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MULTIPLIER SEQUENCES FOR GENERALIZED LAGUERRE BASES

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ABSTRACT. In this paper we present a complete characterization of geometric and linear $L^{(\alpha)}$-multiplier sequences. We also prove that the falling factorial sequence is an $L^{(\alpha)}$-multiplier sequence and show that the set of $L^{(\alpha)}$-multiplier sequences is a proper subset of the set of Hermite-multiplier sequences.

1. Introduction. A set of polynomials $Q = \{q_k\}_{k=0}^{\infty}$ is called *simple* if $\deg q_k = k$ for all k. As such, every simple set of polynomials forms a basis for the polynomial ring $\mathbf{R}[x]$. Given a simple set of polynomials $Q = \{q_k\}_{k=0}^{\infty}$ and a sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$, one can define a linear operator T on $\mathbf{R}[x]$ by declaring $T[q_k(x)] = \gamma_k q_k(x)$ for all k. We call $\{\gamma_k\}_{k=0}^{\infty}$ a *Q-multiplier sequence* if $T[p]$ has only real zeros whenever p has only real zeros. In the case where Q is the standard basis, the terminology 'multiplier sequence' or 'classical multiplier sequence' is used without a reference to Q.

As the terminology indicates, whether a sequence of real numbers is a Q-multiplier sequence depends on the choice of Q. It is known that every Q-multiplier sequence is a classical multiplier sequence if Q is any simple set of polynomials (see Section 2). Given two simple sets of polynomials, in general, it is difficult to decide whether multiplier sequences for one are also multiplier sequences for the other. In this paper, we focus our attention on the simple set of generalized Laguerre polynomials and the corresponding multiplier sequences.

Definition 1.1. The nth generalized Laguerre polynomial is defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \quad (\alpha > -1;\ n = 0, 1, 2, \dots).$$

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We shall use the terminology $L^{(\alpha)}$*-multiplier sequence* instead of 'Laguerre multiplier sequence,' as the latter phrase has been used by other authors with a different meaning (see, for example, [**3**, page 153]).

This paper contains three main results. Theorem 3.5 gives a complete characterization of $L^{(\alpha)}$-multiplier sequences of the form $\{k+a\}_{k=0}^{\infty}$ where $a \in \mathbf{R}$. In Theorem 4.4, we prove that the falling factorial sequence is an $L^{(\alpha)}$-multiplier sequence. Finally, using Theorem 5.6, we show that the set of $L^{(\alpha)}$-multiplier sequences is a proper subset of the set of multiplier sequences for the Hermite polynomials as defined in Section 2.

2. Some classical and newer results. Examples of classical multiplier sequences were demonstrated in the late 1800s by Jensen and Laguerre and, in the early 1900s, all such sequences were completely characterized by Pólya and Schur.

Theorem 2.1 [**9**]. *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of non-negative real numbers. The following are equivalent*:

(1) $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence.*

(2) *For each* n, *the polynomial*

$$T[(1+x)^n] := \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$$

has only real non-positive zeros.

(3) *The series*

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$$

converges in the whole plane, and $\varphi(z)$ *is of the form*

$$c e^{\sigma z} z^m \prod_{k=1}^{\omega} (1 + z_k z)$$

where $c \in \mathbf{R}$, $\sigma \geq 0$, m *is a non-negative integer*, $0 \leq \omega \leq \infty$, $z_k > 0$, *and*

$$\sum_{k=1}^{\omega} z_k < \infty.$$

Definition 2.2. The nth Hermite polynomial is defined by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2} \quad (n = 0, 1, 2, \dots).$$

The set $\{H_n(x)\}_{n=0}^{\infty}$ is a simple set of polynomials, and hence the question whether or not nonconstant Hermite multiplier sequences exist is a meaningful one. Examples of such sequences were demonstrated in the mid 1900s by Turán [**11**] and also in 2001 by Bleecker and Csordas [**1**]. Piotrowski completely characterized all such sequences.

Theorem 2.3 [**8**, Theorem 152]. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers. The following are equivalent.*

(1) $\{\gamma_k\}_{k=0}^{\infty}$ *is a non-trivial Hermite multiplier sequence.*

(2) $\{\gamma_k\}_{k=0}^{\infty}$ *is a non-decreasing multiplier sequence.*

(3) *The series*

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$$

converges in the whole plane and either $\varphi(z)$ or $\varphi(-z)$ is of the form

$$ce^{\sigma z} z^m \prod_{k=1}^{\omega} \left(1 + \frac{z}{z_k}\right)$$

where $c \in \mathbf{R}$, $\sigma \geq 1$, m is a non-negative integer, $0 \leq \omega \leq \infty$, $z_k > 0$, and

$$\sum_{k=1}^{\omega} \frac{1}{z^k} < \infty.$$

Remarkably, every Q-multiplier sequence must be a (classical) multiplier sequence, regardless of the choice of the simple set Q. This result guarantees that every $L^{(\alpha)}$-multiplier sequence must also be a multiplier sequence.

