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STRONG COMMUTATIVITY PRESERVING MAPS ON RINGS

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ABSTRACT. Suppose $\mathcal{R}$ is a unital ring having an idempotent element e which satisfies $a\mathcal{R}e = 0$ implies $a = 0$ and $a\mathcal{R}(1-e) = 0$ implies $a = 0$. In this paper, we aim to characterize the map $f : \mathcal{R} \to \mathcal{R}$, f is surjective and $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{R}$. It is shown that $f(x) = \alpha x + \xi(x)$ for all $x \in \mathcal{R}$, where $\alpha \in \mathcal{Z}(\mathcal{R})$, $\alpha^2 = 1$, and ξ is a map from $\mathcal{R}$ into $\mathcal{Z}(\mathcal{R})$. As an application, a characterization of nonlinear surjective maps preserving strong commutativity on von Neumann algebras with no central summands of type I_1 is obtained.

1. Introduction and main results. Let $\mathcal{R}$ be a ring with center $\mathcal{Z}(\mathcal{R})$. For $x, y \in \mathcal{R}$, we denote $[x, y] = xy - yx$ the commutator of x and y. We say that a map $f : \mathcal{R} \to \mathcal{R}$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for $x, y \in \mathcal{R}$. The problems of characterizing maps that preserve certain subsets or relations had been investigated on various rings and algebras (see [**17**] and the references therein). One of the most studied problems is to describe additive (or linear) bijective maps preserving commutativity (see [**6, 9, 10, 18** and the references therein).

In [**7**], Bell and Daif initiated the study of a certain kind of commutativity preserving map as follows: Let $\mathcal{S}$ be a subset of $\mathcal{R}$. A map $f : \mathcal{S} \to \mathcal{R}$ is called strong commutativity preserving (SCP) on $\mathcal{S}$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{S}$. More precisely, they proved that $\mathcal{R}$ must be commutative if $\mathcal{R}$ is a prime ring and $\mathcal{R}$ admits a derivation or a non-identity endomorphism which is SCP on a right ideal of

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$\mathcal{R}$. Later, additive maps preserving strong commutativity on prime or semiprime rings were investigated in [**8, 11, 14, 15**].

In recent years, more and more mathematicians are interested in discussing nonlinear maps that preserve some properties concerning the Lie product (e.g., [**2–5, 12, 18, 19, 20**]). In [**12**], Dolinar et al. characterized the surjective map $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ with $\operatorname{Lat}[\Phi(A), \Phi(B)] = \operatorname{Lat}[A, B]$ ($\operatorname{Lat} A$ denotes the invariant subspace lattice of A), where X is a Banach space over $\mathbf{F}$ ($\mathbf{C}$ or $\mathbf{R}$).

Obviously, strong commutativity preserving maps must preserve the lattice of Lie products. Hence, by [**12**], the surjective strong commutativity preserving map Φ on $\mathcal{B}(X)$ has the form $\Phi(A) = \phi(A)A + \psi(A)I$ for all $A \in \mathcal{R}$, where ϕ, ψ are maps from $\mathcal{B}(X)$ into $\mathbf{F}$. In [**19**], Qi and Hou proved that every nonlinear surjective strong commutativity map on a unital prime ring $\mathcal{R}$ containing a nontrivial idempotent element is of the form $\Phi(A) = \alpha A + \phi(A)$ for all $A \in \mathcal{R}$, where $\alpha \in \{1, -1\}$, and ϕ is a map from $\mathcal{R}$ into $\mathcal{Z}(\mathcal{R})$. The purpose of this note is to further consider nonlinear surjective strong commutativity preserving maps on general rings. Our main objective is to prove the following.

Theorem 1. *Let $\mathcal{R}$ be a unital ring having an idempotent element e which satisfies*:

(i) $a\mathcal{R}e = 0$ *implies* $a = 0$,

(ii) $a\mathcal{R}(1 - e) = 0$ *implies* $a = 0$.

