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DIFFERENTIABILITY WITH RESPECT TO INITIAL FUNCTIONS FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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ABSTRACT. Initial value problems for first order partial functional almost linear equations with unbounded delay are considered. The functional dependence is represented by the generalized Hale operator with the values in the abstract normed space. The suitable system of axioms for the phase space is given.

A theorem on the global existence of classical solutions and continuous dependence upon initial data is formulated. The proof is based on the method of successive approximations. Finally, a result on the differentiability of solutions with respect to initial functions is proved. Important examples of integral differential equations or differential functional equations with a deviated argument can be obtain by specializing given functions.

1. Introduction. Write $B = (-\infty, 0] \times [-b, b]$, $b \in R^n_+$, $R_+ = [0, +\infty)$. Vectorial inequalities are understood to hold componentwise. For $t \in [0, a]$ where $a > 0$, we put $E_t = (-\infty, t] \times R^n$. Suppose that $z : E_a \to R^k$ and $(t, x) \in E_a$. We consider the function $z_{(t,x)} : B \to R^k$, defined by

$$z_{(t,x)}(s, y) = z(t + s, x + y), \quad (s, y) \in B. \tag{1}$$

Let X be a linear space with the norm $\|\cdot\|_X$ consisting of functions mapping the set B into R^k. We denote by $M_{k\times n}$ the set of all $k \times n$ matrices with real elements. Put $E = [0, a] \times R^n$, and assume that

$$\begin{aligned} f &: E \to M_{k\times n}, & f &= [f_{ij}]_{i=1,\dots,k,j=1,\dots,n},\\ F &: E \times X \to R^k, & F &= (F_1, \dots, F_k), \end{aligned}$$

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$$\psi : E_0 \to R^k, \qquad \psi = (\psi_1, \dots, \psi_k),$$
$$\varphi : E \to R^{n+1}, \qquad \varphi = (\varphi_0, \varphi_1, \dots, \varphi_n),$$

are given functions. Let us denote by $z = (z_1, \dots, z_k)$ an unknown function of the variables (t, x), $x = (x_1, \dots, x_n)$. We consider the weakly coupled system of functional differential equations

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} f_{ij}(t,x)\partial_{x_j} z_i(t,x) = F_i(t,x,z_{\varphi(t,x)}), \quad 1 \le i \le k, \tag{2}$$

with the initial condition

$$z(t,x) = \psi(t,x) \quad \text{for } (t,x) \in E_0. \tag{3}$$

Here X denotes an abstract normed space satisfying suitable axioms. This space is called the *phase space*. Further assumptions on X are given in the next parts of the paper. Our formulation of the differential functional problem is motivated by the general model of ordinary differential functional equations (see [**3**]). The functional variable is represented by the generalized Hale operator $(t, x) \mapsto z_{(t,x)}$ defined by (1). Such an operator for partial equations was first introduced in [**5**] in the case of bounded delay. In the present paper, the function $z_{(t,x)}$ is defined on B, and it is given by the restriction of the unknown function z to the unbounded set $(-\infty, t] \times [x - b, x + b]$. The functional variable in (2) is in the form $z_{\varphi(t,x)}$, and we will require that $\varphi_0(t, x) \le t$, $(t, x) \in E$. In view of that, the functional dependence is of Volterra type. Equations with a deviated argument or integral differential equations can be derived from the model presented (see Remark 2). They have applications in different branches of knowledge. Examples of such applications can be found in [**8, 16**].

We consider classical solutions of the Cauchy problem (2), (3). During this time numerous papers were published concerning various problems for first order partial functional differential equations or systems. The following questions were considered: functional differential inequalities corresponding to initial or mixed problems and their applications, uniqueness of solutions and continuous dependence on initial or initial boundary conditions, existence theory of classical or weak solutions of equations with initial or initial boundary conditions and approximate solutions of functional differential problems. It is not our aim to show a full review of papers concerning the above problems. The monograph [**8**] contains an exposition of the theory of hyperbolic functional differential equations.

Let us mention some of the methods that are adopted for proving existence results for first order partial functional differential equations. The fixed point method is based on the well-known Banach fixed point theorem. By using this method, the first results for quasilinear functional differential equations and Carathéodory solutions of initial or initial boundary value problems were obtained. The Schauder fixed point theorem was used in [**13**] for generalized solutions of almost linear problems with unknown function of two variables. The difference method was used in [**10**] for discussing the existence of Carathéodory solutions for equations with a deviated argument and in [**2**] for a class of functional differential equations with unknown functions of two variables. A general case was not solved.

The method of linearization for initial or initial boundary value problems was used in [**8**]. It consists of linearization of the right-hand side of the equation with respect to partial derivatives of an unknown function. In the second step, a quasilinear system was constructed for an unknown function and for its partial derivatives. The system thus obtained is equivalent to a system of integral functional equations of Volterra type. Classical solutions of integral functional equations led to solutions of original problems.

The method of successive approximations is based on the idea which was first introduced by Wazewski in [**15**] for systems without a functional dependence. The method is very sharp. By means of this method, the first results of classical solutions to functional differential problems were obtained (see [**1, 6, 14**]). We have decided to extend this method to functional differential equations with unbounded delay.

The papers [**7, 9, 11**] initiated the theory of partial functional differential equations with unbounded delay. The authors used general ideas concerning axiomatic approach to equations with unbounded delay which were introduced for ordinary equations (see [**4, 12**]). Sufficient conditions for the existence and uniqueness of Carathéodory solutions or solutions in the Cinquini-Cibrario sense were given. A method of quasilinearization and theorems on integral inequalities were used. All problems considered in those papers have the following property: solutions exist on domains which are local with respect to variable t. In the present paper, we show that there are functional differential equations with unbounded delay such that solutions of suitable initial problems exist on domains which are global with respect to all variables.

The main assumption of existence results in [**7, 9, 11**] is the following: given functions or their partial derivatives, satisfy the global Lipschitz condition. In our paper we adopt nonlinear estimates of the Perron type and suitable inequalities which are local with respect to the functional variable. It is clear that there are differential equations with deviated variables and differential integral equations such that local nonlinear estimates of the Perron type hold and the global Lipschitz condition is not satisfied.

Until now, there have not been any results on the global existence of classical solutions for partial functional differential equations with unbounded delay.

In the paper, we start investigations of differentiability of solutions with respect to initial functions for partial functional differential equations. The monograph [**3**] contains results on differentiability with respect to initial functions for solutions of ordinary functional differential equations.

Let us denote by $\mathcal{J}$ the class of all functions $\psi : E_0 \to R^k$ such that there exists exactly one classical solution $\mathcal{Z}[\psi]$ of problem (2), (3). We give construction of the space X, and we prove that the operator $\psi \mapsto \mathcal{Z}[\psi]$ has the following property: for each $\psi \in \mathcal{J}$, there exists the Fréchet derivative $\partial\mathcal{Z}[\psi]$ of $\mathcal{Z}$ at the point $\psi \in \mathcal{J}$. Moreover, if $\psi, \chi \in \mathcal{J}$ and $v = \partial\mathcal{Z}[\psi]\chi$, then v is a solution of integral functional system generated by (2), (3) and that system is linear.

The paper is organized as follows. The set of axioms and examples of phase spaces are given in Section 2. The sequences of successive approximations are investigated in Section 3. The main existence result and continuous dependence of solutions on initial functions is presented in Section 4. The last part of the paper is concerned with differentiability of solutions with respect to initial functions.