Theorem 2.4 [**8**, Theorem 158]. *Let $Q = \{q_k\}_{k=0}^{\infty}$ be a simple set of polynomials. If the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a Q-multiplier sequence, then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence.*

In general, we will say that an operator T *preserves reality of zeros* if it has the property that $T[p]$ has only real zeros, whenever p has only real zeros. Thus, a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if its corresponding operator preserves reality of zeros. In 2009, as part of their characterization of stability preserving linear operators, Borcea and Brändén completely characterized linear operators which preserve reality of zeros.

Theorem 2.5 [**2**, Theorem 5]. *A linear operator $T : \mathbf{R}[x] \to \mathbf{R}[x]$ preserves reality of zeros if and only if either*:

(1) *T has range of dimension at most two and is of the form $T[f] = \alpha(f)P + \beta(f)Q$, where α and β are linear functionals on $\mathbf{R}[x]$ and P and Q are polynomials with only real interlacing zeros, or*

(2)

$$T[\exp(-xw)] = \sum_{n=0}^{\infty} \frac{(-w)^n T[x^n]}{n!} \in \overline{A},$$

or

(3)

$$T[\exp(xw)] = \sum_{n=0}^{\infty} \frac{w^n T[x^n]}{n!} \in \overline{A},$$

where $\overline{A}$ denotes the set of entire functions in 2 variables which are uniform limits on compact subsets of polynomials in the set

$$A = \{f \in \mathbf{R}[x, w] \mid f(x, w) \neq 0 \textit{ whenever } \operatorname{Im} x > 0 \textit{ and } \operatorname{Im} w > 0\}.$$

With this characterization at hand, the crux of our problem is to find necessary and sufficient conditions on a sequence of real numbers under which the corresponding operator T satisfies one of the conditions (1)–(3) above.

To avoid trivialities, in this paper we consider the identically zero function $f \equiv 0$ to have only real zeros, although this is clearly not the case.

3. Trivial, geometric, and linear sequences. It is straightforward to see that any sequence of the form

$$\{0, 0, \dots, 0, a, b, 0, 0, \dots\} \quad (a, b \in \mathbf{R})$$

is a classical multiplier sequence. Sequences of this form are also Hermite multiplier sequences. In order to demonstrate that sequences of this form are $L^{(\alpha)}$-multiplier sequences, the properties of orthogonal polynomials will be employed.

Lemma 3.1. *Let $\{q_k\}_{k=0}^{\infty}$ be a sequence of orthogonal polynomials. Then, for any $a, b \in \mathbf{R}$ and any integer $n \geq 0$, the polynomial*

$$p(x) = aq_n(x) + bq_{n+1}(x)$$

has only real zeros.

Proof. A sequence of orthogonal polynomials forms a simple set. Thus, p is a polynomial of degree $n+1$. Orthogonal polynomials have only simple real zeros and the zeros of consecutive polynomials are interlacing. Let us denote the zeros of q_n by $x_1 < x_2 < \cdots < x_n$ and those of q_{n+1} by $y_1 < y_2 < \cdots < y_{n+1}$, so that

$$y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n < y_{n+1}.$$

It follows that the sequence

$$\{p(y_1), p(y_2), \dots, p(y_{n+1})\} = \{aq_n(y_1), aq_n(y_2), \dots, aq_n(y_{n+1})\}$$

alternates in sign. By the intermediate value theorem, p must have at least n real zeros. Since $\deg(p) = n+1$ and the non-real zeros of real polynomials come in conjugate pairs, p must have only real zeros. $\square$

It is well known that the generalized Laguerre polynomials form an orthogonal set over the positive real axis with respect to the weight function $x^\alpha e^{-x}$ (recall that in this paper we are only considering $\alpha > -1$). Combining this fact with the preceding lemma, we have proved the following.

Proposition 3.2. *Let $\{\gamma_k\}_{k=0}^\infty$ be a sequence of real numbers. If there exists a natural number n such that $\gamma_k = 0$ for all $k < n$ and all $k > n+1$, then $\{\gamma_k\}_{k=0}^\infty$ is an $L^{(\alpha)}$-multiplier sequence.*

Remark 3.3. We will call sequences of the above form *trivial $L^{(\alpha)}$-multiplier sequences.* Unless stated otherwise, in what follows, we only consider nontrivial $L^{(\alpha)}$-multiplier sequences.

