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# THE CORONA THEOREM FOR MULTIPLIER ALGEBRAS ON WEIGHTED DIRICHLET SPACES

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ABSTRACT. We prove a corona theorem for infinitely many functions from the multiplier algebras on weighted Dirichlet spaces on the unit disk. In addition, explicit estimates on solutions are given.

In this paper we wish to extend the corona theorem on the multiplier algebra on a weighted Dirichlet space, $M(\mathcal{D}_\alpha)$, to infinitely many functions and obtain estimates on *corona problem* solutions. For a finite number of functions, the corresponding theorem is due to Tolokonnikov [**8**]. For infinitely many functions in $H^\infty(\mathbf{D})$, the corona theorem is due to Rosenblum [**7**] and to Tolokonnikov [**8**]. Our methods are in principle close to those of Rosenblum [**7**]. All of these efforts were made possible by Wolff's beautiful proof of Carleson's original corona theorem. (See [**1**]).

This paper uses the basic ideas first used by Trent [**10**] to solve the corresponding problem in Dirichlet space. Later these ideas were used in Costea, Sawyer and Wick [**4**] in their generalization to certain Besov spaces on the unit ball in $\mathbf{C}^n$. Their paper is much more general than the case considered here. However, their formidable estimates on integral operators do not seem to give explicit estimates on corona solutions, which only depend upon $\varepsilon$. Moreover, we can explicitly give a fixed space to which all our multipliers on weighted Dirichlet space extend, namely, the *harmonic weighted Dirichlet space.*

We will establish our notation. $\mathcal{D}_\alpha$ will denote the weighted Dirichlet space on the unit disk, $\mathbf{D}$. That is, for $\alpha \in \mathbf{R}$,

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$$\mathcal{D}_\alpha = \Big\{ f \in \mathrm{Hol}\,(\mathbf{D}) \mid f \text{ is analytic on } D \text{ and for } f(z) = \sum_{n=0}^{\infty} f_n z^n,$$
$$\|f\|^2_{\mathcal{D}_\alpha} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (n+1)^\alpha |f_n|^2 < \infty \Big\}.$$

For a nice account of many interesting properties of Dirichlet space see the survey article of Wu [**12**].

Next we state our main theorem, the corona theorem for the multiplier algebras of weighted Dirichlet space. We refer to this theorem as the $\mathcal{M}(\mathcal{D}_\alpha)$ *Corona theorem*. In this paper, we consider weighted Dirichlet spaces only for $\alpha \in (0,\, 1]$.

$\mathcal{M}(\mathcal{D}_\alpha)$ **Corona theorem.** *For $\alpha \in (0,\, 1]$, there exists a positive number $C_\alpha < \infty$ so that, for any $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D}_\alpha)$ with $\|M_F^C\| \le 1$ and $0 < \varepsilon^2 \le \sum_{j=1}^\infty |f_j(z)|^2$ for all $z \in \mathbf{D}$, there exists a $\{g_j\}_{j=1}^\infty \subseteq \mathcal{M}(\mathcal{D}_\alpha)$ such that*

$$\text{(i)} \quad \sum_{j=1}^{\infty} f_j g_j = 1$$

$$\text{(ii)} \quad \left\| M_G^C \right\| \le \frac{C_\alpha}{\varepsilon^4}.$$

For the case $\alpha = 0$, the $\mathcal{M}(\mathcal{D}_\alpha)$ Corona theorem implies the $H^\infty(\mathbf{D})$ Corona theorem, but with the somewhat weaker estimate of $C_0/\varepsilon^4$ in place of $C_0/\varepsilon^2 \ln(1/\varepsilon^2)$ on the righthand side. For the case $\alpha = 1$, the $\mathcal{M}(\mathcal{D}_\alpha)$ Corona theorem implies the $\mathcal{M}(\mathcal{D})$ Corona theorem with the more precise formula $C(\varepsilon) = C_1/\varepsilon^4$.

Our proof is based on five parts. First, since the reproducing kernel of weighted Dirichlet space has one positive square (or is a complete Nevanlinna-Pick kernel), the commutant lifting theorem comes into play. This reduces the general $\mathcal{M}(\mathcal{D}_\alpha)$ corona problem to the $\mathcal{D}_\alpha$ corona problem, and we may employ Hilbert space methods. This is where our hypothesis that $\alpha \in (0,\, 1]$ enters in, since for $\alpha > 1$, the reproducing kernel for $\mathcal{D}_\alpha$ is not a complete Nevanlinna-Pick kernel. Next, a linear algebra representation allows us to explicitly write down proposed solutions for the $\mathcal{D}_\alpha$ corona problem in the smooth case. Thirdly, we have a series of tedious lemmas that basically say that multipliers on weighted Dirichlet space can be naturally extended to multipliers on (boundary values of) *weighted harmonic Dirichlet space*. Fourthly, estimates based on several equivalent norms and Schur's lemma bound our integral operators for the smooth case. Finally, a compactness argument allows us to remove the smoothness condition to complete the proof.

