M}_b\}$ is a von Neumann algebra on $\mathcal{H}$, where $\mathcal{M}_b := \{x \in \mathcal{M} : \overline{x} \in \mathcal{B}(\mathcal{H})\}$.

The concept of an EW^*-algebra was introduced by Dixon [**6**]. Every EW^*-algebra satisfies the following properties:

$$\mathcal{M}_b\mathcal{D} \subset \mathcal{D},\ \overline{\mathcal{M}_b}'\mathcal{D} \subset \mathcal{D} \ \text{ and } \ \mathcal{M} = \{a \in \mathcal{L}^\dagger(\mathcal{D}) : \overline{a}\eta\overline{\mathcal{M}_b}\}, \tag{2.1}$$

where the symbol $\overline{a}\eta\overline{\mathcal{M}_b}$ means that *the operator* $\overline{a}$ *is affiliated with the von Neumann algebra* $\overline{\mathcal{M}_b}$. This means that $\overline{a}$ commutes with all operators in $(\overline{\mathcal{M}_b})'$, that is, $c\overline{a} \subset \overline{a}c$, for all $c \in (\overline{\mathcal{M}_b})'$.

For constructions of EW^*-algebras, see [**6, 11**].

To define GW^*-algebras, we need some unbounded commutants for an O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ in $\mathcal{H}$, which we recall (see [**2, 13**]).

$$\mathcal{M}'_w := \{a \in \mathcal{B}(\mathcal{H}) : (ax\xi \mid \eta) = (a\xi \mid x^\dagger\eta),\ \forall\ x \in \mathcal{M} \text{ and } \xi, \eta \in \mathcal{D}\}.$$
$$\mathcal{M}'_c := \{a \in \mathcal{L}^\dagger(\mathcal{D}) : ax = xa,\ \forall\ x \in \mathcal{M}\}.$$

The *weak commutant* $\mathcal{M}'_w$ of $\mathcal{M}$ is a closed $*$-invariant subspace of $\mathcal{B}(\mathcal{H})$ with respect to the weak operator topology, but it is not necessarily a von Neumann algebra; this happens when, for instance, $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ [**2,** pp. 60 and 66]. The *unbounded commutant* $\mathcal{M}'_c$ of $\mathcal{M}$ is an O^*-algebra on $\mathcal{D}$. Suppose that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$. Then, an *unbounded bicommutant* $\mathcal{M}''_{wc}$ of $\mathcal{M}$, is given by

$$\mathcal{M}''_{wc} := (\mathcal{M}'_w \restriction_{\mathcal{D}})'_c = \{x \in \mathcal{L}^\dagger(\mathcal{D}) : xa\xi = ax\xi,\ \forall\ a \in \mathcal{M}'_w,\ \xi \in \mathcal{D}\}.$$

$\mathcal{M}''_{wc}$ is a closed O^*-algebra on $\mathcal{D}$ containing $\mathcal{M}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all linear operators x from $\mathcal{D}$ to $\mathcal{H}$ such that $\mathcal{D}(x^*) \supset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a $\dagger$-invariant vector space under the usual algebraic operations and the involution $x \mapsto x^\dagger := x^* \restriction_{\mathcal{D}}$. Moreover, $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ contains $\mathcal{B}(\mathcal{H}) \restriction_{\mathcal{D}}$, and it is complete under the *strong**-*topology* τ_s^* induced by the family of seminorms $\{p_\xi^* : \xi \in \mathcal{D}\}$, where $p_\xi^*(x) := \|x\xi\| + \|x^\dagger\xi\|$, for all $x \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. In [**13,** Proposition 1.7.5], it is shown that

$$\begin{aligned}\mathcal{M}''_{wc} &= \text{ the } \tau_s^* - \text{closure of } (\mathcal{M}'_w)' \lceil_{\mathcal{D}} \text{ in } \mathcal{L}^\dagger(\mathcal{D}) \\ &= \{x \in \mathcal{L}^\dagger(\mathcal{D}) : \overline{x}\eta(\mathcal{M}'_w)'\}.\end{aligned}$$

We can now state the following:

Definition 2.2 ([**13,** Definition 1.7.4]). A closed O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ in $\mathcal{H}$ is called a generalized W^*-algebra (*for brevity* GW^*-algebra), if $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $\mathcal{M}''_{wc} = \mathcal{M}$.

A useful characterization of a GW^*-algebra, that we shall use repeatedly in what follows, is the following [**13,** Proposition 1.7.5]: *A closed O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ is a GW^*-algebra if and only if $\mathcal{M} = \{a \in \mathcal{L}^\dagger(\mathcal{D}) : \overline{a}\eta(\mathcal{M}'_w)'\}$.*

When $\mathcal{M}$ is a GW^*-algebra on $\mathcal{D}$ in $\mathcal{H}$, we shall also use the terminology that $\mathcal{M}$ *is a GW^*-algebra on $\mathcal{D}$ over* (the von Neumann algebra) $(\mathcal{M}'_w)'$.