3.1. Geometric $L^{(\alpha)}$-multiplier sequences. We now consider the geometric sequences $\{r^k\}_{k=0}^\infty$, where $r \in \mathbf{R}$. These sequences are classical multiplier sequences for all nonzero r and are Hermite multiplier sequences if and only if $|r| \geq 1$. In contrast to these results, the only geometric sequence which is an $L^{(\alpha)}$-multiplier sequence is the constant sequence $\{1\}_{k=0}^\infty$.

Proposition 3.4. *The sequence $\{r^k\}_{k=0}^\infty$ is an $L^{(\alpha)}$-multiplier sequence if and only if $r = 1$.*

Proof. Consider the polynomial $p(x) = (x+b)^2$, where $b \in \mathbf{R}$. We can write $p(x)$ as

$$p(x) = 2L_2^{(\alpha)}(x) - 2(\alpha+2+b)L_1^{(\alpha)}(x) + ((\alpha+b)^2 + 3\alpha + 2b + 2)L_0^{(\alpha)}(x).$$

Applying the operator T corresponding to the sequence $\{r^k\}_{k=0}^\infty$, then expanding in terms of the standard basis, we obtain the polynomial

$$\begin{aligned} T[p(x)] = {} & r^2x^2 + (2(\alpha+2+b)r - (2\alpha+4)r^2)x \\ & + 2 + \alpha^2 + 2b + b^2 + \alpha(3+2b) \\ & - 2(2+\alpha+b)(1+\alpha)r + r^2(2+3\alpha+\alpha^2) \end{aligned}$$

with discriminant

$$\Delta = -4r^2(r-1)((2+\alpha)(1-r) + 2b).$$

From this representation, we immediately conclude the following.

(i) If $r = 1$, the discriminant is equal to zero.

(ii) If $r > 1$, large positive values of b result in a negative discriminant.

(iii) If $r < 1$ large negative values of b result in a negative discriminant,

the result follows. □

3.2. Linear $L^{(\alpha)}$-multiplier sequences. In [**8**], Piotrowski shows that the sequence $\{k + a\}_{k=0}^{\infty}$ is not an $L^{(0)}$-multiplier sequence for $a > 1$ or $a < 0$, but that it is an $L^{(0)}$-multiplier sequence if $a = 1$ or $a = 0$. The question whether $\{k+a\}_{k=0}^{\infty}$ is an $L^{(0)}$-multiplier sequence for $0 < a < 1$ is left open. The next theorem settles this question and completely characterizes all linear $L^{(\alpha)}$-multiplier sequences.

Theorem 3.5. *The sequence $\{k + a\}_{k=0}^{\infty}$ is an $L^{(\alpha)}$-multiplier sequence if and only if $0 \le a \le \alpha + 1$.*

Proof. We begin by noting that the polynomials $L_n^{(\alpha)}(x)$ satisfy the following ordinary differential equation (cf. [**10**, page 204]):

$$nL_n^{(\alpha)}(x) = (x - \alpha - 1)L_n^{(\alpha)'}(x) - xL_n^{(\alpha)''}(x). \tag{1}$$

It follows that

$$(a + k)L_k^{(\alpha)}(x) = aL_k^{(\alpha)}(x) + (x - \alpha - 1)L_k^{(\alpha)'}(x) - xL_k^{(\alpha)''}(x).$$

Thus, the action of the sequence $\{k + a\}_{k=0}^{\infty}$ on a polynomial is represented by the operator

$$T := a + (x - \alpha - 1)D - xD^2. \tag{2}$$

Assume $0 \le a \le \alpha + 1$. By the result of Borcea and Brändén (Theorem 2.5), the operator T preserves reality of zeros, provided the polynomial

$$a + (z - \alpha - 1)(-w) - z(-w)^2 = a - w(w + 1)z + w(\alpha + 1)$$

does not vanish whenever $\operatorname{Im} z > 0$ and $\operatorname{Im} w > 0$. Setting the above equation equal to zero and solving for z we obtain

$$z = \frac{w(\alpha+1)+a}{w(w+1)} = (\alpha+1)\frac{w+w_0}{w(w+1)} \quad \left(w_0 = \frac{a}{\alpha+1}\right).$$

Suppose $\operatorname{Im} w > 0$. Since $0 \leq w_0 \leq 1$, we have

$$0 < \arg(w+1) \leq \arg(w+w_0) \leq \arg(w) < \pi,$$

from which we obtain

$$-\pi < -\arg(w) \leq \arg(w+w_0) - \arg(w) - \arg(w+1) \leq -\arg(w+1) < 0.$$

Thus, $\operatorname{Im} z < 0$ whenever $\operatorname{Im} w > 0$, whence $\{k+a\}_{k=0}^{\infty}$ is an $L^{(\alpha)}$-multiplier sequence.