Assume that $f : \mathcal{R} \to \mathcal{R}$ is a surjective map. Then f satisfies $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{R}$ if and only if f is of the form $f(x) = \alpha x + \xi(x)$ for all $x \in \mathcal{R}$, where $\alpha \in \mathcal{Z}(\mathcal{R})$, $\alpha^2 = 1$, and ξ is a map from $\mathcal{R}$ into $\mathcal{Z}(\mathcal{R})$.

It is well known that a von Neumann algebra $\mathcal{M}$ as a ring is semiprime and is prime if and only if it is a factor (i.e., its center consists of scalar multiples of the unit element). We will prove that if $\mathcal{M}$ has no central summands of type I_1, then there exists idempotent $e \in \mathcal{M}$ satisfying conditions (i) and (ii) in Theorem 1. As an application of Theorem 1, we have the following corollary which we think has independent significance.

Corollary 2. *Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. Assume that $f : \mathcal{M} \to \mathcal{M}$ is a surjective map.*

Then f satisfies $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{M}$ if and only if f is of the form $f(x) = \alpha x + \xi(x)$ for all $x \in \mathcal{M}$, where $\alpha \in \mathcal{Z}(\mathcal{M})$, $\alpha^2 = I$, and ξ is a map from $\mathcal{M}$ into $\mathcal{Z}(\mathcal{M})$.

2. Proofs. In all that follows, $\mathcal{R}$ will denote a ring satisfying conditions (i) and (ii) in Theorem 1, and e is the corresponding idempotent element. It is clear that $e \neq 0$, $e \neq 1$. The two-sided Pierce decomposition of $\mathcal{R}$ relative to e takes the form $\mathcal{R} = e\mathcal{R}e + e\mathcal{R}(1-e) + (1-e)\mathcal{R}e + (1-e)\mathcal{R}(1-e)$. We will formally set $e_1 = e$ and $e_2 = 1 - e$, so letting $\mathcal{R}_{ij} = e_i\mathcal{R}e_j$, $i, j = 1, 2$, we may write $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}$. Moreover, an element of the subring $\mathcal{R}_{ij}$ will be denoted by x_{ij}, and for the element $x_{ij} \in \mathcal{R}$, we always mean that $x_{ij} \in \mathcal{R}_{ij}$.

Here we need a classical result on module which can be found in [**1**, Chapter 1, Proposition 4.5]. For the convenience of readers, it is relayed in a lemma.

Lemma 3. *Given a left $\mathcal{R}$-module $\mathcal{M}$, there is a left $\mathcal{R}$-isomorphism $\rho : M \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{M})$ defined by*

$$\rho(x)(a) = ax \quad (x \in \mathcal{M},\ a \in \mathcal{R}).$$

Here, $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{M})$ denotes all $\mathcal{R}$-linear homomorphisms from $\mathcal{R}$ to $\mathcal{M}$.

Proof. For all $a, a', b, b' \in \mathcal{R}$,

$$\rho(x)(ab + a'b') = (ab + a'b')x = a\rho(x)b + a'\rho(x)b',$$

i.e., $\rho(x)$ is an $\mathcal{R}$-homomorphism from $\mathcal{R}$ to $\mathcal{M}$. And it is easy to see that ρ itself is $\mathcal{R}$-linear.

Now ρ is monic for $\rho(x) = 0$ forces $x = \rho(x)1 = 0$. And ρ is epic for, if $f \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{M})$, then $f(a) = af(1) = \rho(f(1))(a)$. The last statement is now easy to check.

Based on Lemma 3, we give the following lemma which plays a crucial role in this paper.