We begin with some notation. Let $E$ be a subset of the complex numbers, $\mathbf{C}$, and let $\mathcal{H}(E)$ denote a Hilbert space of functions on $E$. We will only consider reproducing kernel Hilbert spaces, $\mathcal{H}(E)$. This means that, for every $w \in E$, there exists a unique element of $\mathcal{H}(E)$, denoted by $k_w$, so that

$$\langle f, k_w \rangle_{\mathcal{H}(E)} = f(w) \quad \text{for all } f \in \mathcal{H}(E).$$

We refer to $k_w(\cdot)$ as the reproducing kernel on $\mathcal{H}(E)$.

For any $\phi \in \mathcal{H}(E)$, define $M_\phi$ by $M_\phi(f) = \phi f$, for all $f$ in $\mathcal{H}(E)$. If $\phi f \in \mathcal{H}(E)$ for all $f$ in $\mathcal{H}(E)$, then $\phi$ or $M_\phi$ is called a *multiplier* for $\mathcal{H}(E)$. By the closed graph theorem, $M_\phi \in \mathcal{B}(\mathcal{H}(E))$, the space of bounded operators on $\mathcal{H}(E)$.

We will use the following notation:

$$\mathcal{M}(\mathcal{H}(E)) = \{\phi \in \mathcal{H}(E) : \phi f \in \mathcal{H}(E) \text{ for all } f \in \mathcal{H}(E)\}$$

and

$$M(\mathcal{H}(E)) = \{M_\phi : \phi \in \mathcal{M}(\mathcal{H}(E))\}.$$

Note that, for $\alpha \leq 0$, $\mathcal{M}(\mathcal{D}_\alpha) = H^\infty(\mathbf{D})$. But, for $0 < \alpha \leq 1$, $\mathcal{M}(\mathcal{D}_\alpha) \subsetneqq H^\infty(\mathbf{D})$. In fact, for $f_\alpha(z) = \sum_{n=1}^{\infty}(z^{n^{4m+1}})/(n^{2m\alpha})$, where $m = [1/\alpha] + 1$, $f_\alpha \in H^\infty(\mathbf{D})$, but $f_\alpha$ is not even in $\mathcal{D}_\alpha$.

We will also need a larger Hilbert space than $\mathcal{D}_\alpha$, which we refer to as the *weighted harmonic Dirichlet space*. Let $\alpha \in \mathbf{R}$. We denote the *weighted harmonic Dirichlet space* by $\mathcal{HD}_\alpha$ and define it by:

$$\mathcal{HD}_\alpha = \Big\{ f \in L^2(\mathcal{T}, d\lambda) : f(e^{it}) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{int} \\ \text{and } \|f\|^2_{\mathcal{HD}_\alpha} \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} (1+|n|)^\alpha |\widehat{f}(n)|^2 < \infty \Big\}.$$

Note that, when we consider functions in $\mathcal{HD}_\alpha$, we never consider extensions of the values of functions to the unit disk.

The algebras we consider here are multiplier algebras on weighted Dirichlet spaces and weighted harmonic Dirichlet spaces. For a fixed $\alpha$, we denote the multiplier algebra on $\mathcal{D}_\alpha$ by $\mathcal{M}(\mathcal{D}_\alpha)$ and on $\mathcal{HD}_\alpha$ by $\mathcal{M}(\mathcal{HD}_\alpha)$ and their corresponding operator spaces on $\mathcal{D}_\alpha$ by $M(\mathcal{D}_\alpha)$ and on $\mathcal{HD}_\alpha$ by $M(\mathcal{HD}_\alpha)$, respectively.