For the converse, suppose that $\{k+a\}_{k=0}^{\infty}$ is an $L^{(\alpha)}$-multiplier sequence. The generalized Laguerre polynomials form a simple set of polynomials. Consequently, by Theorem 2.5, any sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ that is an $L^{(\alpha)}$-multiplier sequence is also a classical multiplier sequence. Since, for $a < 0$, the sequence $\{k+a\}_{k=0}^{\infty}$ is not a classical multiplier sequence, we must have $0 \leq a$. To get the upper bound on a, we note that

$$\begin{aligned} T[(x+n)^n] &= a(x+n)^n + (x-\alpha-1)n(x+n)^{n-1} - xn(n-1)(x+n)^{n-2} \\ &= (x+n)^{n-2}[a(x+n)^2 + (x-(\alpha+1))n(x+n) - x(n^2-n)] \\ &= (x+n)^{n-2}[x^2(a+n) + x(2an-n\alpha) + an^2 - n^2(\alpha+1)]. \end{aligned}$$

Calculating the discriminant of the polynomial in the square brackets, we obtain

$$\begin{aligned} \Delta(n) &= n^2[4a^2 - 4a\alpha + \alpha^2 - 4(a+n)(a-(\alpha+1))] \\ &= n^2[\alpha^2 + 4a - 4n(a-(\alpha+1))]. \end{aligned}$$

If $a > (\alpha+1)$, then $\Delta(n) < 0$ for n sufficiently large. Therefore, $T[(x+n)^n]$ will have non-real zeros for large enough n. The proof of Theorem 3.5 is complete. $\square$

4. The falling factorial sequence. The purpose of this section is to prove that, for any natural number n, the falling factorial sequence

$$\left\{\frac{\Gamma(k+1)}{\Gamma(k-n+1)}\right\}_{k=0}^{\infty} = \{k(k-1)\cdots(k-n+1)\}_{k=0}^{\infty}$$

is an $L^{(\alpha)}$-multiplier sequence. To establish this claim, we need several auxiliary results.

Lemma 4.1. *Let δ be the operator defined by $\delta := (x-(\alpha+1))D - xD^2$. Then, for $k \geq 0$, we have*

$$[\delta, D^k] := \delta D^k - D^k\delta = -k(1-D)D^k.$$

Proof. If $k=0$, the result is trivial. Supposing the result holds for all integers up to k, we calculate

$$\begin{aligned}[\delta, D^{k+1}] &= \delta D^{k+1} - D^{k+1}\delta = (\delta D^k)D - D(D^k\delta) \\ &= (\delta D^k)D - D(\delta D^k + k(1-D)D^k) \\ &= \delta D^{k+1} - (\delta D + (1-D)D)D^k - k(1-D)D^{k+1} \\ &= -(k+1)(1-D)D^{k+1},\end{aligned}$$

establishing the desired equality. ∎

Proposition 4.2. *Let δ be the operator defined by $\delta := (x-(\alpha+1))D-xD^2$, and let $L_n^{(\alpha)}(x)$ be the nth generalized Laguerre polynomial. If*

$$\delta(\delta-1)(\delta-2)\cdots(\delta-(n-1)) = \sum_{k=n}^{2n} {}_{2n}q_{k,\alpha}(x)D^k, \tag{3}$$

then

$$\sum_{k=n}^{2n} {}_{2n}q_{k,\alpha}(x)z^k = n!(-1)^n z^n L_n^{(\alpha)}(x-xz). \tag{4}$$

Proof. We proceed by induction on n. If $n=1$, the left hand side of (3) is just $(x-(\alpha+1))D - xD^2$ which, after replacing D^k by z^k, gives $(x-(\alpha+1))z - xz^2 = -zL_1^{(\alpha)}(x-xz)$. Thus, the statement of the proposition holds in the case when $n = 1$. Next we calculate

$$\begin{aligned}\sum_{k=n}^{2n} {}_{2n}q_{k,\alpha}(x)D^k &= \delta(\delta-1)(\delta-2)\cdots(\delta-(n-1)) \\ &= \left(\sum_{k=n}^{2n-2} {}_{2n-2}q_{k,\alpha}(x)D^k\right)(\delta-(n-1)) \\ &= \sum_{k=n}^{2n-2} {}_{2n-2}q_{k,\alpha}(x)(x-(\alpha+1))D^{k+1} \\ &\quad + \sum_{k=n}^{2n-2} {}_{2n-2}q_{k,\alpha}(x)kD^k \\ &\quad - \sum_{k=n}^{2n-2} {}_{2n-2}q_{k,\alpha}(x)kD^{k+1} \\ &\quad - x\sum_{k=n}^{2n-2} {}_{2n-2}q_{k,\alpha}(x)D^{k+2} \\ &\quad - (n-1)\sum_{k=n}^{2n-2} {}_{2n-2}q_{k,\alpha}(x)D^k.\end{aligned}$$