Lemma 4. *For $a_{ii} \in \mathcal{R}$, $i = 1, 2$, if $e_i x a_{ii} = a_{ii} x e_i$ for all $x \in \mathcal{R}$, then there is an element $\lambda \in \mathcal{Z}(\mathcal{R})$ such that $a_{ii} = \lambda e_i$.*

Proof. Since e and $1-e$ have the same property, we only consider the case $i=1$. Let $\mathcal{R}a_{11}\mathcal{R} = \{\sum_{k=1}^{n} x_k a_{11} y_k : x_k, y_k \in \mathcal{R}, n$ is an arbitrary positive integer$\}$, $\mathcal{R}e\mathcal{R}$ can be defined similarly. Then $\mathcal{R}a_{11}\mathcal{R}$ is an ideal of $\mathcal{R}e\mathcal{R}$, and we define $\tau : \mathcal{R}e\mathcal{R} \to \mathcal{R}a_{11}\mathcal{R}$ by $\tau(\sum_{k=1}^{n} x_k e y_k) = \sum_{k=1}^{n} x_k a_{11} y_k$. It is easy to see that τ is an additive map. In order to show τ is well defined, we suppose that $\sum_{k=1}^{n} x_k e y_k = 0$. Consequently, $(\sum_{k=1}^{n} x_k e y_k) z a_{11} = 0$ holds for all $z \in \mathcal{R}$. By our assumption, $(\sum_{k=1}^{n} x_k a_{11} y_k) z e = 0$. This implies that $(\sum_{k=1}^{n} x_k a_{11} y_k)\mathcal{R}e = 0$. From the property of e, $\sum_{k=1}^{n} x_k a_{11} y_k = 0$. This proves τ is well defined.

Clearly, τ is an $\mathcal{R}e\mathcal{R}$-bimodule homomorphism. By Lemma 3, there is an element $\lambda \in \mathcal{R}a_{11}\mathcal{R}$ such that $\tau(x) = \lambda x$, for all $x \in \mathcal{R}e\mathcal{R}$. Hence, $a_{11} = \lambda e$. By the definition of τ, for arbitrary $u \in \mathcal{R}$, $\tau(ux) = u\tau(x)$. Thus, $\lambda u x = u \lambda x$, i.e., $(\lambda u - u\lambda)\mathcal{R}e\mathcal{R} = 0$. Hence, $(\lambda u - u\lambda)\mathcal{R}e = 0$, and so $\lambda u - u\lambda = 0$, $\lambda \in \mathcal{Z}(\mathcal{R})$. □

Proof of Theorem 1. The sufficiency is evident, so we need only prove the necessity. The proof will be completed by checking several claims.

Claim 1. There are elements $\alpha, \mu \in \mathcal{Z}(\mathcal{R})$ such that $f(e) = \alpha e + \mu$, here $\alpha \neq 0$.

For any $x \in \mathcal{R}$, it is easy to check that $[e,[e,[e,x]]] = [e,x]$. So we have $[e,[e,[f(e),f(x)]]] = [f(e),f(x)]$. It follows from the surjectivity of f that $[e,[e,[f(e),x]]] = [f(e),x]$, for all $x \in \mathcal{R}$. Write $f(e) = a_{11} + a_{12} + a_{21} + a_{22}$. Then the above equation becomes

$$\begin{aligned}(1)\quad exa_{11} - exa_{12} + exa_{21} - exa_{22} - a_{11}xe - a_{12}xe + a_{21}xe + a_{22}xe \\ = a_{21}x + a_{22}x - xa_{12} - xa_{22}.\end{aligned}$$

Set $x = x_{12}$ in equation (1). Then we get $x_{12}a_{21} = a_{21}x_{12} = 0$, and so $a_{21}x(1-e) = 0$ for all $x \in \mathcal{R}$. It follows from the property of $1-e$ that $a_{21} = 0$.

Set $x = x_{21}$ in equation (1), and, by a similar argument to the above, one can show that $a_{12} = 0$.

Now set $x = x_{11}$ in equation (1). Then we obtain $x_{11}a_{11} = a_{11}x_{11}$, that is, $exa_{11} = a_{11}xe$ for all $x \in \mathcal{R}$. It follows from Lemma 4 that $a_{11} = \lambda e$ for some $\lambda \in \mathcal{Z}(\mathcal{R})$. Similarly, taking $x = x_{22}$ in equation (1), we then obtain $a_{22} = \mu(1-e)$ for some $\mu \in \mathcal{Z}(\mathcal{R})$. Hence, $f(e) = a_{11} + a_{22} = \lambda e + \mu(1-e) = \alpha e + \mu$, where $\alpha = \lambda - \mu \in \mathcal{Z}(\mathcal{R})$.