**Lemma 1.** *Let $\mathcal{H}(E)$ be a reproducing kernel Hilbert space. If $M_\phi$ is a multiplier for $\mathcal{H}(E)$, then*

$$M^*_\phi(k_\omega) = \overline{\phi(\omega)}\, k_\omega.$$

*Proof.* Let $f \in \mathcal{H}(E)$. Then

$$\begin{aligned} \left\langle f,\, M^*_\phi k_\omega \right\rangle &= \langle M_\phi f,\, k_\omega\rangle = \phi(\omega) f(\omega) \\ &= \left\langle f,\, \overline{\phi(\omega)}\, k_\omega \right\rangle, \quad \text{for all } f \in \mathcal{H}(E). \qquad \square \end{aligned}$$

Thus, $\|\phi\|_{\infty,E} \le \|M_\phi\|$ for any $\phi \in \mathcal{M}(\mathcal{H}(E))$. Similarly, for $\phi_{ij} \in \mathcal{M}(\mathcal{D}_\alpha)$ and the matrix-valued operator $M_{[\phi_{ij}]}$, where

$$M_{[\phi_{ij}]} = \begin{pmatrix} M_{\phi_{11}} & M_{\phi_{12}} & \cdots & \cdots \\ M_{\phi_{21}} & M_{\phi_{22}} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

we have that $\mathcal{M}(\oplus_1^\infty \mathcal{D}_\alpha) \subset H^\infty_{\mathcal{B}(l^2)}(\mathbf{D})$. Note that $M_{[\phi_{ij}]} : \oplus_1^\infty \mathcal{D}_\alpha \to \oplus_1^\infty \mathcal{D}_\alpha$ for each $ij$, $M_{\phi_{ij}} \in \mathcal{B}(\mathcal{D}_\alpha)$ and $\sup_{z\in\mathbf{D}} \|[\phi_{ij}(z)]\|_{\mathcal{B}(l^2)} \le \|M_\phi\|_{\mathcal{B}\oplus_1^\infty \mathcal{D}_\alpha}$.

Next we consider what we call row and column operators for the weighted Dirichlet spaces. These are important operators for our infinite version of the corona theorem on the weighted Dirichlet space.

Let $\{f_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$ and $F = (f_1, f_2, \dots)$. Then define $M_F^R$, a row operator, $M_F^R : \oplus_{n=1}^{\infty} \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ by

$$M_F^R(\{h_j\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty} f_j h_j, \quad \text{for } \{h_j\}_{j=1}^{\infty} \in \bigoplus_{n=1}^{\infty} \mathcal{D}_\alpha, \text{ and}$$

$M_F^C$, a column operator, $M_F^C : \mathcal{D}_\alpha \to \oplus_{n=1}^{\infty} \mathcal{D}_\alpha$ by

$$M_F^C(h) = (f_1 h,\, f_2 h,\, f_3 h, \dots)^T, \quad \text{for } h \in \mathcal{D}_\alpha.$$

The infinite version of Carleson's Corona theorem [**3**] is due to Rosenblum [**7**] and Tolokonnikov [**8**]; the best estimate for the solution is given by Uchiyama [**11**]. (See [**9**].)

$H^\infty(\mathbf{D})$ **Corona theorem.** *Let* $\{f_j\}_{j=1}^{\infty} \subset H^\infty(\mathbf{D})$ *with* $0 < \varepsilon^2 < \sum_{j=1}^{\infty} |f_j(z)|^2 \le 1$ *for all* $z \in \mathbf{D}$*. Then there exists* $\{g_j\}_{j=1}^{\infty} \subset H^\infty(\mathbf{D})$ *such that*

$$\sum_{j=1}^{\infty} f_j g_j = 1 \quad \textit{and} \quad \sup_{z \in \mathbf{D}} \left\{ \sum_{j=1}^{\infty} |g_j(z)|^2 \right\}^{1/2} \le \frac{C_0}{\varepsilon^2} \ln\left(\frac{1}{\varepsilon^2}\right).$$

*Here* $C_0$ *is a universal constant, which can be taken to be* $10$*, if* $\varepsilon^2 < e^{-1}$*.*

Let $\{f_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(H^2(\mathbf{D})) = H^\infty(\mathbf{D})$, and let $F = (f_1, f_2, \dots)$. We denote the row and column operators for the algebra $H^\infty(\mathbf{D})$ by $T_F^R$ and $T_F^C$, respectively.

From part of the pointwise hypothesis of the $H^\infty(\mathbf{D})$ *Corona theorem*, we have that:

$$\left\|T_F^R\right\|^2 = \left\|T_F^C\right\|^2 = \sup\left\{ \sum_{j=1}^{\infty} |f_j(z)|^2 : z \in \mathbf{D} \right\} \le 1.$$

Thus, one part of the pointwise hypothesis of the $H^\infty(\mathbf{D})$ *Corona theorem* gives the boundedness of both of the operators $T_F^R$ and $T_F^C$.