The third equality in this calculation follows from Lemma 4.1. Replacing D^k by z^k in this expression, along with the inductive hypothesis, gives

$$\begin{aligned}&\sum_{k=n}^{2n} {}_{2n}q_{k,\alpha}(x)z^k \\ &\quad = z(x-(\alpha+1))(n-1)!(-1)^{n-1}z^{n-1}L_{n-1}^{(\alpha)}(x-xz) \\ &\qquad + (z-z^2)D_z\left[(n-1)!(-1)^{n-1}z^{n-1}L_{n-1}^{(\alpha)}(x-xz)\right] \\ &\qquad - xz^2(n-1)!(-1)^{n-1}z^{n-1}L_{n-1}^{(\alpha)}(x-xz) \\ &\qquad - (n-1)(n-1)!(-1)^{n-1}z^{n-1}L_{n-1}^{(\alpha)}(x-xz) \\ &\quad = (n-1)!(-1)^{n-1}z^{n-1}\Big\{z(x-(\alpha+1))L_{n-1}^{(\alpha)}(x-xz) \\ &\qquad + (1-z)\left[(n-1)L_{n-1}^{(\alpha)}(x-xz) - zx\frac{d}{dw}\left[L_{n-1}^{(\alpha)}(w)\right]_{w=x-xz}\right]\end{aligned}$$

$$
\begin{aligned}
&- xz^2 L_{n-1}^{(\alpha)}(x - xz) - (n-1)L_{n-1}^{(\alpha)}(x - xz)\Big\} \\
&= (n-1)!(-1)^{n-1}z^{n-1}\Big\{ - z(\alpha + n)L_{n-1}^{(\alpha)}(x - xz) \\
&\quad + z(x - xz)L_{n-1}^{(\alpha)}(x - xz) - z(x - xz)\frac{d}{dw}\left[L_{n-1}^{(\alpha)}(w)\right]_{w=x-xz}\Big\}.
\end{aligned}
$$

Since the generalized Laguerre polynomials satisfy the relations

$$
\begin{aligned}
xDL_n^{(\alpha)}(x) &= nL_n^{(\alpha)}(x) - (\alpha + n)L_{n-1}^{(\alpha)}(x) \\
DL_n^{(\alpha)}(x) &= DL_{n-1}^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x)
\end{aligned}
$$

(see, for example, [**10**, Chapter 12]), it follows that

$$
\begin{aligned}
&(n-1)!(-1)^{n-1}z^{n-1}\Big\{ - z(\alpha + n)L_{n-1}^{(\alpha)}(x - xz) \\
&\quad + z(x - xz)L_{n-1}^{(\alpha)}(x - xz) - z(x - xz)\frac{d}{dw}\left[L_{n-1}^{(\alpha)}(w)\right]_{w=x-xz}\Big\} \\
&= (n-1)!(-1)^{n-1}z^{n-1}\Big\{ - znL_n^{(\alpha)}(x - xz) \\
&\quad + z(x - xz)DL_{n-1}^{(\alpha)}(x - xz) - z(x - xz)L_{n-1}^{(\alpha)}(x - xz) \\
&\quad + z(x - xz)L_{n-1}^{(\alpha)}(x - xz) - z(x - xz)DL_{n-1}^{(\alpha)}(x - xz)\Big\} \\
&= (n-1)!(-1)^{n-1}z^{n-1}(-znL_n^{(\alpha)}(x - xz)) \\
&= n!(-1)^n z^n L_n^{(\alpha)}(x - xz).
\end{aligned}
$$

The proof of Proposition 4.2 is complete. □

The following corollary provides a glimpse of the interplay between the polynomials ${}_{2n}q_{k,\alpha}(x)$: any $n-1$ of the coefficient polynomials of $\delta(\delta - 1)\cdots(\delta - (n-1))$ uniquely determine the nth.