Finally, we still need to prove that $\alpha \neq 0$. On the contrary, if $\alpha = 0$, then $f(e) \in \mathcal{Z}(\mathcal{R})$. Hence, $[e, x] = [f(e), f(x)] = 0$, for all $x \in \mathcal{R}$. So $e \in \mathcal{Z}(\mathcal{R})$. This is contrary to the property of e. The proof is completed. $\square$

Claim 2. For $x, y, z \in \mathcal{R}$, which satisfy that $f(x) = y$ and $f(y) = z$, we can write $x = \sum_{i,j=1}^{2} x_{ij}$, $y = \sum_{i,j=1}^{2} y_{ij}$, and $z = \sum_{i,j=1}^{2} z_{ij}$. Then, for $i \neq j$, $y_{ij} = 0$ implies that $x_{ij} = \alpha z_{ij} = 0$, $\alpha z_{ji} = y_{ji}$ and $x_{ji} = \alpha y_{ji}$.

Case 1. $i = 2$, $j = 1$. In this case, $y_{21} = 0$, so $[e, y] = [e, y_{11} + y_{12} + y_{22}] = [e, y_{12}] = y_{12}$. Hence,

$$[f(e), z] = [\alpha e + \mu, \sum_{i,j=1}^{2} z_{ij}] = \alpha z_{12} - \alpha z_{21} = y_{12}.$$

This implies that $\alpha z_{21} = 0$ and $\alpha z_{12} = y_{12}$. On the other hand,

$$x_{12} - x_{21} = [e, x] = [f(e), y] = [\alpha e + \mu, y] = \alpha y_{12}.$$

Thus, $x_{12} = \alpha y_{12}$ and $x_{21} = 0$.

Case 2. $i = 1, j = 2$. This is similar to Case 1. Note that $[e, y] = -y_{21}$. Therefore,

$$[f(e), z] = [\alpha e + \mu, \sum_{i,j=1}^{2} z_{ij}] = \alpha z_{12} - \alpha z_{21} = -y_{21}.$$

This implies that $\alpha z_{12} = 0$ and $\alpha z_{21} = y_{21}$. On the other hand,

$$x_{12} - x_{21} = [e, x] = [f(e), y] = [\alpha e + \mu, y] = -\alpha y_{21}.$$

Thus, $x_{21} = \alpha y_{21}$ and $x_{12} = 0$. The proof is completed. $\square$

Claim 3. For every $x_{ij} \in \mathcal{R}_{ij}$, $1 \leq i \neq j \leq 2$, there exists $\xi_{ij}(x_{ij}) \in \mathcal{Z}(\mathcal{R})$ such that $\alpha f(x_{ij}) = x_{ij} + \xi_{ij}(x_{ij})$.

Firstly, assume that $i=1$, $j=2$. We take any $x_{12} \in \mathcal{R}_{12}$, and let $f(x_{12}) = \sum_{i,j=1}^{2} a_{ij}$. Since

$$x_{12} = [e, x_{12}] = [f(e), f(x_{12}] = [\alpha e + \mu, \sum_{i,j=1}^{2} a_{ij}] = \alpha a_{12} - \alpha a_{21},$$

we know that $\alpha a_{21} = 0$ and $\alpha a_{12} = x_{12}$.