Consider the Dirichlet space, $\mathcal{D}$. Since $\mathcal{M}(\mathcal{D})$ is strictly contained in $\mathcal{H}^\infty(\mathbf{D})$, $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$ and $\sum_{j=1}^\infty |f_j(z)|^2 < \infty$ need not imply the boundedness of either $M_F^C$ or of $M_F^R$. However, the boundedness of $\|M_F^C\|$ always gives the boundedness of $\|M_F^R\|$ (see [**10**]). Thus, the hypothesis $\sum_{j=1}^\infty |f_j(z)|^2 \le 1$ for $z \in \mathbf{D}$ in the $H^\infty(\mathbf{D})$ Corona theorem is replaced by the condition $\|M_F^C\| \le 1$ in the $\mathcal{M}(\mathcal{D})$ Corona theorem. We do not know if the weaker condition that $\|M_F^R\| \le 1$ suffices for an $\mathcal{M}(\mathcal{D})$ Corona theorem.

$\mathcal{M}(\mathcal{D})$ **Corona theorem.** *Let $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$. Assume that $\|M_F^C\| \le 1$ and $0 < \varepsilon^2 \le \sum_{j=1}^\infty |f_j(z)|^2$ for all $z \in \mathbf{D}$. Then there exist $C(\varepsilon) < \infty$ and $\{g_j\}_{j=1}^\infty \subseteq \mathcal{M}(\mathcal{D})$ such that*:

$$\text{(i)} \quad \sum_{j=1}^\infty f_j g_j = 1$$

$$\text{(ii)} \quad \left\|M_G^C\right\| \le C(\varepsilon).$$

For the proof of the $\mathcal{M}(\mathcal{D})$ Corona theorem, we refer to Trent [**10**].

In this paper, we will prove the $\mathcal{M}(\mathcal{D}_\alpha)$ Corona theorem and give explicit estimates for solutions. We will need several equivalent norms for the weighted Dirichlet spaces.

Recall that for $f \in \mathcal{D}_\alpha$ its norm (by definition) is given by

$$(1) \quad \|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=0}^\infty (n+1)^\alpha \left|f_n\right|^2.$$

Three other norms that are equivalent to the above norm are:

$$(2) \quad \sum_{n=0}^\infty \frac{\Gamma(1-\alpha)n!}{\Gamma(n+1-\alpha)} \left|f_n\right|^2$$

$$(3) \quad \|f\|_{H^2(\mathbf{D})}^2 + \int_{\mathbf{D}} \left|f'(z)\right|^2 (1-|z|^2)^{1-\alpha} dA(z)$$

$$(4) \quad \|f\|^2_{H^2(\mathbf{D})} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|f(e^{it}) - f(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma(t)\, d\sigma(\theta).$$

That (1) is equivalent to (2) can be seen easily, using Stirling's formula. That (1) is equivalent to (3) follows, using the Beta function and Stirling's formula. For completeness, we will directly verify the equivalence between (1) and (4). (4) gives us a norm for function $f$ in the weighted Dirichlet space that is expressed solely in terms of the values of $f$ on the boundary of the unit disk $\mathbf{D}$, $\partial\mathbf{D}$ and has a natural extension to $\mathcal{HD}_\alpha$. This norm will be very useful for our estimates of the size of corona solutions. For the remainder of the paper, any of these norms will be used interchangeably for $\mathcal{D}_\alpha$.

We will use the notation $C_\alpha$ to denote a constant depending only upon $\alpha$.

**Lemma 2.** *There exists a constant $C_\alpha < \infty$, so that for any $f \in \mathcal{D}_\alpha$, with $f(z) = \sum_{n=0}^\infty f_n z^n$, we have*

$$\frac{1}{C_\alpha}\|f\|^2_{\mathcal{D}_\alpha} \le \|f\|^2_{H^2(\mathbf{D})} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|f(e^{it}) - f(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma(t) d\sigma(\theta)$$
$$\le C_\alpha \|f\|^2_{\mathcal{D}_\alpha}.$$

*Proof.* First let's note the following. For $f \in \mathcal{D}_\alpha$ with $f(z) = \sum_{n=0}^\infty f_n z^n$, we have, using (3),

$$\|f\|^2_{H^2(\mathbf{D})} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|f(e^{it}) - f(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma(t)\, d\sigma(\theta)$$
$$= \|f\|^2_{H^2(\mathbf{D})} + \sum_{n=0}^{\infty} |f_n|^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|e^{in\theta} - e^{int}|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma\, d\sigma.$$

Thus, we are done if we show that there exists a constant $E_\alpha < \infty$, depending only upon $\alpha$, such that, for all $N = 1, 2, \ldots,$

$$\frac{1}{E_\alpha}(N+1)^\alpha \le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|e^{iN\theta} - e^{iNt}|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma(t)\, d\sigma(\theta) \le E_\alpha(N+1)^\alpha.$$