Corollary 4.3. *Let* δ *be as in Proposition* 4.2*, and let*

$$
\delta(\delta - 1)\cdots(\delta - (n-1)) = \sum_{k=n}^{2n} {}_{2n}q_{k,\alpha}(x)D^k \quad (x \in \mathbf{R}).
$$

Then

$$\sum_{k=n}^{2n} {}_{2n}q_{k,\alpha}(x) = (-1)^n \prod_{k=1}^{n} (\alpha + k).$$

Proof. By Proposition 4.2, we have

$$\sum_{k=n}^{2n} {}_{2n}q_{k,\alpha}(x) = \sum_{k=n}^{2n} {}_{2n}q_{k,\alpha}(x) z^k \Big|_{z=1} = n!(-1)^n L_n^{(\alpha)}(0).$$

On the other hand, using the generating function (see, for example, [**10**, page 204, formula (4)])

$$\frac{1}{(1-t)^{1+\alpha}} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n \quad (|t| < 1),$$

we see that

$$n!(-1)^n L_n^{(\alpha)}(0) = (-1)^n \prod_{k=1}^{n} (\alpha + k). \qquad \square$$

Theorem 4.4. *For any natural number n, the sequence*

$$\left\{ \frac{\Gamma(k+1)}{\Gamma(k-n+1)} \right\}_{k=0}^{\infty}$$

is an $L^{(\alpha)}$-multiplier sequence.

Proof. Let T be the linear operator defined by

$$T[L_k^{(\alpha)}(x)] = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} L_k^{(\alpha)}(x) = k(k-1)\cdots(k-n+1) L_k^{(\alpha)}(x).$$

Then, from the differential equation (1), we have that $T = \delta(\delta-1)(\delta-2)\cdots(\delta-(n-1))$, where $\delta := (x-(\alpha+1))D - xD^2$ and D denotes differentiation with respect to x. Using the definition of the generalized Laguerre polynomials, we have

$$\begin{aligned} n!(-1)^n z^n L_n^{(\alpha)}(x-xz) & \\ &= n!(-1)^n z^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{(x-xz)^k}{k!} \\ &= n!(-1)^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{x^k}{k!} z^n (1-z)^k. \end{aligned}$$

Thus, by Proposition 4.2,

$$T = n!(-1)^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{x^k}{k!} D^n (1-D)^k,$$

and we have

$$\begin{aligned} &T[\exp(-xw)] \\ &\quad= n!(-1)^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{x^k}{k!} D^n (1-D)^k [\exp(-xw)] \\ &\quad= n!(-1)^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{x^k}{k!} (-w)^n (1+w)^k \exp(-xw) \\ &\quad= n!(-1)^n (-w)^n L_n^{(\alpha)}(x+xw) \exp(-xw). \end{aligned}$$

Note that the polynomial

$$f_m(x,w) = n!(-1)^n (-w)^n L_n^{(\alpha)}(x+xw) \left(1 - \frac{xw}{m}\right)^m \quad (m \in \mathbf{N})$$

converges uniformly on compact subsets to $T[\exp(-xw)]$ as $m \to \infty$. Let $0 < x_1 < x_2 < \cdots < x_n$ be the zeros of $L_n^{(\alpha)}(x)$. Since the generalized Laguerre polynomials have only real simple positive zeros, we see that $f_m(x,w) = 0$ if and only if either: $w = 0$, $x(1+w) = x_k$ or $xw = m$. None of these conditions are satisfied when $\operatorname{Im} x > 0$ and $\operatorname{Im} w > 0$. Therefore, by Theorem 2.5, T preserves reality of zeros and the proof is complete. $\square$

5. Properties of $L^{(\alpha)}$-multiplier sequences.

5.1. Classical properties. There are a number of properties of classical multiplier sequences which carry over to $L^{(\alpha)}$-multiplier sequences.

Lemma 5.1. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be an $L^{(\alpha)}$-multiplier sequence. The following statements hold*:

(i) *If there exist integers $n > m \geq 0$ such that $\gamma_m \neq 0$ and $\gamma_n = 0$, then $\gamma_k = 0$ for all $k \geq n$.*

(ii) *The terms of $\{\gamma_k\}_{k=0}^{\infty}$ are either all of the same sign, or they alternate in sign.*

(iii) *For any $r \in \mathbf{R}$, the sequence $\{r\gamma_k\}_{k=0}^{\infty}$ is also an $L^{(\alpha)}$-multiplier sequence.*

(iv) *The terms of $\{\gamma_k\}_{k=0}^{\infty}$ satisfy Turán's inequality*

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0 \quad (k = 1, 2, 3, \dots).$$

Proof. These claims follow immediately from Theorem 2.4 and the fact the generalized Laguerre polynomials form a simple set of polynomials. For classical multiplier sequences, properties (i) and (ii) have been established ([**5**, pages 341–342], property (iii) is easily verified, and property (iv) was proved by Craven and Csordas (see, for example, [**3**, Theorem 3.5]). $\square$

Remark 5.2. To draw further contrast between $L^{(\alpha)}$-multiplier sequences, Hermite multiplier sequences and classical multiplier sequences, we demonstrate that the following two properties, which hold for multiplier sequences and Hermite multiplier sequences, do *not* hold for $L^{(\alpha)}$-multiplier sequences:

(a) If $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence, then $\{\gamma_k\}_{k=m}^{\infty}$ is a multiplier sequence for any $m \in \mathbf{N}$.