2. Phase spaces. We formulate assumptions on the phase space X. We introduce the following notation. Let the symbol $C(U, V)$

denote the class of all continuous functions from U into V where U and V are metric spaces. For $x \in R^k$, $y \in R^n$ and $C \in M_{k\times n}$ where $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_n)$, $C = [c_{ij}]_{i=1,\dots,k, j=1,\dots,n}$, we put

$$\|x\|_\infty = \max\{|x_i| : 1 \le i \le k\},$$

$$\|y\| = \sum_{j=1}^{n} |y_j|,$$

$$\|C\|_{k\times n} = \max\left\{\sum_{j=1}^{n} |c_{ij}| : 1 \le i \le k\right\}.$$

We will use the symbol $\circ$ to denote the scalar product in R^n. Given a function $w : (-\infty, a] \times [-b, b] \to R^k$ and a point $t \in (-\infty, a]$, we consider $w_{(t,0)} : B \to R^k$ defined by $w_{(t,0)}(s, y) = w(t+x, y)$, $(s, y) \in B$. If w is continuous on $[0, a] \times [-b, b]$, then we write

$$\|w\|_{[t]} = \max\{\, \|w(s, y)\|_\infty : (s, y) \in [0, t] \times [-b, b]\,\}, \quad t \in [0, a].$$

Assumption $H[X]$**.** The space $(X, ||\cdot||_X)$ is a Banach space of functions from B into R^k and

1) there is a $\lambda \in R_+$ such that, for each function $w \in X$, we have

$$\|w(0, x)\|_\infty \le \lambda \|w\|_X, \quad x \in [-b, b],$$

2) if $w : (-\infty, a] \times [-b, b] \to R^k$ is such that $w_{(0,0)} \in X$ and w is continuous on $[0, a] \times [-b, b]$, then $w_{(t,0)} \in X$ for $t \in [0, a]$ and

(i) the function $t \mapsto w_{(t,0)}$ is continuous on $[0, a]$,

(ii) there are $K_1, K_0 \in R_+$ independent of w such that

$$\|w_{(t,0)}\|_X \le K_1 \|w\|_{[t]} + K_0 \|w_{(0,0)}\|_X, \quad t \in [0, a].$$

To complete information for the phase spaces $(X, ||\cdot||_X)$, we give examples of them (see [**9**]).

Example 1. Let X be the class of all functions $w : B \to R^k$ which are bounded and uniformly continuous on B. For $w \in X$, we put

$$||w||_X = \sup\Big\{||w(s, y)||_\infty : (s, y) \in B\Big\}. \tag{4}$$

Then Assumption $H[X]$ is satisfied with $\lambda = K_1 = K_0 = 1$.

Example 2. Let X be the class of all continuous functions $w : B \to R^k$ such that there exists $\lim_{t\to-\infty} w(t,x) = w_0(x)$ uniformly with respect to $x \in [-b,b]$. Then Assumption $H[X]$ is satisfied with the norm defined by (4) and $\lambda = K_1 = K_0 = 1$.

Example 3. Let $\gamma_0 : (-\infty, 0] \to (0,+\infty)$ be a continuous and nonincreasing function. Let X be the class of all continuous functions $w : B \to R^k$ such that

$$\lim_{t\to-\infty} \frac{\|w(t,x)\|_\infty}{\gamma_0(t)} = 0, \quad x \in [-b,b],$$

with the norm of w defined by

$$||w||_X = \sup\left\{ \frac{||w(t,x)||_\infty}{\gamma_0(t)} : (t,x) \in B \right\}.$$

Then Assumption $H[X]$ is satisfied with $\lambda = \gamma_0(0)$, $K_1 = 1/\gamma_0(0)$, $K_0 = 1$.

Example 4. Let $p \geq 1$ be fixed. Denote by Y the class of all functions $w : B \to R^k$ such that

(i) w is continuous on $\{0\} \times [-b,b]$;

(ii) for $x \in [-b,b]$, we have

$$\int_{-\infty}^{0} ||w(s,x)||_\infty^p \, ds < +\infty;$$

(iii) for each $t \in (-\infty, 0]$, the function $w(t,\cdot) : [-b,b] \to R^k$ is continuous.

For $w \in Y$, we define the norm of w by

$$||w||_Y = \max\Big\{ ||w(0,x)||_\infty : x \in [-b,b] \Big\} + \sup\left\{ \left(\int_{-\infty}^{0} ||w(s,x)||_\infty^p \, ds \right)^{1/p} : x \in [-b,b] \right\}.$$

Let us denote by X the closure of Y with the above norm. Then Assumption $H[X]$ is satisfied with $\lambda = 1$, $K_1 = 1$, $K_0 = 1 + a^{1/p}$.

Example 5. Denote by Y the class of all functions $w : B \to R^k$ satisfying the conditions:

(i) w is bounded and it is continuous on $\{0\} \times [-b, b]$;

(ii) for $x \in [-b, b]$, we have

$$I(x) = \sup\left\{ \int_{-(m+1)}^{-m} ||w(s,x)||_\infty \, ds : m \in N \right\} < +\infty;$$

(iii) for each $t \in (-\infty, 0]$, the function $w(t, \cdot) : [-b, b] \to R^k$ is continuous.

For $w \in Y$, we define the norm of w by

$$||w||_Y = \max\Big\{ ||w(0,x)||_\infty : x \in [-b,b] \Big\} + \sup\Big\{ I(x) : x \in [-b,b] \Big\}.$$

Let us denote by X the closure of Y with the above norm. Then Assumption $H[X]$ is satisfied with $\lambda = 1$, $K_1 = 1 + a$, $K_0 = 2$.

Note that, in view of Example 1, our theory also contains the case of bounded delay (i.e., set B is given by $[-b_0, 0] \times [-b, b]$ where $b_0 \geq 0$).

For $t \in [0, a]$ and for functions $z : E_a \to R^k$, $v : E_a \to R^n$ and $u : E_a \to M_{k\times n}$ which are continuous and bounded on E, we write

$$\begin{aligned} \|z\|_{t.k} &= \sup\{\|z(s,y)\|_\infty : (s,y) \in [0,t] \times R^n\}, \\ \|v\|_{t.n} &= \sup\{\|v(s,y)\| : (s,y) \in [0,t] \times R^n\}, \\ \|u\|_{t.k\times n} &= \sup\{\|u(s,y)\|_{k\times n} : (s,y) \in [0,t] \times R^n\}. \end{aligned}$$

Lemma 1. *Suppose that Assumption $H[X]$ is satisfied and $z : E_a \to R^k$, $z = (z_1, \dots, z_k)$. If $z_{(0,x)} \in X$ for $x \in R^n$ and z is continuous on E, then $z_{(t,x)} \in X$, $(t,x) \in E$, and*

$$\|z_{(t,x)}\|_X \leq K_1 \|z\|_{t.k} + K_0 \|z_{(0,x)}\|_X, \quad (t,x) \in E. \tag{5}$$

If we assume additionally that $z_{(t,x)} \in X$ *for* $(t,x) \in [-a,0] \times R^n$ *and there exist derivatives* $\partial_t z$ *and* $\partial_x z = [\partial_{x_j} z_i]_{i=1,\dots,k,j=1,\dots,n}$ *on* E, *and they are continuous and bounded on* E, *then*

$$\begin{aligned}\|z_{(t,x)} - z_{(\overline{t},\overline{x})}\|_X \le K_1\|\partial_t z\|_{t.k} \cdot |t-\overline{t}| + K_1\|\partial_x z\|_{t.k\times n} \\ \cdot \|x-\overline{x}\| + K_0\|z_{(0,x)} - z_{(\overline{t}-t,\overline{x})}\|_X\end{aligned} \tag{6}$$

where $(t,x), (\overline{t},\overline{x}) \in E$, $t > \overline{t}$.

