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DECOMPOSITION FOR A COMPOSITION OPERATOR

M.T. HEYDARI

ABSTRACT. Let φ be an elliptic automorphism of the open unit disc of order k and rotation parameter w, and let C_φ be the composition operator on the Hardy space H^2 induced by φ. Then there are orthogonal projections P_s, $s = 0, 1, \dots, k-1$ with identical norm such that $C_\varphi = \sum_{s=0}^{k-1} w^s P_s$.

1. Introduction. Let $\mathbf{U}$ denote the open unit disc in the complex plane, and the *Hardy space* H^2 the functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ holomorphic in $\mathbf{U}$ such that $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$, with $\widehat{f}(n)$ denoting the nth Taylor coefficient of f. The inner product inducing the norm of H^2 is given by $\langle f, g\rangle := \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$. The inner product of two functions f and g in H^2 may also be computed by integration:

$$\langle f, g\rangle = \frac{1}{2\pi i}\int_{\partial \mathbf{U}} f(z)\overline{g(z)}\,\frac{dz}{z},$$

where $\partial\mathbf{U}$ is positively oriented and f and g are defined almost everywhere on $\partial\mathbf{U}$ via radial limits.

Each holomorphic self map φ of $\mathbf{U}$ induces on H^2 a *composition operator* C_φ defined by the equation $C_\varphi f = f \circ \varphi\,(f \in H^2)$. A consequence of a famous theorem of Littlewood [**6**] asserts that C_φ is a bounded operator (see also [**2, 10**]).

A *conformal automorphism* is a univalent holomorphic mapping of $\mathbf{U}$ onto itself. Each such map is linear fractional and can be represented as a product $w \cdot \alpha_p$, where

$$\alpha_p(z) := \frac{p - z}{1 - \overline{p}z}, (z \in \mathbf{U}),$$

for some fixed $p \in \mathbf{U}$ and $w \in \partial\mathbf{U}$. (See [**9**]).

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The map α_p interchanges the point p and the origin and it is a self-inverse automorphism of $\mathbf{U}$.

Each conformal automorphism is a bijection map from the sphere $\mathbf{C}\bigcup\{\infty\}$ to itself with two fixed points (counting multiplicity). An automorphism is called:

• *elliptic* if it has one fixed point in the disc and one outside the closed disc,

• *hyperbolic* if it has two distinct fixed points on the boundary $\partial\mathbf{U}$, and

• *parabolic* if there is one fixed point of multiplicity 2 on the boundary $\partial\mathbf{U}$.

An elliptic automorphism φ of $\mathbf{U}$ that does not fix the origin must have the form $\varphi = \alpha_p \circ \rho_w \circ \alpha_p$, where

$$\alpha_p(z) := \frac{p-z}{1-\overline{p}z}, \qquad \rho_w(z) = wz \ (z \in \mathbf{U}),$$

for some fixed $p \in \mathbf{U} - \{0\}$ and $w \in \partial\mathbf{U}$. Let us call w the rotation parameter of φ.

Let $k \in \mathbf{N}$, $k \geq 2$, and let w be the kth root of unity. We show that there are orthogonal projections P_s, $s = 0, 1, \dots, k-1$, with identical norm such that $C_\varphi = \sum_{s=0}^{k-1} w^s P_s$.

2. Matrix representation of C_φ. Let φ be a finite order elliptic automorphism of $\mathbf{U}$ of order k, i.e., $\varphi = \alpha_p \circ \rho_w \circ \alpha_p$ and w is the kth root of unity. We begin with $0 < p < 1$. For each nonnegative integer n, let

$$b_n(z) := \frac{\sqrt{1-p^2}}{1-pz}(\alpha_p(z))^n,$$

be the Guyker basis of the Hardy space H^2 (see [**5**] for more details).

We are going to find the matrix representation of the composition operator C_φ with respect to $\{b_n\}$. By a simple computation, it follows that

$$(C_\varphi b_n)(z) = \frac{w^n(1-wp^2-p(1-w)z)}{1-p^2} b_n(z).$$

Hence, for nonnegative integers n and m,

$$\langle C_\varphi b_n, b_m\rangle = \frac{w^n(1-wp^2)}{1-p^2}\langle b_n, b_m\rangle - \frac{p(1-w)w^n}{1-p^2}\langle zb_n, b_m\rangle.$$