Now take the maximal O^*-algebra $\mathcal{L}^\dagger(\mathcal{D})$. Then $\mathcal{L}^\dagger(\mathcal{D})'_w = \mathbf{C1}$, and when $\mathcal{L}^\dagger(\mathcal{D})$ is moreover closed, *it is a GW^*-algebra* with $(\mathcal{L}^\dagger(\mathcal{D})'_w)' = \mathcal{B}(\mathcal{H})$, *but it is clearly not an EW^*-algebra.* If $\mathcal{M}$ is a closed O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$, suppose that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $(\mathcal{M}'_w)'\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{M}$ is an EW^*-algebra with $\overline{\mathcal{M}_b} = (\mathcal{M}'_w)'$ and the GW^*-algebra $\mathcal{M}''_{wc}$ is the maximal EW^*-algebra with $\overline{\mathcal{M}_b} = (\mathcal{M}'_w)'$. It is now evident from the preceding and (2.2) that *every GW^*-algebra, with an identity element and the property $(\mathcal{M}'_w)'\mathcal{D} \subset \mathcal{D}$, is an EW^*-algebra.*

Recall that a classical W^*-algebra is always a C^*-algebra. So, a natural question is *whether a GW^*-algebra, respectively, an EW^*-algebra is a GB^*-algebra.* This is not always true. Conditions under which this happens are presented in [**6, 12**]. More precisely, concerning EW^*-algebras, Zakirov and Chilin [**21,** Theorem 3] proved, in 1991, that a GB^*-algebra, is algebraically $*$-isomorphic to some EW^*-algebra if and only if its bounded part is a W^*-algebra.

In what follows we use a lot of standard results and definitions from the theory of von Neumann algebras. For all these, the reader is referred to [**14, 16, 19**].

3. Commutants. Let $\mathcal{M}$ (respectively, $\mathcal{N}$) be a closed O^*-algebra on $\mathcal{D}$ (respectively, $\mathcal{E}$) in $\mathcal{H}$ (respectively, $\mathcal{K}$) such that $\mathcal{M}'_w$ and $\mathcal{N}'_w$

are von Neumann algebras. A necessary and sufficient condition for $\mathcal{M}'_w$ to be a von Neumann algebra is given by [**2,** Lemma 2.5.6(1)]. In this section, we investigate the weak commutant of the tensor product $\mathcal{M}\widetilde{\otimes}\mathcal{N}$.

Recall that the *tensor product of two von Neumann algebras* $\mathcal{M}_1$, $\mathcal{M}_2$ acting on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively, is denoted by $\mathcal{M}_1 \overset{W^*}{\otimes} \mathcal{M}_2$ (see Section 2) and, by definition, is the von Neumann algebra generated by the *-algebra $\mathcal{M}_1 \otimes \mathcal{M}_2$ (algebraic tensor product of the $*-$algebras $\mathcal{M}_1, \mathcal{M}_2$) acting on the Hilbert space tensor product $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$. In other words, $\mathcal{M}_1 \overset{W^*}{\otimes} \mathcal{M}_2$ is the closure of $\mathcal{M}_1 \otimes \mathcal{M}_2$ in the weak-operator topology of $\mathcal{B}(\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2)$ [**14,** page 812, 11.2.]. According to the celebrated Sakai theorem [**16**], $\mathcal{M}_1 \overset{W^*}{\otimes} \mathcal{M}_2$ has a unique predual Banach space denoted by $(\mathcal{M}_1 \overset{W^*}{\otimes} \mathcal{M}_2)_*$. Blecher and Paulsen as well as Effros and Ruan discovered independently that $(\mathcal{M}_1 \overset{W^*}{\otimes} \mathcal{M}_2)_* = (\mathcal{M}_1)_* \overset{op}{\otimes} (\mathcal{M}_2)_*$, with respect to an isometric isomorphism, where $(\mathcal{M}_1)_*, (\mathcal{M}_2)_*$ are preduals of the von Neumann algebras $\mathcal{M}_1$, $\mathcal{M}_2$, respectively, and $\overset{op}{\otimes}$ is the so-called (completed) *operator-projective tensor product* (see, e.g., [**10,** page x, 3] and the corresponding references therein). By our assumptions for $\mathcal{M}$, $\mathcal{N}$ we clearly obtain that $\mathcal{M}'_w \overset{\mathrm{W}^*}{\otimes} \mathcal{N}'_w \subset (\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w$, but we do not know if the inverse inclusion is also true. Thus, we are led to the following

Question 3.1. *Under what conditions does one have that* $(\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w = \mathcal{M}'_w \overset{W^*}{\otimes} \mathcal{N}'_w$*?*

In this regard, we have

Proposition 3.1. *Suppose that* $(\overline{\mathcal{M}_b})'' = (\mathcal{M}'_w)'$ *and* $(\overline{\mathcal{N}_b})'' = (\mathcal{N}'_w)'$*. Then* $(\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w = \mathcal{M}'_w \overset{W^*}{\otimes} \mathcal{N}'_w$.