(b) If $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence, then $\{(-1)^k\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence.

For property (a), we note that for the simple Laguerre polynomials ($\alpha = 0$), the sequence $\{k+1\}_{k=0}^{\infty}$ is an $L^{(0)}$-multiplier sequence, but $\{k+1\}_{k=1}^{\infty} = \{k+2\}_{k=0}^{\infty}$ is not (see Theorem 3.5).

For property (b), we note again that $\{k+1\}_{k=0}^{\infty}$ is an $L^{(0)}$-multiplier sequence. We now show that $\{(-1)^k(k+1)\}_{k=0}^{\infty}$ is not. The polynomial

$$p(x) = (x-10)^2 = 82L_0^{(0)}(x) + 16L_1^{(0)}(x) + 2L_2^{(0)}(x)$$

has only real zeros, while

$$3 \cdot 82L_0^{(0)}(x) - 2 \cdot 16L_1^{(0)}(x) + 1 \cdot 2L_2^{(0)}(x) = 3x^2 + 20x + 56$$

has two non-real zeros.

5.2. Monotonicity of $L^{(\alpha)}$-multiplier sequences. The main result obtained in this section is that, if a classical multiplier sequence with non-negative terms is an $L^{(\alpha)}$-multiplier sequence, then it must be non-decreasing, and hence must also be a Hermite multiplier sequence (see Theorem 2.1). The converse is not true for $L^{(\alpha)}$-multiplier sequences since, for example, $\{r^k\}_{k=0}^{\infty}$ for $r > 1$ is not an $L^{(\alpha)}$-multiplier sequence (see Proposition 3.4). Thus, we will establish the proper inclusion

$$\left\{\text{non-negative } L^{(\alpha)}\text{-multiplier sequences}\right\} \subsetneq \{\text{Hermite multiplier sequences}\}.$$

The next lemma contains two simple but useful results. The first one essentially says that, if a polynomial has only simple real zeros and one makes a small perturbation to the coefficients, then the resulting polynomial also has only real zeros. Similarly, if we begin with a polynomial which has some non-real zeros, then any small perturbation of the coefficients will result in another polynomial which has some non-real zeros.

Lemma 5.3. *Let p and q be real polynomials with deg$(q) <$ deg(p).*

(i) *If p has only simple real zeros, then there exists $\varepsilon > 0$ such that $p(x) + bq(x)$ has only real zeros whenever $|b| < \varepsilon$.*

(ii) *If p has some non-real zeros then there exists $\varepsilon > 0$ such that $p(x) + bq(x)$ has some non-real zeros whenever $|b| < \varepsilon$.*

Proof. (i) Suppose no such ε exists. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of real numbers converging to zero such that, for each n, the real polynomial $p_n(x) = p(x) + b_n q(x)$ has some non-real zeros. The polynomials p_n converge uniformly on compact subsets of $\mathbf{C}$ to p. By Hurwitz's theorem, the zeros of p must be limits of the zeros of p_n, contradicting the fact that the zeros of p are all real and simple.

(ii) The proof is similar to that of (i), the only difference being that the contradiction lies in the fact that a non-real number cannot be the limit of a sequence of real numbers. $\square$

Lemma 5.4. *For $n \geq 2$ and $b \in \mathbf{R}$, define*

$$f_{n,b,\alpha}(x) := L_n^{(\alpha)}(x) + bL_{n-2}^{(\alpha)}(x),$$

and

$$E_n := \{b \in \mathbf{R} \mid f_{n,b,\alpha}(x) \quad \text{has only real zeros}\}.$$

Then $\max(E_n)$ *exists and is a positive real number.*

Proof. By Lemma 5.3 (i), there exists an $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq E_n$. In particular, E_n is nonempty and $\max(E_n)$, if it exists, is positive. It now suffices to show that E_n is closed and bounded above.

Suppose $t \in (\mathbf{R} \setminus E_n)$. Then, by part (ii) of Lemma 5.3, there exists a $\delta > 0$ such that

$$f_{n,t,\alpha}(x) + bL_{n-2}^{(\alpha)}(x) = L_n^{(\alpha)}(x) + (t + b)L_{n-2}^{(\alpha)}(x)$$

has non-real zeros whenever $|b| < \delta$. That is to say, $(b - \delta, b + \delta) \subseteq (\mathbf{R} \setminus E_n)$. Whence, $\mathbf{R} \setminus E_n$ is open and, therefore, E_n is closed.