Proof. Let $w : (-\infty,a]\times[-b,b] \to R^k$ be given by $w(s,y) = z(s,x+y)$ where $x \in R^n$ is fixed. Then $w_{(t,0)} = z_{(t,x)}$, $t \in [0,a]$. It follows from Assumption $H[X]$ that $z_{(t,x)} \in X$ and (5) holds. To prove (6), suppose that $(t,x), (\overline{t},\overline{x}) \in E$, $t > \overline{t}$ and $\widetilde{z} : E_a \to R^k$ is defined by $\widetilde{z}(s,y) = z(s+\overline{t}-t, y+\overline{x}-x)$, $(s,y) \in E_a$. Then $\widetilde{z}_{(t,x)} = z_{(\overline{t},\overline{x})}$. It follows from (5) that

$$\|z_{(t,x)} - z_{(\overline{t},\overline{x})}\|_X = \|(z-\widetilde{z})_{(t,x)}\|_X \le K_1\|z-\widetilde{z}\|_{t.k} + K_0\|(z-\widetilde{z})_{(0,x)}\|_X,$$

and thus we obtain assertion (6). The proof of Lemma 1 is complete. □

3. The sequences of successive approximations. We formulate assumptions on the given functions. For a function $w : E \to M_{k\times n}$ or $w : E_a \to M_{k\times n}$ where $w = [w_{ij}]_{i=1,\dots,k,j=1,\dots,n}$, we write

$$w_{[i]} = (w_{i1},\dots,w_{in}), \quad 1 \le i \le k,$$

and

$$w_{\{j\}} = (w_{1j},\dots,w_{kj}), \quad 1 \le j \le n.$$

Let the symbol $CL(X,R)$ denote the class of all continuous and linear operators defined on X and taking values in R. The norm in the space $CL(X,R)$ generated by the norm $\|\cdot\|_X$ in X will be denoted by the same symbol, $\|\cdot\|_X$. For $W = (W_1,\dots,W_k)$ where $W_i \in CL(X,R)$, $1 \le i \le k$, we write

$$\|W\|_\infty^* = \max\{\|W_i\|_X : 1 \le i \le k\}.$$

Let us use the symbol $\mathcal{M}$ to denote the class of all functions $\alpha : R_+ \to R_+$ such that α is continuous, nondecreasing, $\alpha(0) = 0$ and $\alpha(t+\tau) \le \alpha(t) + \alpha(\tau)$ for $t, \tau \in R_+$.

Assumption $H[f, F]$**.** The functions $f : E \to M_{k\times n}$, $F : E \times X \to R^k$ are continuous, Assumption $H[X]$ is satisfied and

1) there is an $A_0 \in R_+$ such that

$$\|f_{[i]}(t,x) - f_{[i]}(t,\overline{x})\| \le A_0 \|x - \overline{x}\| \quad \text{on } E, \quad 1 \le i \le k;$$

2) there exists a $\sigma : [0, a] \times R_+ \to R_+$ such that σ is continuous and nondecreasing with respect to the second variable and, for each $\xi \in R_+$, there is on $[0, a]$ the maximal solution of the Cauchy problem

$$\omega'(t) = K_1 \sigma(t, \omega(t)), \quad \omega(0) = \xi;$$

3) the estimate

$$\|F(t,x,w)\|_\infty \le \sigma(t, \|w\|_X)$$

is satisfied on $E \times X$.

Assumption $H[\varphi]$**.** The function $\varphi : E \to R^{n+1}$ is continuous and

1) $\varphi_0(t,x) \le t$ for $(t,x) \in E$;

2) there exist on E the derivatives $\partial_{x_i}\varphi_j$, $1 \le i,j \le n$, which are continuous on E and there is a $B_0 \in R_+$ such that

$$|\partial_{x_i}\varphi_j(t,x)| \le B_0 \quad \text{on } E, \quad 1 \le i,j \le n.$$

Now we define the set of initial functions for the problem (2), (3). Let $\mathcal{J}$ be the class of all $\psi \in C(E_0, R^k)$ such that

1) $\psi_{(t,x)} \in X$ for $(t,x) \in E_0$ and

$$\|\psi\|^* = \sup\{\|\psi_{(t,x)}\|_X : (t,x) \in E_0\} < +\infty;$$

2) there exist the derivatives $\partial_t\psi$, $\partial_x\psi = [\partial_{x_j}\psi_i]_{i=1,\dots,k,j=1,\dots,n}$, which are continuous and $(\partial_t\psi)_{(t,x)} \in X$, $(\partial_{x_j}\psi)_{(t,x)} \in X$ for $(t,x) \in E_0$, $1 \le j \le n$, and

$$\|\partial_t\psi\|^* < +\infty,$$
$$\|\partial_x\psi\|^{**} = \sup\left\{ \sum_{j=1}^{n} \|(\partial_{x_j}\psi)_{(t,x)}\|_X : (t,x) \in E_0 \right\} < +\infty;$$

3) there is an $\widetilde{\alpha} \in \mathcal{M}$ such that

$$\left\|(\partial_t\psi)_{(t,x)} - (\partial_t\psi)_{(\overline{t},\overline{x})}\right\|_X + \sum_{j=1}^{n} \|(\partial_{x_j}\psi)_{(t,x)} - (\partial_{x_j}\psi)_{(\overline{t},\overline{x})}\|_X \le \widetilde{\alpha}(|t-\overline{t}| + \|x-\overline{x}\|),$$

where $(t,x), (\overline{t},\overline{x}) \in E_0$;

4) ψ satisfies (2) for $t = 0$, $x \in R^n$.

Consider the Cauchy problem

$$\eta'(\tau) = f_{[i]}(\tau, \eta(\tau)), \quad \eta(t) = x, \tag{7}$$

where $(t,x) \in E$ and $1 \le i \le k$. Let us denote by $g_i(\cdot, t, x)$ the solution of (7). The function g_i is called the *ith characteristic* of system (2). It follows from Assumption $H[f,F]$ that, for each $(t,x) \in E$ and $1 \le i \le k$, the function $g_i(\cdot,t,x)$ is the unique solution of (7) defined on $[0,a]$.

For an initial function $\chi \in \mathcal{J}$ and for a function $\zeta \in C([0,a], R_+)$, we set

$$\mathcal{C}_\chi = \left\{ z \in C(E_a, R^k) : z = \chi \text{ on } E_0 \right\},$$
$$\mathcal{C}_\chi[\zeta] = \left\{ z \in \mathcal{C}_\chi : \|z_{(t,x)}\|_X \le \zeta(t),\ (t,x) \in E \right\}.$$

Let us fix the initial function $\psi \in \mathcal{J}$. We define the additional classes of functions

$$\mathcal{C}_{\partial_t\psi} = \left\{ v_0 \in C(E_a, R^k) : v_0 = \partial_t\psi \text{ on } E_0 \right\},$$
$$\mathcal{C}_{\partial_x\psi_i} = \left\{ v \in C(E_a, R^n) : v = \partial_x\psi_i \text{ on } E_0 \right\}, \quad 1 \le i \le k.$$

Suppose that Assumptions $H[f,F]$ and $H[\varphi]$ are satisfied. Write

$$P_i[z](\tau,t,x) = \big(\tau, g_i(\tau,t,x), z_{\varphi(\tau,g_i(\tau,t,x))}\big), \quad 1 \le i \le k.$$

For $z \in \mathcal{C}_\psi$, we define $\mathcal{F}[z] = (\mathcal{F}_1[z], \dots, \mathcal{F}_k[z])$ in the following way:

$$\begin{gathered}\mathcal{F}[z](t,x) = \psi(t,x) \text{ on } E_0,\\ \mathcal{F}_i[z](t,x) = \psi_i(0, g_i(0,t,x)) + \int_0^t F_i(P_i[z](\tau,t,x))\,d\tau\\ \text{on } E, \quad 1 \le i \le k.\end{gathered}$$

Let the function $\gamma : [0,a] \to R_+$ be the maximal solution of the problem

$$\omega'(t) = K_1\sigma(t,\omega(t)), \qquad \omega(0) = (K_0 + \lambda K_1)\|\psi\|^*. \tag{8}$$

The above-defined operator $\mathcal{F}$ has the following property.