Proof. Let $a \in (\mathcal{M}'_w)'$ and $b \in (\mathcal{N}'_w)'$. By our assumptions, there exist nets $\{x_\alpha\}$ in $\mathcal{M}_b$ and $\{y_\beta\}$ in $\mathcal{N}_b$ such that $x_\alpha \to a$ and $y_\beta \to b$ with respect to the strong*-topology τ_s^* . Thus, if $C \in (\mathcal{M} \otimes \mathcal{N})'_w$, it follows that

$$\begin{aligned}(C(a\otimes b)(\xi_1\otimes\eta_1)\mid\xi_2\otimes\eta_2) &= \lim_{\alpha,\beta}(C(x_\alpha\otimes y_\beta)(\xi_1\otimes\eta_1)\mid\xi_2\otimes\eta_2)\\ &= \lim_{\alpha,\beta}(C(\xi_1\otimes\eta_1)\mid(x_\alpha^\dagger\otimes y_\beta^\dagger)(\xi_2\otimes\eta_2))\\ &= (C(\xi_1\otimes\eta_1)\mid(a^*\otimes b^*)(\xi_2\otimes\eta_2)).\end{aligned}$$

for all $\xi_1,\xi_2\in\mathcal{D}$ and $\eta_1,\eta_2\in\mathcal{E}$. This implies that (see also [**19,** Theorem IV.5.9]) $C\in((\mathcal{M}'_w)'\overset{\mathrm{W}^*}{\otimes}(\mathcal{N}'_w)')'=(\mathcal{M}'_w)''\overset{\mathrm{W}^*}{\otimes}(\mathcal{N}'_w)''=\mathcal{M}'_w\overset{\mathrm{W}^*}{\otimes}\mathcal{N}'_w$. □

Corollary 3.2. *If $\mathcal{M}$ and $\mathcal{N}$ are EW^*-algebras, then $(\mathcal{M}\otimes\mathcal{N})'_w=\mathcal{M}'_w\overset{W^*}{\otimes}\mathcal{N}'_w$.*

Proof. From [**11,** Definition 2.7, Proposition 2.9], we have that $\overline{\mathcal{M}_b}=(\mathcal{M}'_w)'$ and $\overline{\mathcal{N}_b}=(\mathcal{N}'_w)'$. Moreover, $\overline{\mathcal{M}_b}=(\mathcal{M}_b)''$ and $\overline{\mathcal{N}_b}=(\mathcal{N}_b)''$, so the result follows from Proposition 3.1. □

4. GW^*-tensor products. Throughout the present section, $\mathcal{M}$ (respectively, $\mathcal{N}$) are closed O*-algebras on $\mathcal{D}$ (respectively, $\mathcal{E}$) in $\mathcal{H}$ (respectively, $\mathcal{K}$) such that $\mathcal{M}'_w\mathcal{D}\subset\mathcal{D}$ and $\mathcal{N}'_w\mathcal{E}\subset\mathcal{E}$. The latter inclusions occur whenever $\mathcal{M}$ (respectively, $\mathcal{N}$) is a GW^*-algebra (see Definition 2.2). Moreover, by the latter inclusions, $\mathcal{M}'_w$ and $\mathcal{N}'_w$ are von Neumann algebras [**2,** Proposition 2.3.5]; therefore, the same is true for their commutants.

Note that, in general, we do not know whether the inclusion

$$(\mathcal{M}\otimes\mathcal{N})'_w(\mathcal{D}\widetilde{\otimes}\mathcal{E})\subset\mathcal{D}\widetilde{\otimes}\mathcal{E}\tag{4.1}$$

is true. So let us suppose that (4.1) is valid. Then, the same is, of course, true for $\mathcal{M}\widetilde{\otimes}\mathcal{N}$, consequently (ibid) $((\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w)'$ is a von Neumann algebra, so that we may put

$$\mathcal{M}\overset{GW^*}{\otimes}\mathcal{N}=\{x\in\mathcal{L}^\dagger(\mathcal{D}\widetilde{\otimes}\mathcal{E}):\overline{x}\eta((\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w)'\}.\tag{4.2}$$

Then $\mathcal{M}\overset{\mathrm{GW}^*}{\otimes}\mathcal{N}$ is a GW^*-algebra on $\mathcal{D}\widetilde{\otimes}\mathcal{E}$ over $((\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w)'$ according to [**13,** Proposition 1.7.5]. On the other hand, from the same reference as before, it follows that $\mathcal{M}\overset{GW^*}{\otimes}\mathcal{N}$ is the strong*-closure of

$((\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w)' \lceil_{\mathcal{D}\widetilde{\otimes}\mathcal{E}}$, which coincides with $(\mathcal{M}\otimes\mathcal{N})''_{wc}$ (see Section 2). Thus, we may set the following

Definition 4.1. $\mathcal{M} \overset{\mathrm{GW}^*}{\otimes} \mathcal{N}$ is said to be a *GW^*-tensor product* of the closed O^*-algebras $\mathcal{M}$ and $\mathcal{N}$, as above.

Because of (4.1), the preceding GW^*-tensor product $\mathcal{M} \overset{GW^*}{\otimes} \mathcal{N}$ does not always exist. So we define another GW^*-tensor product of the above closed O^*-algebras $\mathcal{M}$ and $\mathcal{N}$, which always exists. Indeed, put

$$(4.3)\qquad (\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)' = \{x \in \mathcal{L}^\dagger(\mathcal{D}\widetilde{\otimes}\mathcal{E}) : \overline{x}\eta(\mathcal{M}'_w)' \overset{\mathrm{W}^*}{\otimes} (\mathcal{N}'_w)'\}.$$

Note that if $(\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w = \mathcal{M}'_w \overset{\mathrm{W}^*}{\otimes} \mathcal{N}'_w$, then $(\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)' = \mathcal{M} \overset{\mathrm{GW}^*}{\otimes} \mathcal{N}$, since by the commutation theorem of the W^*-tensor products **[19,** Theorem IV.5.9] we have that $(\mathcal{M}'_w)' \overset{\mathrm{W}^*}{\otimes} (\mathcal{N}'_w)' = (\mathcal{M}'_w \overset{\mathrm{W}^*}{\otimes} \mathcal{N}'_w)' = ((\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w)'$.