To show that E_n is bounded above, we consider the $(n-2)$nd derivative of $f_{n,b,\alpha}$. A calculation shows that

$$\frac{d^{n-2}}{dx^{n-2}} f_{n,b,\alpha}(x) = \frac{1}{2}x^2 - (n + \alpha)x + \frac{(n + \alpha)(n + \alpha - 1)}{2} + b.$$

Thus, $d^{n-2}/(dx^{n-2})f_{n,b,\alpha}(x)$, and therefore $f_{n,b,\alpha}(x)$, has some non-real zeros whenever b is sufficiently large. $\square$

Proposition 5.5. *Suppose that $\{\gamma_k\}_{k=0}^{\infty}$ is a non-trivial $L^{(\alpha)}$-multiplier sequence. Then there exists an $m \in \mathbf{Z}$ such that $\gamma_k = 0$ for all $k < m$ and $\gamma_k \neq 0$ for all $k \geq m$.*

Proof. Since $\{\gamma_k\}_{k=0}^{\infty}$ is a non-trivial multiplier sequence, there is at least one $k \in \mathbf{Z}$ such that $\gamma_k \neq 0$ (see the remark after Proposition 3.2). Let m be the minimal index such that $\gamma_m \neq 0$. It is easy to see that γ_{m+1} and γ_{m+2} are non-zero, for if either of them were zero, in light of Lemma 5.1, we would have to conclude that $\{\gamma_k\}_{k=0}^{\infty}$ is a trivial multiplier sequence. Suppose now that there exists a $n > m+2$ such that $\gamma_n = 0$. By Lemma 5.4, there are constants a_m, a_{m+2} such that the polynomial

$$\widetilde{q}(x) = a_m \gamma_m L_m^{(\alpha)}(x) + a_{m+2}\gamma_{m+2} L_{m+2}^{(\alpha)}(x)$$

has some non-real zeros. On the other hand, by Lemma 5.3, there exists an a_n such that

$$\begin{aligned} q(x) &= a_m L_m^{(\alpha)}(x) + a_{m+2} L_{m+2}^{(\alpha)}(x) + a_n L_n^{(\alpha)}(x) \\ &= a_n \left(L_n^{(\alpha)}(x) + \frac{a_m}{a_n} L_m^{(\alpha)}(x) + \frac{a_{m+2}}{a_n} L_{m+2}^{(\alpha)}(x) \right) \end{aligned}$$

has only real zeros. Applying the $L^{(\alpha)}$-multiplier sequence $\{\gamma_k\}_{k=0}^{\infty}$ to $q(x)$, we obtain the polynomial $\widetilde{q}(x)$, a contradiction. Hence, $\gamma_k \neq 0$ for all $k \geq m$, and the proof is complete. $\square$

Theorem 5.6. *If the sequence of non-negative real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is a non-trivial $L^{(\alpha)}$-multiplier sequence, then $\gamma_k \leq \gamma_{k+1}$ for all $k \geq 0$.*

Proof. Let T_L denote the operator associated to the $L^{(\alpha)}$-multiplier sequence $\{\gamma_k\}_{k=0}^{\infty}$. Suppose $n \geq 2$ and that $\gamma_{n-2} \neq 0$. By Proposition 5.5, we have $\gamma_n \neq 0$. Using the notation of Lemma 5.4, the function

$$f_{n,\beta_n^*,\alpha}(x) = L_n^{(\alpha)}(x) + \beta_n^* L_{n-2}^{(\alpha)}(x) \quad (\beta_n^* = \max(E_n))$$

has only real zeros. It follows that

$$\begin{aligned}T_L[f_{n,\beta_n^*,\alpha}(x)] &= \gamma_n L_n^{(\alpha)}(x) + \gamma_{n-2}\beta_n^* L_{n-2}^{(\alpha)}(x)\\ &= \gamma_n\left(L_n^{(\alpha)}(x) + \frac{\gamma_{n-2}}{\gamma_n}\beta_n^* L_{n-2}^{(\alpha)}(x)\right)\end{aligned}$$

also has only real zeros. By Lemma 5.4, we must have $\gamma_{n-2}/(\gamma_n)\beta_n^* \leq \beta_n^*$, which gives $0 < \gamma_{n-2}/(\gamma_n) \leq 1$. On the other hand, by Lemma 5.1, we have

$$\gamma_{n-1}^2 - \gamma_n\gamma_{n-2} \geq 0, \quad (n \geq 2),$$

which means $(\gamma_{n-1}/\gamma_{n-2})^2 \geq \gamma_n/(\gamma_{n-2}) \geq 1$. In other words, $\gamma_{n-1} \geq \gamma_{n-2}$, and the proof is complete. $\square$

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