Lemma 2. *If Assumptions $H[f,F]$ and $H[\varphi]$ are satisfied, then the operator $\mathcal{F}$ maps the set $\mathcal{C}_\psi[\gamma]$ into itself.*

Proof. For $z \in \mathcal{C}_\psi[\gamma]$, we have

$$|\mathcal{F}_i[z](t,x)| \le \lambda\|\psi\|^* + \int_0^t \sigma(\tau, \|z_{\varphi(\tau,g_i(\tau,t,x))}\|_X)\,d\tau,$$

$(t,x) \in E,\ 1 \le i \le k$. Thus,

$$\|\mathcal{F}[z]_{(t,x)}\|_X \le (K_0 + \lambda K_1)\|\psi\|^* + K_1\int_0^t \sigma(\tau,\gamma(\tau))\,d\tau = \gamma(t)$$

on E, and the proof is complete. □

Assume $H[f,F]$ and $H[\varphi]$. For fixed $\eta_\psi > \gamma(a)$ where γ is the maximal solution of (8), we denote by $X[\eta_\psi]$ the set of all $w \in X$ such that $\|w\|_X \le \eta_\psi$.

In the sequel we will need the following additional assumptions.

Assumption $H_\psi[f,F]$**.** The functions $f : E \to M_{k\times n}$ and $F : E \times X \to R^k$ satisfy Assumption $H[f,F]$, and

1) there is an $\widetilde{A} \in R_+$ such that $|f_{ij}(t,x)| \leq \widetilde{A}$ on E, $1 \leq i,j \leq n$, and there exist on E the derivatives

$$\partial_x f_{[\nu]} = [\partial_{x_j} f_{\nu i}]_{i,j=1,\dots,n}, \quad 1 \leq \nu \leq k,$$

and $\partial_x f_{[\nu]}$ is continuous on E, $1 \leq \nu \leq k$;

2) the derivatives $\partial_x F = [\partial_{x_j} F_i]_{i=1,\dots,k,j=1,\dots,n}$ exist on $E \times X[\eta_\psi]$,

3) the Fréchet derivatives $\partial_w F(t,x,w) = (\partial_w F_1(t,x,w), \dots, \partial_w F_k(t,x,w))$ exist for $(t,x,w) \in E \times X[\eta_\psi]$ and $\partial_w F_i(t,x,w) \in CL(X,R)$ where $1 \leq i \leq k$, $(t,x,w) \in E \times X[\eta_\psi]$;

4) the functions $\partial_x F$ and $\partial_w F$ are continuous on $E \times X[\eta_\psi]$, and there is a $\beta \in \mathcal{M}$ such that, for each $c \in R_+$, we have

$$\int_1^\infty \beta(c \cdot 2^{-\tau})\, d\tau < +\infty \tag{9}$$

and the estimates

$$\|\partial_x F(t,x,w) - \partial_x F(t,x,\overline{w})\|_{k\times n} \leq \beta(\|w-\overline{w}\|_X),$$
$$\|\partial_w F(t,x,w) - \partial_w F(t,x,\overline{w})\|_\infty^* \leq \beta(\|w-\overline{w}\|_X)$$

are satisfied for $(t,x) \in E$, $w, \overline{w} \in X[\eta_\psi]$;

5) there is an $A \in R_+$ such that

$$\|\partial_x F(t,x,w)\|_{k\times n} \leq A, \qquad \|\partial_w F(t,x,w)\|_\infty^* \leq A \quad \text{on } E \times X[\eta_\psi].$$

Remark 1. If the function $\beta : R_+ \to R_+$ is given by

$$\beta(s) = L_0\, s^\alpha, \quad s \in R_+,$$

where $L_0 \in R_+$ and $\alpha \in (0,1]$, then it satisfies the required conditions, and then assumption 4) of $\mathrm{H}_\psi[f,F]$ becomes the following Holder conditions:

$$\|\partial_x F(t,x,w) - \partial_x F(t,x,\overline{w})\|_{k\times n} \leq L_0 \|w-\overline{w}\|_X^\alpha,$$
$$\|\partial_w F(t,x,w) - \partial_w F(t,x,\overline{w})\|_\infty^* \leq L_0 \|w-\overline{w}\|_X^\alpha,$$

where $(t,x) \in E$, $w, \overline{w} \in X[\eta_\psi]$.

Remark 2. We give comments on particular cases of problem (2), (3). Consider the function $\widetilde{F} : E \times R^k \to R^k$, $\widetilde{F} = (\widetilde{F}_1, \dots, \widetilde{F}_k)$. Write

$$F(t,x,w) = \widetilde{F}(t,x, \int_B w(\tau,s)\, d\tau\, ds).$$

Then (2) with $\varphi(t,x) = (t,x)$, $(t,x) \in E$, reduces to the differential integral system

$$\partial_t z_i(t,x) + \sum_{j=1}^n f_{ij}(t,x)\partial_{x_j} z_i(t,x) = \widetilde{F}_i(t,x, \int_B z(t+\tau, x+s)\, d\tau\, ds),$$
$$1 \le i \le k.$$

For the above $\widetilde{F}$, we put

$$F(t,x,w) = \widetilde{F}(t,x,w(0,0)). \tag{10}$$

Then (2) is a system with deviated variables

$$\partial_t z_i(t,x) + \sum_{j=1}^n f_{ij}(t,x)\partial_{x_j} z_i(t,x) = \widetilde{F}_i(t,x,z(\varphi(t,x))), \tag{11}$$

$1 \le i \le k$.

Suppose that $\varphi(t,x) = (t,x)$ for $(t,x) \in E$. Then (2) reduces to the functional differential system

$$\partial_t z_i(t,x) + \sum_{j=1}^n f_{ij}(t,x)\partial_{x_j} z_i(t,x) = F_i(t,x,z_{(t,x)}), \quad 1 \le i \le k. \tag{12}$$

There are the following motivations for investigations of (2) and (3) instead of (12) and (3). Differential systems with deviated variables are obtained from (12) in the following way. Write

$$F(t,x,w) = \widetilde{F}(t,x,w(\varphi(t,x) - (t,x))). \tag{13}$$

Then system (2) is equivalent to (11). It is clear that, under natural assumptions on $\widetilde{F}$, the function F given by (10) satisfies Assumption $H_\psi[f,F]$. Note that the function F given by (13) does not satisfy $H_\psi[f,F]$. More precisely, the derivatives $\partial_x F = [\partial_{x_j} F_i]_{i=1,\dots,k,j=1,\dots,n}$ do not exist on $E \times X[\eta_\psi]$.

With the above motivation we consider problem (2), (3) instead of (12), (3).