So, let $\beta = (1+\alpha)/2$, and consider

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{e^{iNt}-e^{iN\theta}}{(e^{it}-e^{i\theta})^{\beta}}\right|^2 d\sigma\, d\sigma = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{1-e^{iNs}}{(1-e^{is})^{\beta}}\right|^2 d\sigma(s)\, d\sigma(\theta)$$
$$= \int_{-\pi}^{\pi}\left|\frac{1-e^{iNs}}{(1-e^{is})^{\beta}}\right|^2 d\sigma(s).$$

We want to show:

$$\int_{-\pi}^{\pi}\left|\frac{1-e^{iNs}}{(1-e^{is})^{\beta}}\right|^2 d\sigma(s) \sim (N+1)^{2\beta-1}.$$

First, we observe that the $n$th Fourier coefficient of $1/(1-e^{is})^{\beta}$ is $\Gamma(n+\beta)/\Gamma(\beta)n!$ for $n \geq 0$ and 0 otherwise. So, for $N > 1$,

$$\int_{-\pi}^{\pi}\left|\frac{1-e^{iNs}}{(1-e^{is})^{\beta}}\right|^2 d\sigma(s) = \underbrace{\sum_{n=0}^{N-1}\left(\frac{\Gamma(n+\beta)}{\Gamma(\beta)n!}\right)^2}_{a(N)}$$
$$+ \underbrace{\sum_{n=N}^{\infty}\left(\frac{\Gamma(n+\beta)}{\Gamma(\beta)n!} - \frac{\Gamma(n+N+\beta)}{\Gamma(\beta)(n+N)!}\right)^2}_{b(N)}.$$

By Stirling's formula, there exist $C_\beta, N_\beta < \infty$, so that, for all $n > N_\beta$,

$$\frac{1}{C_\beta}n^{\beta-1} \leq \frac{\Gamma(n+\beta)}{\Gamma(\beta)n!} \leq C_\beta n^{\beta-1}.$$

Thus, for $N > N_\beta$,

$$\frac{1}{C_\beta^2}\int_{N_\beta}^{N-1} x^{2\beta-2}dx \leq a(N) \leq C_\beta^2 \int_{N_\beta-1}^{N} x^{2\beta-2}dx + C_\beta^2 N^{2\beta-1}.$$

We conclude that there exists a finite $D_\beta$ so that, for all $N = 1, 2, \ldots,$

$$\frac{1}{D_\beta}N^{2\beta-1} \leq a(N) \leq D_\beta N^{2\beta-1}.$$

As for the second term,

$$b(N)=\sum_{n=N}^{\infty}\left(\frac{\Gamma(n+\beta)}{\Gamma(\beta)n!}\right)^2\left(1-\left(1-\frac{(1-\beta)}{n+1}\right)\cdots\left(1-\frac{(1-\beta)}{n+N}\right)\right)^2.$$

Now

$$\begin{aligned}1-\prod_{j=1}^{N}\left(1-\frac{1-\beta}{n+j}\right)&=1-e^{\sum_{j=1}^{N}\ln[1-((1-\beta)/n+j)]}\\&\le 1-e^{-\sum_{j=n+1}^{n+N}[(1-\beta)/j][1/(1-(1-\beta)/j)]}\\&\le 1-e^{-\sum_{j=n+1}^{n+N}[(1-\beta)/\beta](1/j)}\\&\le 1-e^{-[(1-\beta)/\beta]\ln[(n+N)/n]}\\&\le 1-\left(1+\frac{N+1}{n-1}\right)^{-(1-\beta)/\beta}\\&\le\frac{N+1}{n-1}\left(\frac{1-\beta}{\beta}\right).\end{aligned}$$

Estimating, we derive that

$$b(N)\le D_\beta^2\left(\frac{1-\beta}{\beta}\right)^2\sum_{n=N}^{\infty}n^{2\beta-2}\frac{(N+1)^2}{(n-1)^2}.$$

So there exists $E_\beta<\infty$, so that $b(N)\le E_\beta N^{2\beta-1}$. This completes our proof. □

Next we consider $\oplus_1^\infty\mathcal{D}_\alpha$, $l^2$-valued Dirichlet spaces. Let $f\in\oplus_1^\infty\mathcal{D}_\alpha$; then

$$\|f\|_{\mathcal{D}_\alpha}^2\sim\int_{-\pi}^{\pi}\|f\|^2\,d\sigma+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left\|f(e^{it})-f(e^{i\theta})\right\|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}\,d\sigma\,d\sigma.$$

We also need an extension of the weighted Dirichlet spaces, weighted harmonic Dirichlet spaces, $\mathcal{HD}_\alpha$ and $\oplus_1^\infty\mathcal{HD}_\alpha$ ($l^2$ valued), which are defined on the boundary of $\mathbf{D}$.