Concerning (4.3) we have the following

Proposition 4.2. *$(\mathcal{M}'_w)' \overset{GW^*}{\otimes} (\mathcal{N}'_w)'$ is a GW^*-algebra on $\mathcal{D}\widetilde{\otimes}\mathcal{E}$ over $(\mathcal{M}'_w)' \overset{W^*}{\otimes} (\mathcal{N}'_w)'$ containing $\mathcal{M}\widetilde{\otimes}\mathcal{N}$. Furthermore, $(\mathcal{M}'_w)' \overset{GW^*}{\otimes} (\mathcal{N}'_w)'$ is the strong*-closure of $((\mathcal{M}'_w)' \overset{W^*}{\otimes} (\mathcal{N}'_w)') \lceil_{\mathcal{D}\widetilde{\otimes}\mathcal{E}}$ in $\mathcal{L}^\dagger(\mathcal{D}\widetilde{\otimes}\mathcal{E})$, which coincides with $((\mathcal{M}'_w \overset{W^*}{\otimes} \mathcal{N}'_w)|_{\mathcal{D}\widetilde{\otimes}\mathcal{E}})'_c$ (see Section 2).*

Proof. By the very definitions it is easily checked that the assumptions $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $\mathcal{N}'_w\mathcal{E} \subset \mathcal{E}$ imply that $(\mathcal{M}'_w \overset{\mathrm{W}^*}{\otimes} \mathcal{N}'_w)(\mathcal{D}\widetilde{\otimes}\mathcal{E}) \subset \mathcal{D}\widetilde{\otimes}\mathcal{E}$. Hence, $\mathcal{M}\widetilde{\otimes}\mathcal{N}$ and $\mathcal{M}'_w \overset{\mathrm{W}^*}{\otimes} \mathcal{N}'_w$ have the same domain. From this and the definitions involved, it follows readily that $\mathcal{M}\widetilde{\otimes}\mathcal{N} \subset (\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)'$, which since $\mathcal{M}\widetilde{\otimes}\mathcal{N}$ is closed implies that $(\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)'$ is closed too. The assertion now follows from **[13,** Proposition 1.7.5], (4.1) and the discussion after it. □

Thus, we may state

Definition 4.3. $(\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)'$ is said to be a *GW^*-tensor product* defined by $(\mathcal{M}'_w)'$ and $(\mathcal{N}'_w)'$.

Furthermore, we may proceed to a more general definition of a GW^*-tensor product, as follows: Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be von Neumann algebras on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Suppose that there exists a dense subspace $\mathcal{F}$ of the Hilbert space $\mathcal{H}\bar{\otimes}\mathcal{K}$ such that $(\mathcal{M}_0 \overset{W^*}{\otimes} \mathcal{N}_0)'\mathcal{F} \subset \mathcal{F}$. Then

$$\mathcal{A} := \{x \in \mathcal{L}^\dagger(\mathcal{F}) : \overline{x}\eta\, \mathcal{M}_0 \overset{W^*}{\otimes} \mathcal{N}_0\}$$

is an O^*-algebra on $\mathcal{F}$ in $\mathcal{H}\bar{\otimes}\mathcal{K}$. The closure of the O^*-algebra $\mathcal{A}$ is a GW^*-algebra on $\widetilde{\mathcal{F}} = \bigcap_{x\in\mathcal{A}} \mathcal{D}(\overline{x})$ in $\mathcal{H}\bar{\otimes}\mathcal{K}$ over $\mathcal{M}_0 \overset{\mathrm{W}^*}{\otimes} \mathcal{N}_0$, and it is denoted by $\mathcal{M}_0 \underset{\mathcal{F}}{\overset{GW^*}{\otimes}} \mathcal{N}_0$. So, we are led to

Definition 4.4. $\mathcal{M}_0 \underset{\mathcal{F}}{\overset{GW^*}{\otimes}} \mathcal{N}_0$ is said to be a *GW^*-tensor product* defined by $\mathcal{M}_0$, $\mathcal{N}_0$ and $\mathcal{F}$.

It is natural now to try to find out the connection among the three GW^*-tensor products given by Definitions 4.1, 4.3 and 4.4. We can see that, under suitable conditions, all three GW^*-tensor products coincide. Indeed, let $\mathcal{M}$ and $\mathcal{N}$ be closed O*-algebras on $\mathcal{D}$ and $\mathcal{E}$, respectively, such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $\mathcal{N}'_w\mathcal{E} \subset \mathcal{E}$. Then, taking in place of $\mathcal{M}_0$ and $\mathcal{N}_0$ the von Neumann algebras $(\mathcal{M}'_w)'$ and $(\mathcal{N}'_w)'$, respectively, and in the place of $\mathcal{F}$, $\mathcal{D}\widetilde{\otimes}\mathcal{E}$, Definitions 4.3 and 4.4 imply

$$(\mathcal{M}'_w)' \overset{GW^*}{\otimes} (\mathcal{N}'_w)' = (\mathcal{M}'_w)' \underset{\mathcal{D}\widetilde{\otimes}\mathcal{E}}{\overset{GW^*}{\otimes}} (\mathcal{N}'_w)'.$$