If $w \in X^n$ and $y \in R^n$ where $w = (w_1, \dots, w_n)$, $y = (y_1, \dots, y_n)$, then we define $w \circ y \in X$ by $w \circ y = w_1 y_1 + \dots + w_n y_n$ and $\partial_w F_i(P) w \in R^n$ with $P \in E \times X$, $1 \le i \le k$, by

$$\partial_w F_i(P) w = (\partial_w F_i(P) w_1, \dots, \partial_w F_i(P) w_n).$$

Now we are ready to define the sequences of successive approximations $\{z^{(m)}\}$, $\{u_0^{(m)}\}$, $\{u^{(m)}\}$. For $m \ge 0$, the functions $z^{(m)}, u_0^{(m)} : E_a \to R^k$, $u^{(m)} : E_a \to M_{k\times n}$ are such that $z^{(m)} = (z_1^{(m)}, \dots, z_k^{(m)})$, $u_0^{(m)} = (u_{01}^{(m)}, \dots, u_{0k}^{(m)})$, $u^{(m)} = [u_{ij}^{(m)}]_{i=1,\dots,k,j=1,\dots,n}$. Let $\widetilde{\psi} : E_a \to R^k$ be a function of class C^1 such that $\widetilde{\psi} \in \mathcal{C}_\psi[\gamma]$, $\partial_t \widetilde{\psi} \in \mathcal{C}_{\partial_t \psi}$ and $\partial_x \widetilde{\psi}_i \in \mathcal{C}_{\partial_x \psi_i}$, $1 \le i \le k$. If $m = 0$, then we put

$$z^{(0)} = \widetilde{\psi}, \quad u_0^{(0)} = \partial_t \widetilde{\psi}, \qquad u_{[i]}^{(0)} = \partial_x \widetilde{\psi}_i, \quad 1 \le i \le k. \tag{14}$$

Suppose that $z^{(m)}$, $u_0^{(m)}$ and $u^{(m)}$ are known functions. Then

$$z^{(m+1)} = \mathcal{F}[z^{(m)}], \tag{15}$$

and, for each $1 \le i \le k$, the function $u_{[i]}^{(m+1)} : E_a \to R^n$ is the solution of the equation

$$v = \mathcal{G}_i^{(m)}[v], \tag{16}$$

where

$$\mathcal{G}_i^{(m)}[v](t,x) = \partial_x \psi_i(t,x) \quad \text{on } E_0,$$
$$\mathcal{G}_i^{(m)}[v](t,x) = -\int_0^t v(\tau, g_i(\tau,t,x)) \partial_x f_{[i]}(\tau, g_i(\tau,t,x))\, d\tau + H_i^{(m)}(t,x)$$

on E. The functions $H_i^{(m)} : E \to R^n$, $1 \le i \le k$, are given by

$$\begin{aligned} H_i^{(m)}(t,x) &= \partial_x \psi_i(0, g_i(0,t,x)) \\ &\qquad + \int_0^t \{\partial_x F_i(P_i[z^{(m)}](\tau,t,x)) \\ &\qquad + \partial_w F_i(P_i[z^{(m)}](\tau,t,x))[u_0^{(m)}, u^{(m)}]_{\varphi(\tau,y)} \partial_x \varphi(\tau,y)\}\, d\tau, \end{aligned}$$

where $y = g_i(\tau,t,x)$ and $[u_0^{(m)}, u^{(m)}]_{\varphi(\tau,y)} \partial_x \varphi(\tau,y) \in X^n$ is defined as follows:

$$[u_0^{(m)}, u^{(m)}]_{\varphi(\tau,y)} \partial_x \varphi(\tau,y) = (w_1, \dots, w_n)$$

and

$$w_l = (u_0^{(m)})_{\varphi(\tau,y)} \partial_{x_l} \varphi_0(\tau,y) + \sum_{j=1}^n (u_{\{j\}}^{(m)})_{\varphi(\tau,y)} \partial_{x_l} \varphi_j(\tau,y), \quad 1 \le l \le n.$$

The function $u_0^{(m+1)} : E_a \to R^k$ is such that $u_0^{(m+1)} = \partial_t \psi$ on E_0 and

$$u_{0i}^{(m+1)}(t,x) = F_i(t,x,(z^{(m)})_{\varphi(t,x)}) - f_{[i]}(t,x) \circ u_{[i]}^{(m+1)}(t,x) \tag{17}$$

on E, $1 \le i \le k$.

Now we prove important properties of the above-defined sequences. We put

$$\mu(t) = A_1 e^{A_2 t} \quad \text{and} \quad \mu_0(t) = \sigma(t,\gamma(t)) + \widetilde{A}\mu(t),$$

where $t \in [0,a]$, and the constants $A_1, A_2 \in R_+$ are given by

$$\begin{aligned} A_1 &= Aa + (\lambda + aAB_0K_0)\|\partial_x \psi\|^{**} + aAB_0K_0\|\partial_t \psi\|^*, \\ A_2 &= A_0 + AB_0K_1(\widetilde{A} + 1). \end{aligned}$$

We denote by $\mathcal{C}_{\partial_x \psi_i}(\mu)$, $1 \le i \le k$, the set of all $v \in \mathcal{C}_{\partial_x \psi_i}$ such that $\|v(t,x)\| \le \mu(t)$, $(t,x) \in E$. Let the symbol $\mathcal{C}_{\partial_t \psi}(\mu_0)$ denote the set of all $v_0 \in \mathcal{C}_{\partial_t \psi}$ such that $\|v_0(t,x)\|_\infty \le \mu_0(t)$, $(t,x) \in E$.

Lemma 3. *If Assumptions $H_\psi[f,F]$ and $H[\varphi]$ are satisfied, then for each $m \ge 0$, we have*

(I_m) the functions $z^{(m)}$, $u_0^{(m)}$ and $u^{(m)}$ are defined on E_a, $z^{(m)} \in \mathcal{C}_\psi[\gamma]$, $u_0^{(m)} \in \mathcal{C}_{\partial_t \psi}(\mu_0)$ and $u_{[i]}^{(m)} \in \mathcal{C}_{\partial_x \psi_i}(\mu)$, $1 \le i \le k$;

(II_m) there exist on E_a the derivatives $\partial_t z^{(m)}$, $\partial_x z_i^{(m)}$, $1 \le i \le k$, and $\partial_t z^{(m)}(t,x) = u_0^{(m)}(t,x)$, $\partial_x z_i^{(m)}(t,x) = u_{[i]}^{(m)}(t,x)$, $(t,x) \in E_a$, $1 \le i \le k$.

Proof. To prove Lemma 3, we use induction with respect to m. It follows from the definition (14) that conditions (I_0) and (II_0) are satisfied.

We assume conditions (I_m) and (II_m) for fixed $m \ge 0$. Function $z^{(m+1)}$ is defined by (15), and it follows from Lemma 2 that $z^{(m+1)} \in \mathcal{C}_\psi[\gamma]$.

We prove that there exist $u^{(m+1)} : E_a \to M_{k\times n}$ and $u_{[i]}^{(m+1)} \in \mathcal{C}_{\partial_x \psi_i}(\mu)$, $1 \le i \le k$. Fix $i \in \{1, \ldots, k\}$. If $v \in \mathcal{C}_{\partial_x \psi_i}(\mu)$, then

$$\|\mathcal{G}_i^{(m)}[v](t,x)\| \le A_1 + A_2 \int_0^t \mu(\tau)\, d\tau = \mu(t)$$

on E, and thus $\mathcal{G}_i^{(m)} : \mathcal{C}_{\partial_x \psi_i}(\mu) \to \mathcal{C}_{\partial_x \psi_i}(\mu)$. For $v \in \mathcal{C}_{\partial_x \psi_i}(\mu)$, we put

$$[|v|] = \max\{\|v\|_{t.n} e^{-2At} : t \in [0,a]\}.$$

If $v, \overline{v} \in \mathcal{C}_{\partial_x \psi_i}(\mu)$, then

$$\|\mathcal{G}_i^{(m)}[v](t,x) - \mathcal{G}_i^{(m)}[\overline{v}](t,x)\| \le [|v - \overline{v}|] \int_0^t A e^{2A\tau} d\tau \le \frac{1}{2}[|v - \overline{v}|] e^{2At},$$

where $(t,x) \in E$, and consequently,

$$[|\mathcal{G}_i^{(m)}[v] - \mathcal{G}_i^{(m)}[\overline{v}]|] \le \frac{1}{2}[|v - \overline{v}|].$$

It follows from the Banach fixed point theorem that there exists in $\mathcal{C}_{\partial_x \psi_i}(\mu)$ exactly one solution $u_{[i]}^{(m+1)}$ of equation (16).