Note also that, for $f \in \mathcal{HD}_\alpha$ (and, similarly, for $f \in \oplus_1^\infty \mathcal{HD}_\alpha$), the norms below are equivalent as in the lemma:

$$(1)\quad \left(f(e^{it}) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{int}\right) \|f\|^2_{\mathcal{HD}_\alpha} \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} (1+|n|)^\alpha |\widehat{f}(n)|^2$$

$$(2)\quad \|f\|^2_{H^2(\mathbf{D})} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|f(e^{it}) - f(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma\, d\sigma.$$

We show that boundedness of the "column operators" implies the boundedness of the "row operators." So our hypothesis on the column operator in the Corona theorem controls the size of the row operator in the estimates to come.

**Lemma 3.** *Let $M_F^C \in B(\mathcal{D}_\alpha, \oplus_1^\infty \mathcal{D}_\alpha)$. Then $M_F^R \in B(\oplus_1^\infty \mathcal{D}_\alpha, \mathcal{D}_\alpha)$ and*

$$\|M_F^R\| \le \sqrt{10}\|M_F^C\|.$$

*Proof.* Take $\|M_F^C\| \le 1$. Note that, since $\sup_{z\in\mathbf{D}} \sum_{j=1}^\infty |f_j(z)|^2 \le \|M_F^C\|^2 \le 1$, we have

$$\sum_{j=1}^{\infty} |f_j(z)|^2 = F(z)F(z)^* \le 1, \quad \text{for all } z \in \mathbf{D}.$$

Now let $\underline{U} = \{u_k\}_{k=1}^\infty \in \oplus_1^\infty \mathcal{D}_\alpha$. Then

$$\begin{aligned}\left\|M_F^R(\{u_k\}_{k=1}^\infty)\right\|^2 &= \left\|\sum_{k=1}^\infty f_k u_k\right\|^2 \\ &= \int_{\partial\mathbf{D}} \left|\sum_{k=1}^\infty f_k u_k\right|^2 d\sigma \\ &\quad + \int_{\mathbf{D}} \left|\left(\sum_{k=1}^\infty f_k u_k\right)'\right|^2 (1-|z|^2)^{1-\alpha}\, dA.\end{aligned}$$

First note that

$$\begin{aligned}\int_{\partial\mathbf{D}} \left|\sum_{k=1}^\infty f_k u_k\right|^2 d\sigma &\le \int_{\partial\mathbf{D}} \sum_{k=1}^\infty |f_k|^2 \sum_{k=1}^\infty |u_k|^2 d\sigma \\ &\le \int_{\partial\mathbf{D}} \sum_{k=1}^\infty |u_k|^2 d\sigma = \|\underline{U}\|_\sigma^2 .\end{aligned}$$

Second, if we set $\omega(z) = (1-|z|^2)^{1-\alpha}$, then

$$\begin{aligned}\int_{\mathbf{D}} \left|\left(\sum_{k=1}^{\infty} f_k u_k\right)'\right|^2 (1-|z|^2)^{1-\alpha}\, dA \\ &= \int_{\mathbf{D}} \left|\left(\sum_{k=1}^{\infty} f_k' u_k + f_k u_k'\right)\right|^2 \omega(z)\, dA \\ &\leq 2\int_{\mathbf{D}} \left(\left|\sum_{k=1}^{\infty} f_k' u_k\right|^2 + \left|\sum_{k=1}^{\infty} f_k u_k'\right|^2 \right)\omega(z)\, dA \\ &\leq 2\int_{\mathbf{D}} \left(\left|\sum_{k=1}^{\infty} f_k' u_k\right|^2 + \sum_{k=1}^{\infty} |u_k'|^2\right)\omega(z)\, dA.\end{aligned}$$

Therefore,

$$\left\| M_F^R(\{u_k\}_{k=1}^{\infty})\right\|^2 \leq 2\,\|\underline{U}\|_{\mathcal{D}_\alpha}^2 + 2\int_{\mathbf{D}} \left(\left|\sum_{k=1}^{\infty} f_k' u_k\right|^2\right)\omega(z)\, dA.$$

But

$$\begin{aligned}\left|\sum_{k=1}^{\infty} f_k' u_k\right|^2 &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |f_k' u_k f_j' u_j| \\ &\leq \frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (|f_k' u_j|^2 + |f_j' u_k|^2) \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |f_k' u_j|^2 \\ &\leq 2\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} |f_k u_j'|^2 + \sum_{k=1}^{\infty} |(f_k u_j)'|^2\right).\end{aligned}$$