Furthermore, if we assume $(\mathcal{M}\widetilde{\otimes}\mathcal{N})'_w = \mathcal{M}'_w \overset{\mathrm{W}^*}{\otimes} \mathcal{N}'_w$, then using again the commutation theorem for W^*-tensor products, as well as (4.2) and (4.3), we obtain

$$(\mathcal{M}'_w)' \overset{\mathrm{GW}^*}{\otimes} (\mathcal{N}'_w)' = (\mathcal{M}'_w)' \underset{\mathcal{D}\widetilde{\otimes}\mathcal{E}}{\overset{GW^*}{\otimes}} (\mathcal{N}'_w)'.$$

Summing up, from Definition 4.1, Proposition 3.1 and Corollary 3.2, we conclude:

Proposition 4.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be closed O^*-algebras on $\mathcal{D}$ and $\mathcal{E}$, respectively. If $\overline{\mathcal{M}_b}' = \mathcal{M}'_w$ and $\overline{\mathcal{N}_b}' = \mathcal{N}'_w$, in particular if $\mathcal{M}$ and $\mathcal{N}$*

are EW^*-algebras, then

$$\mathcal{M} \overset{GW^*}{\otimes} \mathcal{N} = (\mathcal{M}'_w)' \overset{GW^*}{\otimes} (\mathcal{N}'_w)' = (\mathcal{M}'_w)' \underset{\mathcal{D}\tilde{\otimes}\mathcal{E}}{\overset{GW^*}{\otimes}} (\mathcal{N}'_w)'.$$

So we may say that, in the class of EW^*-algebras, as well as in the context of closed O^*-algebras described in Proposition 4.5, the GW^*-*tensor product exists and it is unique.*

5. Structure of properly W^*-infinite GW^*-algebras. If $\mathcal{M}_0$ is a properly infinite von Neumann algebra on a Hilbert space $\mathcal{H}$, it is known that $\mathcal{M}_0 \cong \mathcal{M}_0 \overset{W^*}{\otimes} \mathcal{B}(\mathcal{K})$ (with respect to a $*$-isomorphism), for every separable Hilbert space $\mathcal{K}$ [**20,** Appendix C, Theorem]. In this section, we try to extend this W^*-tensor product result to the context of GW^*-tensor product algebras.

Let $\mathcal{M}$ be a closed O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$, and $\mathcal{K}$ a separable Hilbert space with an orthonormal basis $\{\omega_n\}$. The mapping

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n \ni (\xi_n) \longmapsto \sum_{n=1}^{\infty} \xi_n \otimes \omega_n \in \mathcal{H}\bar{\otimes}\mathcal{K}$$

establishes a canonical identification between the Hilbert space direct sum $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ and the Hilbert space tensor product $\mathcal{H}\bar{\otimes}\mathcal{K}$, where $\mathcal{H}_n = \mathcal{H}$, for all $n \in \mathbf{N}$. In what follows, we denote by $\mathbf{1}_{\mathcal{H}}$, $\mathbf{1}_{\mathcal{K}}$ the identity operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Then, for every $x \in \mathcal{M}$, we have

$$\mathcal{D}((x\otimes\mathbf{1}_{\mathcal{K}})^*) = \Big\{ \sum_{n=1}^{\infty} \xi_n\otimes\omega_n : \xi_n \in \mathcal{D}(x^*),\ n \in \mathbf{N},\ \sum_{n=1}^{\infty} \|x^*\xi_n\|^2 < \infty \Big\},$$

with

$$(x \otimes \mathbf{1}_{\mathcal{K}})^* \Big(\sum_{n=1}^{\infty} \xi_n \otimes \omega_n \Big) = \sum_{n=1}^{\infty} x^*\xi_n \otimes \omega_n.$$

Also, for every $x \in \mathcal{M}$, we have

$$\mathcal{D}(\overline{x \otimes \mathbf{1}_{\mathcal{K}}}) = \Big\{ \sum_{n=1}^{\infty} \xi_n \otimes \omega_n : \xi_n \in \mathcal{D}(\overline{x}),\ n \in \mathbf{N},\ \sum_{n=1}^{\infty} \|\overline{x}\xi_n\|^2 < \infty \Big\},$$

with

$$(\overline{x \otimes \mathbf{1}_{\mathcal{K}}})\left(\sum_{n=1}^{\infty} \xi_n \otimes \omega_n\right) = \sum_{n=1}^{\infty} \overline{x}\xi_n \otimes \omega_n.$$

Therefore, we obtain

$$\begin{aligned} \mathcal{D}\widetilde{\otimes}\mathcal{K} &= \bigcap_{x\in\mathcal{M}} \mathcal{D}(\overline{x \otimes \mathbf{1}_{\mathcal{K}}}) \\ &= \left\{\sum_{n=1}^{\infty} \xi_n \otimes \omega_n : \xi_n \in \mathcal{D},\ n \in \mathbf{N},\ \sum_{n=1}^{\infty} \|\overline{x}\xi_n\|^2 < \infty,\ \forall\ x \in \mathcal{M}\right\}. \end{aligned} \tag{5.1}$$

Furthermore, by the properties of the weak commutant we see that since any $c \in (\mathcal{M}\widetilde{\otimes}\mathcal{B}(\mathcal{K}))'_w$ commutes with $\mathbf{1}_{\mathcal{H}} \otimes \epsilon_{i,j}$, $i, j \in \mathbf{N}$, where $\{\epsilon_{i,j}\}$ is a matrix unit in $\mathcal{B}(\mathcal{K})$, it follows that $c \in \mathcal{M}'_w \overset{W^*}{\otimes} \mathbf{C}\mathbf{1}_{\mathcal{K}}$. Hence, we are led to the following