By definition (17), we obtain $u_0^{(m+1)} \in \mathcal{C}_{\partial_t \psi_i}(\mu_0)$, and the condition (I_{m+1}) is proved.

To verify (II_{m+1}), we write

$$\begin{aligned}&\Delta_i(t,x;h_0,h)\\&= z_i^{(m+1)}(t+h_0,x+h)-z_i^{(m+1)}(t,x)-u_{0i}^{(m+1)}(t,x)h_0-u_{[i]}^{(m+1)}(t,x)\circ h,\end{aligned}$$

and we prove that, for each $(t,x)\in E$, there is an $\varepsilon\in\mathcal{M}$ such that

$$|\Delta_i(t,x;h_0,h)| \le (|h_0|+\|h\|)\,\varepsilon(|h_0|+\|h\|) \tag{18}$$

where $h_0\in[-t,a-t]$, $h\in R^n$, $1\le i\le k$.

Let us fix $i\in\{1,\dots,k\}$. Suppose that $(t,x)\in E$, $h_0\in[-t,a-t]$ and $h\in R^n$. For simplicity, we write $\overline{t}=t+h_0$ and $\overline{x}=x+h$. It follows from (15) and (16) that

$$\begin{aligned}&\Delta_i(t,x;h_0,h)\\&=\mathcal{F}_i[z^{(m)}](\overline{t},\overline{x})-\mathcal{F}_i[z^{(m)}](t,x)-u_{0i}^{(m+1)}(t,x)h_0-\mathcal{G}_i^{(m)}[u_{[i]}^{(m+1)}](t,x)\circ h.\end{aligned}$$

Write

$$Q_i^{(m)}(s,\tau,t,x;h_0,h)=(1-s)P_i[z^{(m)}](\tau,t,x)+sP_i[z^{(m)}](\tau,\overline{t},\overline{x}).$$

By using the Hadamard mean value theorem we obtain

$$\begin{aligned}F_i(P_i[z^{(m)}](\tau,\bar{t},\bar{x}))&-F_i(P_i[z^{(m)}](\tau,t,x))\\&=\int_0^1\partial_xF_i(Q_i^{(m)}(s,\tau,t,x;h_0,h))\\&\quad\circ[g_i(\tau,\overline{t},\overline{x})-g_i(\tau,t,x)]\,ds\\&\quad+\int_0^1\partial_wF_i(Q_i^{(m)}(s,\tau,t,x;h_0,h))[(z^{(m)})_{\varphi(\tau,g_i(\tau,\overline{t},\overline{x}))}\\&\quad-(z^{(m)})_{\varphi(\tau,g_i(\tau,t,x))}]\,ds.\end{aligned}$$

Then the expression $\Delta_i(t,x;h_0,h)$ can be written in the following form:

$$\Delta_i(t,x;h_0,h)=\sum_{j=1}^4\Delta_i^{(j)},$$

where

$$\begin{aligned}
\Delta_i^{(1)} &= \psi_i(0, g_i(0, \overline{t}, \overline{x})) - \psi_i(0, g_i(0, t, x)) \\
&\quad - \partial_x \psi_i(0, g_i(0, t, x)) \circ [g_i(0, \overline{t}, \overline{x}) - g_i(0, t, x)], \\
\Delta_i^{(2)} &= \int_0^t \int_0^1 [\partial_x F_i(Q_i^{(m)}(s, \tau, t, x; h_0, h)) \\
&\quad - \partial_x F_i(P_i[z^{(m)}](\tau, t, x))] \circ (\overline{y} - y)\, ds\, d\tau \\
&\quad + \int_0^t \int_0^1 [\partial_w F_i(Q_i^{(m)}(s, \tau, t, x; h_0, h)) \\
&\quad - \partial_w F_i(P_i[z^{(m)}](\tau, t, x))][(z^{(m)})_{\varphi(\tau, \overline{y})} - (z^{(m)})_{\varphi(\tau, y)}]\, ds\, d\tau, \\
\Delta_i^{(3)} &= \int_0^t \partial_w F_i(P_i[z^{(m)}](\tau, t, x))[(z^{(m)})_{\varphi(\tau, \overline{y})} - (z^{(m)})_{\varphi(\tau, y)} \\
&\quad - [u_0^{(m)}, u^{(m)}]_{\varphi(\tau, y)} \partial_x \varphi(\tau, y) \circ (\overline{y} - y)]\, d\tau, \\
\Delta_i^{(4)} &= \partial_x \psi_i(0, g_i(0, t, x)) \circ [g_i(0, \overline{t}, \overline{x}) - g_i(0, t, x) - h] \\
&\quad + \int_0^t \{\partial_x F_i(P_i[z^{(m)}](\tau, t, x)) \circ (\overline{y} - y - h) \\
&\quad + u_{[i]}^{(m+1)}(\tau, y) \partial_x f_{[i]}(\tau, y) \circ h \\
&\quad + \partial_w F_i(P_i[z^{(m)}](\tau, t, x))[u_0^{(m)}, u^{(m)}]_{\varphi(\tau, y)} \partial_x \varphi(\tau, y) \\
&\quad \circ (\overline{y} - y - h)\}\, d\tau \\
&\quad + \int_t^{\overline{t}} F_i(P_i[z^{(m)}](\tau, \overline{t}, \overline{x}))\, d\tau - u_{0i}^{(m+1)}(t, x) h_0,
\end{aligned}$$

and

$$y = g_i(\tau, t, x), \quad \overline{y} = g_i(\tau, \overline{t}, \overline{x}).$$

We transform the last expression $\Delta_i^{(4)}$. The relation

$$\begin{aligned}
\overline{y} - y - h &= \int_\tau^t [f_{[i]}(\xi, g_i(\xi, t, x)) - f_{[i]}(\xi, g_i(\xi, \overline{t}, \overline{x}))]\, d\xi \\
&\quad - \int_t^{\overline{t}} f_{[i]}(\xi, g_i(\xi, \overline{t}, \overline{x}))\, d\xi,
\end{aligned}$$

implies that

$$\begin{aligned}\Delta_i^{(4)} = \int_0^t &\{u_{[i]}^{(m+1)}(\tau, y)\partial_x f_{[i]}(\tau, y) \circ (\overline{y} - y) \\ &+ [f_{[i]}(\tau, y) - f_{[i]}(\tau, \overline{y})] \circ V_i(\tau)\}\, d\tau \\ &- \int_t^{\overline{t}} \{f_{[i]}(\tau, \overline{y}) \circ V_i(t) + F_i(P_i[z^{(m)}](\tau, \overline{t}, \overline{x}))\}\, d\tau \\ &- u_{0i}^{(m+1)}(t, x)h_0,\end{aligned}$$

where

$$\begin{aligned}V_i(\tau) = {} & \partial_x\psi_i(0, g_i(0, t, x)) \\ &+ \int_0^\tau \{\partial_x F_i(P_i[z^{(m)}](\xi, t, x)) \\ &- u_{[i]}^{(m+1)}(\xi, g_i(\xi, t, x))\partial_x f_{[i]}(\xi, g_i(\xi, t, x)) \\ &+ \partial_w F_i(P_i[z^{(m)}](\xi, t, x))[u_0^{(m)}, u^{(m)}]_{\varphi(\xi, g_i(\xi, t, x))}\partial_x\varphi(\xi, g_i(\xi, t, x))\}\, d\xi.\end{aligned}$$

The characteristics satisfy the following relations $g_i(\tau, \xi, g_i(\xi, t, x)) = g_i(\tau, t, x)$, $(t, x) \in E$, $\xi, \tau \in [0, a]$. Thus, we get

$$\begin{aligned}\Delta_i^{(4)} = \int_0^t & u_{[i]}^{(m+1)}(\tau, y) \circ [f_{[i]}(\tau, y) - f_{[i]}(\tau, g_i(\tau, t, \overline{x})) \\ &- (g_i(\tau, t, x) - g_i(\tau, t, \overline{x}))\partial_x f_{[i]}(\tau, g_i(\tau, t, x))]\, d\tau \\ &+ \int_t^{\overline{t}} [f_{[i]}(t, x) - f_{[i]}(\tau, \overline{y})] \circ u_{[i]}^{(m+1)}(t, x)\, d\tau \\ &+ \int_t^{\overline{t}} [F_i(P_i[z^{(m)}](\tau, \overline{t}, \overline{x})) - F_i(t, x, (z^{(m)})_{\varphi(t,x)})]\, d\tau.\end{aligned}$$

Now it follows easily from Assumptions $H_\psi[f, F]$ and $H[\varphi]$ and from Lemma 1 that there is an $\varepsilon \in \mathcal{M}$ such that condition (18) is satisfied.