For any $b = b_{11} + b_{12} + b_{21} + b_{22} \in \mathcal{R}$, by the surjectivity of f, there exists an element $y = y_{11} + y_{12} + y_{21} + y_{22} \in \mathcal{R}$ such that $f(y) = b$. Since $\alpha[b, f(x_{12})] = \alpha[y, x_{12}]$, we have

$$\begin{aligned}(2)\quad & \alpha(b_{11}a_{11} + b_{11}a_{12} + b_{12}a_{22} + b_{21}a_{11} + b_{21}a_{12} + b_{22}a_{22} \\ & \quad - a_{11}b_{11} - a_{11}b_{12} - a_{12}b_{21} - a_{12}b_{22} - a_{22}b_{21} - a_{22}b_{22}) \\ & \qquad = \alpha(y_{11}x_{12} + y_{21}x_{12} - x_{12}y_{21} - x_{12}y_{22}).\end{aligned}$$

Multiplying both sides of equation (2) by e, we get

$$\alpha(b_{11}a_{11} - a_{11}b_{11} - a_{12}b_{21}) = -\alpha x_{12}y_{21}.$$

One can take $b_{21} = 0$. By Claim 2, $\alpha y_{21} = 0$. Thus, $\alpha(b_{11}a_{11} - a_{11}b_{11}) = 0$. That is, $eb(\alpha a_{11}) = (\alpha a_{11})be$ for all $b \in \mathcal{R}$. It follows from Lemma 4 that there exists an element $\xi_{12}(x_{12}) \in \mathcal{Z}(\mathcal{R})$ such that

$$\alpha a_{11} = \xi_{12}(x_{12})e.$$

Multiplying by e and $1-e$ from the right and the left, respectively, in equation (2), we get $\alpha(b_{21}a_{11} - a_{22}b_{21}) = 0$. So $b_{21}\xi_{12}(x_{12})e - \alpha a_{22}b_{21} = 0$, i.e., $(\xi_{12}(x_{12}) - \alpha a_{22})b_{21} = 0$. This implies $(\xi_{12}(x_{12}) - \alpha a_{22})(1-e)be = 0$ for all $b \in \mathcal{R}$. So

$$\alpha a_{22} = \xi_{12}(x_{12})(1-e).$$

Hence, we can obtain

$$\begin{aligned}\alpha f(x_{12}) &= \alpha(a_{11} + a_{12} + a_{21} + a_{22}) \\ &= \xi_{12}(x_{12})e + x_{12} + \xi_{12}(x_{12})(1-e) \\ &= x_{12} + \xi_{12}(x_{12}).\end{aligned}$$

Similarly, for any $x_{21} \in \mathcal{R}_{21}$, one can show that $\alpha f(x_{21}) = x_{21} + \xi_{21}(x_{21})$ for some $\xi(x_{21}) \in \mathcal{Z}(\mathcal{R})$. The proof is completed. $\square$

Claim 4. For every $x_{ii} \in \mathcal{R}_{ii}$, $i = 1, 2$, there is $\xi_{ii}(x_{ii}) \in \mathcal{Z}(\mathcal{R})$ such that $f(x_{ii}) = \alpha x_{ii} + \xi_{ii}(x_{ii})$.

We need only check the case $i = 1$; the other case can be treated similarly.

Take any $x_{11} \in \mathcal{R}_{11}$, and let $f(x_{11}) = \sum_{i,j=1}^{2} a_{ij}$.

For any $b_{12} \in \mathcal{R}_{12}$, by Claim 3, we have

$$[b_{12}, f(x_{11})] = \alpha[f(b_{12}), f(x_{11})] = \alpha[b_{12}, x_{11}] = -\alpha x_{11} b_{12}.$$

It follows that

$$b_{12}a_{21} + b_{12}a_{22} - a_{11}b_{12} - a_{21}b_{12} = -\alpha x_{11} b_{12}. \tag{3}$$