In [**5**], Guyker established that the matrix representation for C_φ relative to $\{b_n\}$ is lower triangular with diagonal entries $[1, \varphi'(p), \varphi'(p)^2, \ldots]$. So we need to determine matrix entries on and below the matrix diagonal. If $n = m$, $\langle zb_m, b_m\rangle = p$. Now let $m > n$. We have

$$\begin{aligned}\langle (1-pz)b_{n+1}, b_m\rangle &= \left\langle (1-pz)\frac{\sqrt{1-p^2}}{1-pz}(\alpha_p(z))^{n+1}, b_m\right\rangle \\ &= \langle (p-z)b_n, b_m\rangle \\ &= p\langle b_n, b_m\rangle - \langle zb_n, b_m\rangle.\end{aligned}$$

Hence, we have the following recursion formula

$$\langle zb_n, b_m\rangle = p\langle b_n, b_m\rangle + p\langle zb_{n+1}, b_m\rangle - \langle b_{n+1}, b_m\rangle.$$

By solving this recursion formula we have

$$\begin{aligned}\langle zb_n, b_m\rangle &= (p)^{m-n-1}(p\langle zb_m, b_m\rangle - 1) \\ &= (p^2-1)p^{m-n-1},\end{aligned}$$

and hence

$$\langle C_\varphi b_n, b_m\rangle = \begin{cases} 0 & \text{if } m < n; \\ w^n & \text{if } m = n; \\ w^n(1-w)p^{m-n} & \text{if } m > n. \end{cases}$$

Therefore, the matrix representation of the composition operator C_φ with respect to the Guyker basis $\{b_n\}$ of H^2 is

$$C_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ (1-w)p & w & 0 & 0 & 0 & 0 & \cdots \\ (1-w)p^2 & w(1-w)p & w^2 & 0 & 0 & 0 & \cdots \\ (1-w)p^3 & w(1-w)p^2 & w^2(1-w)p & w^3 & 0 & 0 & \cdots \\ (1-w)p^4 & w(1-w)p^3 & w^2(1-w)p^2 & w^3(1-w)p & w^4 & 0 & \cdots \\ (1-w)p^5 & w(1-w)p^4 & w^2(1-w)p^3 & w^3(1-w)p^2 & w^4(1-w)p & w^5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence, the (i, j) element of this matrix is:

$$a_{i,j} := \begin{cases} w^{j-1} & \text{if } i = j; \\ w^{j-1}(1-w)p^{i-j} & \text{if } i > j; \\ 0 & \text{if } i < j, \end{cases}$$

where the indices i, j vary over the positive integers.

3. Main results. Before coming to the main course of this section, we need the following definition and results which can be found in [**3, 4**].

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$, i.e., $T \in \mathcal{L}(\mathcal{H})$. The *numerical range* of T is the set

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

in the complex plane, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$. Also the numerical radius of T is the

$$w(T) := \sup\{|z| : z \in W(T)\}.$$

Some properties of the numerical range follow easily from the definition. For one thing, the numerical range is unchanged under the unitary equivalence of operators: $W(T) = W(U^*TU)$ for any unitary U. It also behaves nicely under the operation of taking the adjoint of an operator: $W(T^*) = \{\overline{z} : z \in W(T)\}$. One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toeplitz-Hausdorff theorem. Another important property of $W(T)$ is that its closure contains the spectrum of the operator, i.e., $\sigma(T) \subseteq \overline{W(T)}$. $W(T)$ is a connected set and, in the finite-dimensional case, is compact. Also, $w(T) \leq \|T\|$.

We denote by $\mathcal{F}(T)$ the family of all functions which are analytic on some neighborhood of $\sigma(T)$.

For an open-and-closed subset τ of $\sigma(T)$, there is a function $f \in \mathcal{F}(T)$ which is identically one on τ and which vanishes on the rest of $\sigma(T)$. We put $E(\tau) = f(T)$. $E(\tau)$ is called the *spectral projection corresponding* to τ. If τ consists of the single point λ, the symbol $E(\lambda)$ will be used instead of $E(\{\lambda\})$.

Proposition 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ and $f \in \mathcal{F}(T)$ such that $f(T) = 0$ and $f'(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$. Then $\sigma(T)$ is a finite set and T is a scalar-type spectral operator of the form $\sum_{r=1}^{n} \lambda_r F(\lambda_r)$, where $F(\lambda_r)$ is the spectral projection corresponding to the open-and-closed subset λ_r of $\sigma(T)$.*

These facts will help in discussing and proving the main result below.