In view of (18), the derivatives $\partial_t z^{(m+1)}$, $\partial_x z_i^{(m+1)}$, $1 \le i \le k$, exist on E and $\partial_t z^{(m+1)} = u_0^{(m+1)}$, $\partial_x z_i^{(m+1)} = u_{[i]}^{(m+1)}$, $1 \le i \le k$. The proof of Lemma 3 is complete. $\square$

Remark 3. If we assume that $\varphi_0 : E \to R$ does not depend on x, then we do not need to consider the sequence $\{u_0^{(m)}\}$, and we can omit the proof of existence $\partial_t z^{(m)}$.

4. Global existence of the solution. Now we formulate the theorem on the global existence of a classical solution $\mathcal{Z}[\psi] : E_a \to R^k$ of the problem (2), (3) with the initial function $\psi \in \mathcal{J}$.

Theorem 1. *Suppose that $\psi \in \mathcal{J}$ and Assumptions $H_\psi[f, F]$ and $H[\varphi]$ are satisfied. Then there exists on E_a a unique classical solution $\mathcal{Z}[\psi]$ of problems* (2), (3). *Moreover, there is an $\varepsilon_\psi > 0$ such that, for each $\overline{\psi} \in \mathcal{J}$ satisfying the condition $\|\overline{\psi} - \psi\|^* < \varepsilon_\psi$, we have*

$$\|\mathcal{Z}[\psi] - \mathcal{Z}[\overline{\psi}]\|_{t.k} \le L\|\psi - \overline{\psi}\|^*, \quad t \in [0, a], \tag{19}$$

with some constant $L \in R_+$.

Proof. We prove the uniform convergence of the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$. Let the scalar sequences $\{Z_m\}$ and $\{U_m\}$ be given by

$$Z_m(t) = \|z^{(m)} - z^{(m-1)}\|_{t.k} \quad \text{and} \quad U_m(t) = \|u^{(m)} - u^{(m-1)}\|_{t.k\times n},$$

where $t \in [0, a]$, $m \ge 1$. Put

$$[|Z_m|] = \max\{Z_m(t)e^{-2AK_1 t} : 0 \le t \le a\}, \quad m \ge 1.$$

Since

$$Z_{m+1}(t) \le AK_1 \int_0^t Z_m(\tau)\, d\tau, \quad t \in [0, a],\ m \ge 1,$$

we have

$$[|Z_{m+1}|] \le \frac{1}{2}[|Z_m|], \quad m \ge 1.$$

The above recursive inequality gives the estimate

$$[|Z_m|] \le \frac{1}{2^m} c_0, \quad m \ge 1, \tag{20}$$

where $c_0 \in R_+$ is such that $[|Z_1|] \le c_0$. Therefore, $\lim_{m\to\infty} Z_m(t) = 0$ uniformly on $[0, a]$, and there is a $\widetilde{z} \in \mathcal{C}_\psi[\gamma]$ such that

$$\widetilde{z}(t, x) = \lim_{m\to\infty} z^{(m)}(t, x) \quad \text{uniformly on } E.$$

Moreover, $\widetilde{z}$ satisfies the integral functional equation $z = \mathcal{F}[z]$.

Now we prove that $\lim_{m\to\infty} U_m(t) = 0$ uniformly on $[0,a]$. It follows that

$$u_{[i]}^{(m+1)}(t,x) - u_{[i]}^{(m)}(t,x) = H_i^{(m)}(t,x) - H_i^{(m-1)}(t,x) \\ - \int_0^t (u_{[i]}^{(m+1)} - u_{[i]}^{(m)})(\tau, g_i(\tau,t,x))\partial_x f_{[i]}(\tau, g_i(\tau,t,x))\, d\tau \\ \text{on } E, \quad m \ge 1,\ 1 \le i \le k.$$

In view of definition (17), we have

$$\|u_0^{(m)} - u_0^{(m-1)}\|_{t.k} \le AK_1 Z_{m-1}(t) + \widetilde{A} U_m(t), \quad m \ge 1.$$

There is a $C_1 \in R_+$ such that

$$\|u_{[i]}^{(m+1)}(t,x) - u_{[i]}^{(m)}(t,x)\| \le C_1[\beta(c_m) + c_{m-1} + \int_0^t U_m(\tau)\, d\tau] \\ + A\int_0^t U_{m+1}(\tau)\, d\tau,$$

where $m \ge 1$ and $c_j = K_1 Z_j(a)$, $j \ge 0$. We conclude from the Gronwall inequality that there are $C, C_0 \in R_+$ such that

$$U_{m+1}(t) \le C\int_0^t U_m(\tau)\, d\tau + C_0(\beta(c_m) + c_{m-1}), \quad t \in [0,a],\ m \ge 1.$$

Write

$$[|U_m|] = \max\{U_m(t)e^{-2Ct} : 0 \le t \le a\}, \quad m \ge 1.$$

Consequently,

$$[|U_{m+1}|] \le \frac{1}{2}[|U_m|] + C_0(\beta(c_m) + c_{m-1}), \quad m \ge 1.$$

Thus, for $m \ge 1$, we obtain

$$[|U_m|] \le S_m \quad \text{where} \quad S_m = \frac{1}{2^{m-1}}[|U_1|] + C_0 \sum_{j=1}^{m-1} \frac{1}{2^{m-1-j}}(\beta(c_j) + c_{j-1}).$$

Estimate (20) and assumption (9) imply that

$$\sum_{j=1}^{\infty}(\beta(c_j)+c_{j-1})<+\infty.$$

Then we get

$$\sum_{j=1}^{\infty}[|U_j|]\leq\sum_{j=1}^{\infty}S_j<+\infty.$$

Finally, the sequence $\{U_m\}$ uniformly tends to zero and there is a $\widetilde{u}_{[i]}\in\mathcal{C}_{\partial_x\psi_i}(\mu)$ such that

$$\widetilde{u}_{[i]}(t,x)=\lim_{m\to\infty}u_{[i]}^{(m)}(t,x)\quad\text{uniformly on } E,\quad 1\leq i\leq k.$$

It follows from Lemma 3 that $\partial_x\widetilde{z}_i$, $1\leq i\leq k$, exist on E and $\partial_x\widetilde{z}_i=\widetilde{u}_{[i]}$.

We prove that $\widetilde{z}$ is the solution of (2), (3). Let $(t,x)\in E$ and $y=g_i(0,t,x)$, $1\leq i\leq k$. Then $x=g_i(t,0,y)$, and generally $g_i(\tau,t,x)=g_i(\tau,0,y)$ for $\tau\in[0,t]$. Thus,

$$\widetilde{z}(t,g_i(t,0,y))=\psi_i(0,y)+\int_0^t F_i(\tau,g_i(\tau,0,y),\widetilde{z}_{\varphi(\tau,g_i(\tau,0,y))})\,d\tau,\ 1\leq i\leq k.$$

By differentiating the above relations with respect to t and taking x instead of $g_i(t,0,y)$, we obtain that $\widetilde{z}$ satisfies (2) on E.