Lemma 5.1. *With $\mathcal{M}$ and $\mathcal{K}$ as before, we have that $(\mathcal{M}\widetilde{\otimes}\mathcal{B}(\mathcal{K}))'_w = \mathcal{M}'_w \overset{W^*}{\otimes} \mathbf{C}\mathbf{1}_{\mathcal{K}}$.*

By Lemma 5.1, $(\mathcal{M}\widetilde{\otimes}\mathcal{B}(\mathcal{K}))'_w(\mathcal{D}\widetilde{\otimes}\mathcal{K}) \subset \mathcal{D}\widetilde{\otimes}\mathcal{K}$. Thus, Section 4 implies that $\mathcal{M} \overset{GW^*}{\otimes} \mathcal{B}(\mathcal{K})$ is well defined and it is a GW^*-algebra on $\mathcal{D}\widetilde{\otimes}\mathcal{K}$ over $(\mathcal{M}'_w)' \overset{W^*}{\otimes} \mathcal{B}(\mathcal{K}) = ((\mathcal{M}\widetilde{\otimes}\mathcal{B}(\mathcal{K}))'_w)'$ (again from Lemma 5.1).

Now let $\mathcal{M}$ be a GW^*-algebra on $\mathcal{D}$ in $\mathcal{H}$. Suppose that the von Neumann algebra $(\mathcal{M}'_w)'$ is properly infinite, that is, there exists a sequence $\{e_n\}$ of mutually orthogonal projections in $(\mathcal{M}'_w)'$ with $e_n \sim \mathbf{1}_{\mathcal{H}}$, for all $n \in \mathbf{N}$ and $\sum_{n=1}^{\infty} e_n = \mathbf{1}_{\mathcal{H}}$. Since $e_n \sim \mathbf{1}_{\mathcal{H}}$, for all $n \in \mathbf{N}$, there exists a sequence $\{v_n\}$ of partial isometries in $(\mathcal{M}'_w)'$ such that

$$e_n = v_n^* v_n \quad \text{and} \quad \mathbf{1}_{\mathcal{H}} = v_n v_n^*,\ \forall\ n \in \mathbf{N}. \tag{5.2}$$

We may now ask the following

Question 5.2. *Is $\mathcal{M}$ $*$-isomorphic to $\mathcal{M} \overset{GW^*}{\otimes} \mathcal{B}(\mathcal{K})$, for every separable Hilbert space $\mathcal{K}$?*

In this regard, we set

Definition 5.2. A GW^*-algebra $\mathcal{M}$ is called *properly W^*-infinite* if the von Neumann algebra $(\mathcal{M}'_w)'$ is properly infinite.

According to the comments after Definition 2.2, whenever $\mathcal{D}$ with the graph topology is complete, the *GW^*-algebra $\mathcal{L}^\dagger(\mathcal{D})$ is properly W^*-infinite, when* $\dim \mathcal{H} = \infty$.

Now, for the case where $\mathcal{D}$ is a Fréchet domain (i.e., $\mathcal{D}$, with the graph topology, is a Fréchet locally convex space), we have

Theorem 5.3. *Let $\mathcal{M}$ be a properly W^*-infinite GW^*-algebra on $\mathcal{D}$ in $\mathcal{H}$. Suppose that the graph topology $t_{\mathcal{M}}$ on $\mathcal{D}$ is defined by a sequence $\{\|\cdot\|_{t_n} : t_n \in \mathcal{M}\}$ of seminorms such that $v_n\overline{t_k} \subset \overline{t_k}v_n$, for all $k, n \in \mathbf{N}$, where the family $\{v_n\}$ is as in* (5.2)*. Then,*

$$\mathcal{M} = \mathcal{M} \overset{GW^*}{\otimes} \mathcal{B}(\mathcal{K}),$$

with respect to a $*$-isomorphism, for every separable Hilbert space $\mathcal{K}$.

Proof. From our assumption, it follows that, for all $x \in \mathcal{M}$ and for all $m \in \mathbf{N}$, there exists $\gamma > 0$ such that

$$\|x\xi\| \leq \gamma \sum_{k=1}^{m} \|\xi\|_{t_k}, \quad \forall\ \xi \in \mathcal{D}. \tag{5.3}$$

We clearly have that $\mathcal{D} = \bigcap_{x\in\mathcal{M}} \mathcal{D}(\overline{x}) \subset \bigcap_{k=1}^{\infty} \mathcal{D}(\overline{t_k})$, where the equality is due to the fact that $\mathcal{M}$ is closed (see Section 2). Moreover, from the condition $v_n\overline{t_k} \subset \overline{t_k}v_n$, for all $k, n \in \mathbf{N}$, we conclude that $v_n\xi \in \bigcap_{k=1}^{\infty} \mathcal{D}(\overline{t_k})$, for all $n \in \mathbf{N}$ and $\xi \in \mathcal{D}$. Hence (see Section 2), $v_n \restriction_{\mathcal{D}} \in \mathcal{M}$, for all $n \in \mathbf{N}$. Now let $\mathcal{K}$ be a separable Hilbert space with an orthonormal basis $\{\omega_n\}$, and let $\{\epsilon_{ij}\}$ be a matrix unit in $\mathcal{B}(\mathcal{K})$. Put

$$u_k = \sum_{i=1}^{k} v_i \otimes \epsilon_{i1}, \quad \forall\ k \in \mathbf{N}.$$