Theorem 3.2. *Let φ be an elliptic automorphism of $\mathbf{U}$ with rotation parameter w. If w is the kth root of unity, then there are orthogonal projections P_s, $s = 0, 1, \dots, k-1$, with identical norm such that $C_\varphi = \sum_{s=0}^{k-1} w^s P_s$.*

Proof. We have already seen that $\varphi = \alpha_p \circ \rho_w \circ \alpha_p$, for some fixed $p \in \mathbf{U} - \{0\}$. Now let $f(z) = z^k - 1$. Then $f \in \mathcal{F}(C_\varphi)$, $f(C_\varphi) = 0$ and $f'(\lambda) \neq 0$ for all $\lambda \in \sigma(C_\varphi) = \{1, w, w^2, \dots, w^{k-1}\}$. Hence, we will apply the previous proposition. For each $s = 0, 1, \dots, k-1$, let $f_s(z) = (1/k)\sum_{t=0}^{k-1}(z^t/w^{ts})$. Then $P_s := f_s(C_\varphi)$ is the spectral projection corresponding to w^s, and C_φ is a scalar-type spectral operator of the form $\sum_{s=0}^{k-1} w^s P_s$. Also $P_s P_r = 0$ for $s \neq r$. Thus, it only remains to show that $\|P_s\| = \|P_r\|$.

Let $0 < p < 1$ and $A^{(t)} = [a_{ij}^{(t)}]$ be the matrix of the C_φ^t. Then for each (i, j), we have

$$a_{ij}^{(t)} = \begin{cases} 0 & \text{if } i < j; \\ w^{t(j-1)} & \text{if } i = j; \\ w^{t(j-1)}(1 - w^t)p^{i-j} & \text{if } i > j. \end{cases}$$

Then the elements of the matrix of the idempotent $P_s = [p_{ij}^{(s)}]$ are

$$p_{ij}^{(s)} = \frac{1}{k}\sum_{t=0}^{k-1} \frac{a_{ij}^{(t)}}{w^{ts}} = \begin{cases} 0 & \text{if } i < j; \\ \frac{1}{k}\sum_{t=0}^{k-1} \frac{w^{t(i-1)}}{w^{ts}} & \text{if } i = j; \\ \frac{1}{k}\sum_{t=0}^{k-1} \frac{w^{t(j-1)}(1-w^t)p^{i-j}}{w^{ts}} & \text{if } i > j. \end{cases}$$

$$= \begin{cases} 1 & \text{if } i = j \text{ and } (i-1 \equiv_k s); \\ p^{i-j} & \text{if } i > j \text{ and } (j-1 \equiv_k s); \\ -p^{i-j} & \text{if } i > j \text{ and } (j \equiv_k s); \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $x = (x_1, x_2, \dots)$ with $\|x\| = 1$. Then

$$x^* P_s x = \sum_{t=0}^{\infty} \Bigg[\Bigg(\sum_{i=tk+s+2}^{\infty} p_{i,kn+s+1}^{(s)} x_i \Bigg) \overline{x_{tk+s+1}} + \Bigg(\sum_{i=tk+s+2}^{\infty} p_{i,kn+s}^{(s)} x_i \Bigg) \overline{x_{tk+s}} + p_{tk+s+1,tk+s+1}^{(s)} |x_{tk+s+1}|^2 \Bigg].$$

So

$$x^* P_s x = \sum_{t=0}^{\infty} \Bigg[\Bigg(\sum_{i=tk+s+2}^{\infty} p^{i-tk-s-1} x_i \Bigg) \overline{x_{tk+s+1}} + \Bigg(\sum_{i=tk+s+2}^{\infty} -p^{i-tk-s} x_i \Bigg) \overline{x_{tk+s}} + |x_{tk+s+1}|^2 \Bigg].$$

Define $y = (0, x_1, x_2, \dots)$. Then

$$\begin{aligned} x^* P_s x &= \sum_{t=0}^{\infty} \Bigg[\Bigg(\sum_{i=tk+s+2}^{\infty} p^{i-tk-s-1} y_{i+1} \Bigg) \overline{y_{tk+s+2}} \\ &\quad + \Bigg(\sum_{i=tk+s+2}^{\infty} -p^{i-tk-s} y_{i+1} \Bigg) \overline{y_{tk+s+1}} + |y_{tk+(s+1)+1}|^2 \Bigg] \\ &= \sum_{t=0}^{\infty} \Bigg[\Bigg(\sum_{i=tk+(s+1)+1}^{\infty} p^{i-tk-(s+1)-1} y_i \Bigg) \overline{y_{tk+(s+1)+1}} \\ &\quad + \Bigg(\sum_{i=tk+(s+1)+1}^{\infty} -p^{i-tk-(s+1)} y_i \Bigg) \overline{y_{tk+(s+1)}} + |y_{tk+(s+1)+1}|^2 \Bigg] \\ &= y^* P_{s+1} y. \end{aligned}$$