Assume now that $\varepsilon_\psi>0$ is such that for $\overline{\psi}\in\mathcal{J}$ and $\|\overline{\psi}-\psi\|^*<\varepsilon_\psi$, the maximal solution $\overline{\gamma}$ of the problem

$$\omega'(t)=K_1\sigma(\,t,\omega(t)\,),\qquad \omega(0)=(K_0+\lambda K_1)\|\overline{\psi}\|^*,$$

satisfies the condition $\overline{\gamma}(a)<\eta_\psi$. If $z=\mathcal{Z}[\psi]$ and $\overline{z}=\mathcal{Z}[\overline{\psi}]$, then

$$\|z-\overline{z}\|_{t.k}\leq(\lambda+AK_0)\|\psi-\overline{\psi}\|^*+AK_1\int_0^t\|z-\overline{z}\|_{\tau.k}d\tau,\quad t\in[0,a].$$

By using the Gronwall inequality we get (19) with $L=(\lambda+AK_0)e^{a\tilde{A}}$. This finishes the proof. $\square$

5. Differentiability of the solution with respect to initial functions. It is our aim to prove that the Fréchet derivative $\partial\mathcal{Z}[\psi]$ exists at the point $\psi \in \mathcal{J}$. Suppose that $\chi \in \mathcal{J}$. For $v \in \mathcal{C}_\chi$, we define $\Lambda[v] = (\Lambda_1[v], \dots, \Lambda_k[v])$ in the following way:

$$\begin{aligned}\Lambda[v](t,x) &= \chi(t,x) \quad \text{on } E_0,\\ \Lambda_i[v](t,x) &= \chi_i(0, g_i(0,t,x)\\ &\qquad + \int_0^t \partial_w F_i(\tau, g_i(\tau,t,x), \mathcal{Z}[\psi]_{\varphi(\tau,g_i(\tau,t,x))}) v_{\varphi(\tau,g_i(\tau,t,x))}\, d\tau\end{aligned}$$

on E, $1 \le i \le k$.

Lemma 4. *If $\psi \in \mathcal{J}$ and Assumptions $H_\psi[f,F]$ and $H[\varphi]$ are satisfied, then the integral functional equation*

$$v = \Lambda[v] \tag{21}$$

has exactly one solution $v^* \in \mathcal{C}_\chi[\vartheta]$ where $\vartheta(t) = (K_0 + \lambda K_1)\|\chi\|^* e^{K_1 A t}$, $t \in [0,a]$.

Proof. The operator Λ maps the set $\mathcal{C}_\chi[\vartheta]$ into itself. For $v \in \mathcal{C}_\chi[\vartheta]$, we put

$$[|v|] = \sup\{\|v_{(t,x)}\|_X \cdot e^{-2K_1 A t} : (t,x) \in E\}.$$

Then

$$[|\Lambda[v] - \Lambda[\overline{v}]|] \le \frac{1}{2}[|v - \overline{v}|]$$

where $v, \overline{v} \in \mathcal{C}_\chi[\vartheta]$. Thus, there exists exactly one fixed point $v^* \in \mathcal{C}_\chi[\vartheta]$ of equation (21).

Now we are ready to prove the main theorem of this paper.

Theorem 2. *If $\psi \in \mathcal{J}$ and Assumptions $H_\psi[f,F]$ and $H[\varphi]$ are satisfied, then the Fréchet derivative $\partial\mathcal{Z}[\psi]$ exists, and $v^* = \partial\mathcal{Z}[\psi]\chi$ with $\chi \in \mathcal{J}$, $\chi = (\chi_1, \dots, \chi_k)$, is the solution of equation* (21).

Proof. Let us fix $\psi \in \mathcal{J}$, and let $\varepsilon_\psi > 0$ be given as in Theorem 1. Assume that $\chi \in \mathcal{J}$ and $s \in R$ is such that $s \ne 0$, $\|s\chi\|^* < \varepsilon_\psi$.

Then there exist on E_a the solutions $\mathcal{Z}[\psi]$ and $\mathcal{Z}[\psi + s\chi]$. We define $\delta_s : E_a \to R^k$, $\delta_s = (\delta_{s.1}, \dots, \delta_{s.k})$, as follows:

$$\delta_s(t,x) = \chi(t,x) \quad \text{on } E_0,$$

$$\delta_s(t,x) = \frac{1}{s}(\mathcal{Z}[\psi + s\chi](t,x) - \mathcal{Z}[\psi](t,x)) \quad \text{on } E.$$

The function δ_s satisfies the following integral functional equation

$$v = \Gamma_s[v]$$

where $\Gamma_s[v] = (\Gamma_{s.1}[v], \dots, \Gamma_{s.k}[v])$, $v \in \mathcal{C}_\chi$, is given by formulas

$$\Gamma_s[v](t,x) = \chi(t,x) \quad \text{on } E_0,$$

$$\begin{aligned}\Gamma_{s.i}[v](t,x) &= \chi_i(0, g_i(0,t,x)) \\ &\quad + \int_0^t \int_0^1 \partial_w F_i(Q_{s.i}[\psi,\chi](\xi,\tau,t,x)) v_{\varphi(\tau, g_i(\tau,t,x))}\, d\xi\, d\tau \\ &\qquad\qquad\qquad \text{on } E,\ 1 \le i \le k,\end{aligned}$$

and

$$\begin{aligned}&Q_{s.i}[\psi,\chi](\xi,\tau,t,x) \\ &\quad = (\tau, g_i(\tau,t,x), (1-\xi)\mathcal{Z}[\psi]_{\varphi(\tau,g_i(\tau,t,x))} + \xi\mathcal{Z}[\psi + s\chi]_{\varphi(\tau,g_i(\tau,t,x))}).\end{aligned}$$

We conclude from Lemma 4 that exactly one solution $v^* \in \mathcal{C}_\chi$ of equation (21) exists. Then the function $r_s : E_a \to R^k$, $r_s = (r_{s.1}, \dots, r_{s.k})$, defined by $r_s = v^* - \delta_s$ satisfies the relations

$$r_s(t,x) = 0 \quad \text{on } E_0,$$

$$\begin{aligned}r_{s.i}(t,x) = \int_0^t \int_0^1 &[\partial_w F_i(\tau, g_i(\tau,t,x), \mathcal{Z}[\psi]_{\varphi(\tau,g_i(\tau,t,x))}) \\ &- \partial_w F_i(Q_{s.i}[\psi,\chi](\xi,\tau,t,x))](v^*)_{\varphi(\tau,g_i(\tau,t,x))}\, d\xi\, d\tau \\ &- \int_0^t \int_0^1 \partial_w F_i(Q_{s.i}[\psi,\chi](\xi,\tau,t,x))(r_s)_{\varphi(\tau,g_i(\tau,t,x))}\, d\tau \\ \text{on } E,& \quad 1 \le i \le k.\end{aligned}$$

Thus, there is a $c^* \in R_+$ such that

$$\|r_s\|_{t.k} \le c^* \beta(\, K_1 \,\|\mathcal{Z}[\psi + s\chi] - \mathcal{Z}[\psi]\,\|_{t.k} + K_0 \,\|s\chi\|^* \,) \\ + AK_1 \int_0^t \|r_s\|_{\tau.k} \, d\tau, \quad t \in [0, a].$$

It follows from the Gronwall inequality and from assertion (19) of Theorem 1 that

$$\|r_s\|_{t.k} \le e^{AK_1 a} c^* \beta(|s|(K_1 L + K_0)\|\chi\|^*), \quad t \in [0, a].$$

Therefore,

$$\lim_{s \to 0} \delta_s(t, x) = v^*(t, x) \quad \text{uniformly on } E,$$

and the proof is complete. □

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