Multiplying equation (3) by $1 - e$ from the two sides, we have $a_{21}b_{12} = 0$. That is, $a_{21}b(1 - e) = 0$ for all $b \in \mathcal{R}$. Hence,

$$a_{21} = 0.$$

Multiplying equation (3) by e from the left and by $1 - e$ from the right, respectively, we get $b_{12}a_{22} - a_{11}b_{12} = -\alpha x_{11}b_{12}$, that is, $b_{12}a_{22} = (a_{11} - \alpha x_{11})b_{12}$. For any $b_{11} \in \mathcal{R}_{11}$, since $b_{11}b_{12} \in \mathcal{R}_{12}$, replacing b_{12} with $b_{11}b_{12}$, we have $b_{11}b_{12}a_{22} = (a_{11} - \alpha x_{11})b_{11}b_{12}$. Hence, $b_{11}(a_{11} - \alpha x_{11})b_{12} = (a_{11} - \alpha x_{11})b_{11}b_{12}$. That is, $(b_{11}(a_{11} - \alpha x_{11}) - (a_{11} - \alpha x_{11})b_{11})b_{12} = 0$. This implies that $(b_{11}(a_{11} - \alpha x_{11}) - (a_{11} - \alpha x_{11})b_{11})b(1 - e) = 0$ for all $b \in \mathcal{R}$ and so $b_{11}(a_{11} - \alpha x_{11}) = (a_{11} - \alpha x_{11})b_{11}$. It follows that $eb(a_{11} - \alpha x_{11}) = (a_{11} - \alpha x_{11})be$ for all $b \in \mathcal{R}$. From Lemma 4, we have

$$a_{11} - \alpha x_{11} = \xi_{11}(x_{11})e \tag{4}$$

for some $\xi_{11}(x_{11}) \in \mathcal{Z}(\mathcal{R})$.

For any $b_{21} \in \mathcal{R}_{21}$, by Claim 3, we have

$$[b_{21}, f(x_{11})] = \alpha[f(b_{21}), f(x_{11})] = \alpha[b_{21}, x_{11}] = \alpha b_{21} x_{11}.$$

It follows that

$$b_{21}a_{12} + b_{21}a_{11} - a_{22}b_{21} - a_{12}b_{21} = \alpha b_{21} x_{11}. \tag{5}$$

Multiplying e from both sides of equation (5), we have $a_{12}b_{21} = 0$, that is, $a_{12}be = 0$ for all $b \in \mathcal{R}$. This implies

$$a_{12} = 0.$$

Multiplying equation (5) by e from the right and by $1 - e$ from the left, respectively, we get $b_{21}a_{11} - a_{22}b_{21} = \alpha b_{21}x_{11}$. That is, $b_{21}(a_{11} - \alpha x_{11}) = a_{22}b_{21}$. By (4), we obtain $b_{21}\xi_{11}(x_{11}) = a_{22}b_{21}$, i.e., $(a_{22} - \xi_{11}(x_{11}))b_{21} = 0$. Therefore, $(a_{22} - \xi_{11}(x_{11}))(1 - e)be = 0$ for all $b \in \mathcal{R}$. Thus, we get

$$a_{22} = \xi_{11}(x_{11})(1 - e).$$

Hence, $f(x_{11}) = a_{11} + a_{22} = \alpha x_{11} + \xi_{11}(x_{11})$. The proof is completed. □

Claim 5. $\alpha^2 = 1$. Consequently, $f(x_{ij}) = \alpha x_{ij} + \xi_{ij}(x_{ij})$, $1 \leq i \neq j \leq 2$, for some $\xi_{ij}(x_{ij}) \in \mathcal{Z}(\mathcal{R})$.

For any $x_{12} \in \mathcal{R}_{12}$ and $x_{21} \in \mathcal{R}_{21}$, by the definition of f and Claim 3, we have $[x_{12}, x_{21}] = [\alpha f(x_{12}), \alpha f(x_{21})] = \alpha^2[x_{12}, x_{21}]$. It follows that $(\alpha^2 - 1)[x_{12}, x_{21}] = 0$, which implies that $(\alpha^2 - 1)x_{12}x_{21} = (\alpha^2 - 1)x_{21}x_{12} = 0$ for all $x_{12} \in \mathcal{R}_{12}$ and $x_{21} \in \mathcal{R}_{21}$. Thus, $(\alpha^2 - 1)ey(1 - e)xe = 0$ for all $x, y \in \mathcal{R}$. By the property of $e, 1 - e$. we obtain $(\alpha^2 - 1)e = 0$. Similarly, from $(\alpha^2 - 1)x_{21}x_{12} = 0$, one can get $(\alpha^2 - 1)(1 - e) = 0$. Therefore, $\alpha^2 = 1$. Using Claim 3, we obtain the desired result. The proof is completed. □

Claim 6. There exists a map $\xi : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ such that $f(x) = \alpha x + \xi(x)$. That is, Theorem 1 holds true.