Hence,

$$\begin{aligned}\left\| M_F^R(\{u_k\}_{k=1}^{\infty})\right\|^2 &\leq 2\,\|\underline{U}\|_{\mathcal{D}_\alpha}^2 \\ &\quad + 2\int_{\mathbf{D}} \left(\left|\sum_{k=1}^{\infty} f_k' u_k|^2\right)\omega(z)\, dA\end{aligned}$$

$$
\begin{aligned}
&\leq 2\,\|\underline{U}\|^2_{\mathcal{D}_\alpha} \\
&\quad + 4\sum_{j=1}^{\infty}\int_{\mathbf{D}}\left(\sum_{k=1}^{\infty}|f_k u_j'|^2 + \sum_{k=1}^{\infty}|(f_k u_j)'|^2\right)\omega(z)\,dA \\
&\leq 6\,\|\underline{U}\|^2_{\mathcal{D}_\alpha} \\
&\quad + 4\sum_{j=1}^{\infty}\left\|M_F^C(u_j)\right\|^2_{\oplus_1^\infty \mathcal{D}_\alpha} \\
&\leq 10\,\|\underline{U}\|^2_{\mathcal{D}_\alpha}.
\end{aligned}
$$

Therefore, $\|M_F^R\| \leq \sqrt{10}\|M_F^C\|$. □

We prove the $\mathcal{M}(\mathcal{D}_\alpha)$ Corona theorem by establishing the following two theorems:

**Theorem A.** *Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$. Assume that $\|M_F^C\| \leq 1$ and $0 < \varepsilon^2 \leq \sum_{j=1}^{\infty}|f_j(z)|^2$ for all $z \in \mathbf{D}$. Then there exists a $C_\alpha < \infty$, such that $(\varepsilon^8/C_\alpha^2)I \leq M_F^R(M_F^R)^* \leq I$.*

**Theorem B.** *Assume that $\delta^2 I \leq M_F^R(M_F^R)^* \leq I$. Then there exists a $\{g_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$ such that*

$$
\begin{aligned}
&\text{(i)} \quad \sum_{j=1}^{\infty} f_j g_j = 1 \\
&\text{(ii)} \quad \left\|M_G^C\right\| \leq 1/\delta.
\end{aligned}
$$

*Theorems A and B* with $\delta = [\varepsilon^4/C_\alpha]^{-1}$ complete the $D_\alpha$ Corona theorem. Before proving Theorem B, we first state some facts about $\mathcal{D}_\alpha$.

$\mathcal{D}_\alpha$ is a reproducing kernel Hilbert space with reproducing kernel given by

$$
k_\omega(z) = \sum_{n=0}^{\infty}\frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)n!}z^n\overline{\omega}^n = \frac{1}{(1-z\overline{\omega})^{1-\alpha}}, \quad \text{for all } z,\omega \in \mathbf{D}.
$$

(Of course, here we use the norm $\|f\|^2_{\mathcal{D}_\alpha} = \sum_{n=0}^{\infty}(\Gamma(1-\alpha)n!)/(\Gamma(n+1-\alpha))|f_n|^2$.) Furthermore, the reproducing kernel $k_\omega(z)$ has one positive square, i.e.,

$$1 - \frac{1}{k_\omega(z)} = 1 - (1 - z\overline{\omega})^{1-\alpha} = \sum_{n=1}^{\infty} C_n \overline{\omega}^n z^n,$$
$$\text{where } C_n \geq 0 \quad \text{for all } n.$$

That $C_n \geq 0$ for all $n \geq 1$ follows easily from a calculus argument, since $0 \leq 1 - \alpha \leq 1$.

In general, we say that a reproducing kernel $k_w(z) \in \mathcal{H}(E)$ has *one positive square* if there exists a $b_n$ in $\mathcal{H}(E)$ such that

$$\frac{1}{k_w(z)} = 1 - \sum_{1}^{\infty} b_n(z)\overline{b_n(w)} \quad \text{for all } z,\, w \in E.$$

The fact that $\mathcal{D}_\alpha$ is a reproducing kernel Hilbert space with one positive square makes it possible to use a corresponding *commutant lifting theorem* (CLT) to prove Theorem B. We first state the commutant lifting theorem for $\mathcal{D}_\alpha$. For more details refer to Ball, Trent and Vinnikov [**2**]. In particular, the case of finitely many functions in the subsequent theorem appears as Example 2 in [**2**].