Then, using (5.1) and (5.3), we have that, for all $x \in \mathcal{M}$, $\xi =$

$\sum_{j=1}^{\infty} \xi_j \otimes \omega_j \in \mathcal{D}\widetilde{\otimes}\mathcal{K}$ and $k > l$

$$\|(x \otimes \mathbf{1}_{\mathcal{K}})(u_k\xi - u_l\xi)\|^2 = \left\| \sum_{j=1}^{\infty} \sum_{i=l+1}^{k} xv_i\xi_j \otimes \epsilon_{i1}\omega_j \right\|^2 = \left\| \sum_{i=l+1}^{k} xv_i\xi_1 \otimes \omega_1 \right\|^2$$

(since $\varepsilon_{i1}\omega_j = \omega_1$ for $j = 1$, and 0 for $j \neq 1$)

$$= \left\| \sum_{i=l+1}^{k} xv_i\xi_1 \right\|^2 \leq \sum_{i=l+1}^{k} \|xv_i\xi_1\|^2$$
$$\leq \text{(by (5.3))} \sum_{i=l+1}^{k} \gamma^2 \sum_{j=1}^{m} (\|v_i\xi_1\| + \|t_jv_i\xi_1\|)^2$$
$$\leq 2\gamma^2 \sum_{j=1}^{m} \sum_{i=l+1}^{k} (\|v_i\xi_1\|^2 + \|v_it_j\xi_1\|^2)$$
$$= 2\gamma^2 \sum_{j=1}^{m} \sum_{i=l+1}^{k} (\|e_i\xi_1\|^2 + \|e_it_j\xi_1\|^2) \longrightarrow 0, \quad \text{as } l \to \infty.$$

By similar calculations, we have that for all $x \in \mathcal{M}$

$$\|(x \otimes \mathbf{1}_{\mathcal{K}})(u_k^*\xi - u_l^*\xi)\|^2 = \left\| \sum_{j=1}^{\infty} \sum_{i=l+1}^{k} xv_i^\dagger\xi_j \otimes \epsilon_{1i}\omega_j \right\|^2 = \left\| \sum_{i=l+1}^{k} xv_i^\dagger\xi_i \right\|^2$$

(since $\varepsilon_{1i}\omega_j = \omega_i$ for $j = i$, and 0 for $j \neq i$)

$$\leq 2\gamma^2 \sum_{j=1}^{m} \sum_{i=l+1}^{k} (\|\xi_i\|^2 + \|t_j\xi_i\|^2) \longrightarrow 0, \quad \text{as } l \to \infty.$$

Therefore, there exists $u \in (\mathcal{M}'_w)' \overset{\mathrm{W}^*}{\otimes} \mathcal{B}(\mathcal{K})$ such that $\lim_{k\to\infty} u_k = u$ with respect to the strong* topology τ_s^*. It follows easily from (5.1) that $u(\mathcal{D}\widetilde{\otimes}\mathcal{K}) \subset \mathcal{D}\widetilde{\otimes}\mathcal{K}$, and $u^*(\mathcal{D}\widetilde{\otimes}\mathcal{K}) \subset \mathcal{D}\widetilde{\otimes}\mathcal{K}$. Hence, $u \restriction_{\mathcal{D}\widetilde{\otimes}\mathcal{K}} \in \mathcal{M} \overset{GW^*}{\otimes} \mathcal{B}(\mathcal{K})$. We shall just use the symbol u for the restriction of u to $\mathcal{D}\widetilde{\otimes}\mathcal{K}$. It is now easy to see that

$$u^*u = \mathbf{1}_{\mathcal{H}} \otimes \epsilon_{11} \text{ and } uu^* = \mathbf{1}_{\mathcal{H}} \otimes \mathbf{1}_{\mathcal{K}}. \tag{5.4}$$

Then the map

$$\sigma : \mathcal{M} \overset{GW^*}{\otimes} \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{M}\widetilde{\otimes}\mathbf{C}\epsilon_{11} : \widetilde{x} \mapsto u^\dagger \widetilde{x} u,$$

as seen by (5.4), is a bijective $*$-homomorphism. Therefore, σ is a $*$-isomorphism. Since $\mathcal{M}\widetilde{\otimes}\mathbf{C}\epsilon_{11} \cong \mathcal{M}$, it follows that $\mathcal{M} \cong \mathcal{M} \overset{GW^*}{\otimes} \mathcal{B}(\mathcal{K})$, for every separable Hilbert space $\mathcal{K}$. □

Corollary 5.4. *Let $\mathcal{M}$ be a properly W^*-infinite GW^*-algebra on $\mathcal{D}^\infty(\overline{h}) = \bigcap_{n=1}^\infty \mathcal{D}(\overline{h}^n)$ in $\mathcal{H}$, where $h \in \mathcal{M}$ and $\overline{h}$ is a positive self-adjoint operator on $\mathcal{H}$ such that $v_n\overline{h} \subset \overline{h}v_n$ for all $n \in \mathbf{N}$, where the family $\{v_n\}$ is as in* (5.2). *Then $\mathcal{M}$ is $*$-isomorphic to $\mathcal{M} \overset{GW^*}{\otimes} \mathcal{B}(\mathcal{K})$, for every separable Hilbert space $\mathcal{K}$.*