For any $x, y, z \in \mathcal{R}$, we have

$$\begin{aligned}
&[f(x + y) - f(x) - f(y), f(z)] \\
&\qquad = [f(x + y), f(z)] - [f(x), f(z)] - [f(y), f(z)] \\
&\qquad = [x + y - x - y, z] = 0.
\end{aligned}$$

By the surjectivity of f, there is $\delta_{xy} \in \mathcal{Z}(\mathcal{R})$ such that $f(x + y) = f(x) + f(y) + \delta_{xy}$.

By Claims 4 and 5, we have $f(x_{ij}) = \alpha x_{ij} + \xi_{ij}(x_{ij})$ for all $x_{ij} \in \mathcal{R}_{ij}$, $1 \le i, j \le 2$. Now, for every $x \in \mathcal{R}$, we can write $x = x_{11} + x_{12} + x_{21} + x_{22}$, and so

$$\begin{aligned} f(x) &= f(x_{11}) + f(x_{12}) + f(x_{21}) + f(x_{22}) + \delta_x \\ &= \alpha(x_{11} + x_{12} + x_{21} + x_{22}) \\ &\quad + \xi_{11}(x_{11}) + \xi_{12}(x_{12}) + \xi_{21}(x_{21}) + \xi_{22}(x_{22}) + \delta_x = \alpha x + \xi(x), \end{aligned}$$

where $\xi(x) = \xi_{11}(x_{11}) + \xi_{12}(x_{12}) + \xi_{21}(x_{21}) + \xi_{22}(x_{22}) + \delta_x$. Clearly, ξ is a map from $\mathcal{R}$ into $\mathcal{Z}(\mathcal{R})$. $\square$

Before embarking on the proof of Corollary 2, we need some notation and terminology about von Neumann algebras. A von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator I. For $A \in \mathcal{M}$, the central carrier of A, denoted by $\overline{A}$, is the intersection of all central projections P (central self-adjoint idempotent) such that $PA = A$. It is well known that the central carrier of A is the projection with the range $[\mathcal{M}A(H)]$, the closed linear span of $\{MA(x) \mid M \in \mathcal{M}, x \in H\}$. For each self-adjoint operator $A \in \mathcal{M}$, we define the core of A, denoted by $\underline{A}$, to be $\sup\{S \in \mathcal{Z}_{\mathcal{M}} \mid S = S^*, S \le A\}$. Clearly, one has $A - \underline{A} \ge 0$. Further, if $S \in \mathcal{Z}_{\mathcal{M}}$ and $A - \underline{A} \ge S \ge 0$, then $S = 0$. If P is a projection it is clear that $\underline{P}$ is the largest central projection $\le P$. We call a projection core-free if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$; here $\overline{I - P}$ denotes the central carrier of $I - P$. We use [**13**] as a general reference for the theory of von Neumann algebras.

Proof of Corollary 2. By [**16**, Lemma 4], we know that if the von Neumann algebra $\mathcal{M}$ has no central summands of type I_1, then each nonzero central projection of $\mathcal{M}$ is the central carrier of a core-free projection of $\mathcal{M}$. Hence, there is a core-free projection P such that $\overline{P} = I$. This implies that $[\mathcal{M}P(H)] = H$. Hence, $A\mathcal{M}P = 0$ implies $A = 0$. By the definitions of the core and central carrier, P is core-free if and only if $\overline{I - P} = I$. Hence, $A\mathcal{M}(I - P) = 0$ implies $A = 0$. Now Theorem 1 can be applied directly. $\square$

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