Hence, $W(P_s) \subseteq W(P_{s+1})$. Therefore, we have

$$W(P_s) \subseteq W(P_{s+1}) \subseteq W(P_{s+2}) \subseteq \cdots \subseteq W(P_{s+k}) = W(P_s).$$

Therefore, for each $s, r = 0, 1, 2, \dots, k-1$,

$$W(P_s) = W(P_r).$$

Since, for any idempotent operator E, we have $w(E) = (1 + \|E\|)/2$ where $w(E)$ is the numerical radius of E [**8**], then $\|P_s\| = \|P_r\|$. Now the proof is completed for $0 < p < 1$. Let $p \in \mathbf{U}$ and C_φ be the composition operator induced by $\varphi = \alpha_p \circ \rho_w \circ \alpha_p$. Let $\eta \in \partial \mathbf{U}$ be such that $p = \eta|p|$. A direct computation implies that $C_\varphi = C_{\rho_{\eta^{-1}}} \circ C_\phi \circ C_{\rho_\eta}$, where $\phi = \alpha_{|p|} \circ \rho_w \circ \alpha_{|p|}$. Let $C_\phi = \sum_{s=0}^{k-1} w^s P_s$ be the decomposition of the C_ϕ and $q_s := C_{\rho_{\eta^{-1}}} \circ P_s \circ C_{\rho_\eta}$. Then $C_\varphi = \sum_{s=0}^{k-1} w^s q_s$ and q_s, $s = 0, 1, \dots, k-1$ satisfy Theorem 3.2. $\square$

The importance of Theorem 3.2 is not the decomposition, but the identity $\|P_s\| = \|P_r\|$ for all r, s. Only decomposition, without considering equality of norm of projections, can be proved for any k-order operator in the first paragraph of the proof of Theorem 3.2. The identity of norms of projections is an important and special property of the composition operator that is proved using its matrix representation in a suitable basis and some properties of its numerical range.

Remark 1. In [**1**], Bourdon and Shapiro have considered the question of when the numerical range of a composition operator is a disc centered at the origin and have shown that this happens whenever the inducing map is a non elliptic conformal automorphism of the unit disc. They also have shown that the numerical range of an elliptic automorphism with order 2 is an ellipse with focus at ± 1. However, for the elliptic automorphisms with finite order $k > 2$, this is still an open problem.

Under the conditions of Theorem 3.2, suppose that $z \in W(C_\varphi)$, so there is $x = (x_1, x_2, \dots)$ with $\|x\| = 1$ such that $z = x^* C_\varphi x$. If $y = (0, x_1, x_2, \dots)$, then

$$\begin{aligned} wz &= w|x_1|^2 + \sum_{i=2}^{\infty} \left(\sum_{j=1}^{i-1} w^{j-1}(1-w)p^{i-j} x_j + w^{i-1} x_i \right) \overline{x_i} \\ &= w|x_1|^2 + \sum_{i=3}^{\infty} \left(\sum_{j=2}^{i-1} w^{j-2}(1-w)p^{i-j} x_{j-1} + w^{i-2} x_{i-1} \right) \overline{x_{i-1}} \end{aligned}$$

$$= w|y_2|^2 + \sum_{i=2}^{\infty}\left(\sum_{j=1}^{i-1} w^{j-1}(1-w)p^{i-j}y_j + w^{i-1}y_i\right)\overline{y_i}$$
$$= y^*C_\varphi y.$$

Hence, $wz \in W(C_\varphi)$, and therefore $wW(C_\varphi) \subseteq W(C_\varphi)$. This implies that $W(C_\varphi) = w^k W(C_\varphi) \subseteq w^{k-1}W(C_\varphi) \subseteq w^{k-2}W(C_\varphi) \subseteq \cdots \subseteq wW(C_\varphi) \subseteq W(C_\varphi)$.

So $w^j W(C_\varphi) = W(C_\varphi)$ for $j = 1, 2, \ldots, k-1$. Then $W(C_\varphi)$ is not an ellipse if $k > 2$.

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Department of Mathematics, College of Sciences, Yasouj University, Yasouj, 75914, Iran
Email address: heydari@mail.yu.ac.ir