Proof. It is easily shown that $\mathcal{D}^\infty(\overline{h})$ is a Fréchet space under the locally convex topology τ defined by the sequence $\{\|\cdot\|_{\overline{h}^n} : n \in \mathbf{N}\}$ of seminorms, where $\|\xi\|_{\overline{h}^n} = \|\xi\| + \|\overline{h}^n\xi\|$, $\xi \in \mathcal{D}^\infty(\overline{h})$, $n \in \mathbf{N}$. By using the spectral decomposition of $\overline{h}$, we have that $\overline{h}^n \restriction_{\mathcal{D}^\infty(\overline{h})} = h^n \in \mathcal{M}$, for all $n \in \mathbf{N}$. This implies that the topology τ is weaker than the graph topology $t_{\mathcal{M}}$. On the other hand, since every element x of $\mathcal{M}$ is a closed linear map from the Fréchet space $\mathcal{D}^\infty(\overline{h})[\tau]$ to the Hilbert space $\mathcal{H}$, x is continuous by the closed graph theorem. This means that $t_{\mathcal{M}}$ is weaker than τ. Hence, $t_{\mathcal{M}} = \tau$, and so the graph topology $t_{\mathcal{M}}$ is induced by the sequence $\{\|\cdot\|_{h^n} : n \in \mathbf{N}\}$ of seminorms defined above. Furthermore, the assumption $v_n\overline{h} \subset \overline{h}v_n$ for all $n \in \mathbf{N}$ implies that $v_n\overline{h^k} \subset \overline{h^k}v_n$ for all $k, n \in \mathbf{N}$. Thus the assertion of Corollary 5.4 now follows from Theorem 5.3. □

Corollary 5.4 gives assumptions under which the condition $v_n\overline{t_k} \subset \overline{t_k}v_n$, for all $k, n \in \mathbf{N}$, of Theorem 5.3 is fulfilled. An example satisfying the assumptions of Corollary 5.4 is exhibited next.

Example 5.5. Let $\mathcal{H}$ be a Hilbert space. Then the algebra $\mathcal{B}(\mathcal{H})$ is a properly infinite von Neumann algebra, so that (see discussion before (5.2)) there exists a sequence $\{e_n\}$ of mutually orthogonal projections on $\mathcal{H}$ and a sequence of partial isometries v_n in $\mathcal{B}(\mathcal{H})$ such that $e_n = v_n^* v_n$ and $\mathbf{1}_{\mathcal{H}} = v_n v_n^*$, for all $n \in \mathbf{N}$. Let h be a positive self-adjoint unbounded operator in $\mathcal{H}$ affiliated with $\{v_n; n \in \mathbf{N}\}'$. Then (see Corollary 5.4, for the notation), $v_n \mathcal{D}^\infty(h) \subset \mathcal{D}^\infty(h)$, $n \in \mathbf{N}$. We retain the symbol v_n for the restriction $v_n \upharpoonright_{\mathcal{D}^\infty(h)}$. Let $h = \int \lambda de_h(\lambda)$ be the spectral resolution of h and $\mathcal{M}_0$ a von Neumann algebra on $\mathcal{H}$ containing $\{v_n, e_h(\lambda);\ n \in \mathbf{N}, \lambda \in [0, \infty)\}$. Now put

$$GW^*(\mathcal{M}_0, \mathcal{D}^\infty(h)) \equiv \{x \in \mathcal{L}^\dagger(\mathcal{D}^\infty(h));\ \overline{x} \eta \mathcal{M}_0\}.$$

Since $ch \subset hc$, for each $c \in \mathcal{M}_0'$, we have that $\mathcal{M}_0' \mathcal{D}^\infty(h) \subset \mathcal{D}^\infty(h)$. Furthermore, retaining the symbol h for the restriction $h \upharpoonright_{\mathcal{D}^\infty(h)}$, we have that $h \in GW^*(\mathcal{M}_0, \mathcal{D}^\infty(h))$, whence it follows that $GW^*(\mathcal{M}_0, \mathcal{D}^\infty(h))$ is a self-adjoint, properly W^*-infinite GW^*-algebra on $\mathcal{D}^\infty(h)$, over $\mathcal{M}_0$, satisfying the condition $v_n h \subset h v_n$, for all $n \in \mathbf{N}$.

In particular, $GW^*(\{v_n, e_h(\lambda)\}'', \mathcal{D}^\infty(h))$ is a properly W^*-infinite GW^*-algebra on $\mathcal{D}^\infty(h)$ over $\{v_n, e_h(\lambda)\}''$, and $\mathcal{L}^\dagger(\mathcal{D}^\infty(h))$ is a properly W^*-infinite GW^*-algebra on $\mathcal{D}^\infty(h)$ over $\mathcal{B}(\mathcal{H})$.

Remark 1. In a forthcoming paper [**7**], GW^*-tensor products are used for the definition and study of crossed products of unbounded operator algebras.

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Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece
Email address: **fragoulop@math.uoa.gr**

Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan
Email address: **a-inoue@fukuoka-u.ac.jp**

Department of Mathematics and Applied Mathematics, Nelson Mandela Metropolitan University, Summerstrand Campus South, Port Elizabeth, 6031, South Africa
Email address: **weigt.martin@gmail.com**