**Commutant lifting theorem for $\mathcal{D}_\alpha$.** *Let $M_*$ and $N_*$ be invariant subspaces for $M_z^*$ on $\oplus_1^M \mathcal{D}_\alpha$ and $\oplus_1^N \mathcal{D}_\alpha$, respectively, where $1 \leq M,\, N \leq \infty$. Assume that $X^* \in B(M_*, N_*)$ satisfies $X^* M_z^*|_{M_*} = M_z^*|_{N_*} X^*$. Then there exists a $Y^* \in B(\oplus_1^M \mathcal{D}_\alpha, \oplus_1^N \mathcal{D}_\alpha)$ so that*:

(i) $Y^*|_{M_*} = X^*$ *($X^*$ lifts to $Y^*$),*

(ii) $Y^* M_z^* = M_z^* Y^*$ *($Y$ is in the commutant of $M_z$)*

(iii) $\|Y^*\| = \|X^*\|$ *(norms are preserved).*

Using the CLT, we can proceed to the proof of Theorem B.

*Proof of Theorem* B. Define $M = \infty$, $N = 1$, $M_* = \operatorname{range}(M_F^R)^*$, $N_* = \mathcal{D}_\alpha$. Let $X^* = [M_F^R(M_F^R)^*]^{-1}(M_F^R)$. (Note that $X^* : \oplus_1^M \mathcal{D}_\alpha \to \mathcal{D}_\alpha$.)

$$\begin{aligned} \|X^*\|^2 &= \left\|[M_F^R(M_F^R)^*]^{-1} M_F^R (M_F^R)^* [[M_F^R(M_F^R)^*]^{-1}]^*\right\| \\ &= \left\|[[M_F^R(M_F^R)^*]^{-1}]\right\|. \end{aligned}$$

From $\delta^2 I \le M_F^R (M_F^R)^* \le I$, it follows that

$$I \le [M_F^R (M_F^R)^*]^{-1} \le 1/\delta^2\, I.$$

Thus, $\|X^*\| \le 1/\delta$. Now consider

$$\begin{aligned} M_z M_F^R(\{u_j\}) &= \sum_{j=1}^{\infty} f_j u_j \\ &= \sum_{j=1}^{\infty} f_j (z u_j) = M_F^R(\{z u_j\}) = M_F^R(M_z(\{u_j\})). \end{aligned}$$

This shows that

$$(M_z)^*(M_F^R)^* \;=\; (M_F^R)^*(M_z)^*.$$

Let $u \in \mathcal{D}_\alpha$; then

$$X^* {M_z}^*((M_F^R)^* u) = X^*(M_F^R)^* {M_z}^* u = {M_z}^* u$$

and

$$X^* {M_z}^*((M_F^R)^* = {M_z}^*(X^*(M_F^R)^*).$$

So we have

$$X^* {M_z}^*|_{M_*} = {M_z}^*|_{N_*} X^*.$$

Thus, by the CLT there exist $Y^* \in B(\oplus_1^M \mathcal{D}_\alpha, \mathcal{D}_\alpha)$ satisfying (i), (ii) and (iii).

By (ii), $Y$ has entries in $\mathcal{M}(D_\alpha)$, say $g_i \in \mathcal{M}(D_\alpha)$. So $Y = M_G^C$.

By (i), $Y^*(M_F^R)^* = I$, which implies $M_F^R M_G^C = I$.

Finally, (iii) implies $\|M_G^C\| = \|Y\| = \|X\| \le 1/\delta$. □

Next we prove that Theorem A is equivalent to Theorem A′.

**Theorem A′.** *Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$. Assume that $\|M_F^C\| \le 1$ and $0 < \varepsilon^2 \le \sum_{j=1}^{\infty} |f_j(z)|^2$ for all $z \in \mathbf{D}$. Then there exists a $C_\alpha < \infty$ such that, for all $h \in \mathcal{D}_\alpha$, there exists a $\mathbf{U}_h \in \oplus_1^\infty \mathcal{D}_\alpha$ with*

$$\text{(i)} \quad M_F^R(\mathbf{U}_h) = h$$

$$\text{(ii)} \quad \|\mathbf{U}_h\|_{\oplus_1^\infty \mathcal{D}_\alpha} \le \frac{C_\alpha}{\varepsilon^4} \|h\|_{\mathcal{D}_\alpha}.$$

*Proof.* (Equivalence between Theorem A and Theorem A′.) Note the conclusion of Theorem A is

$$\delta^2 I \leq M_F^R (M_F^R)^* \leq I.$$

Thus, by Douglas's lemma, there exists a $C^* : \oplus_1^\infty \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ such that

$$\text{(i)} \quad C^* M_F^{R^*} = I \quad \text{and} \quad \text{(ii)} \quad \|C\| \leq \delta^{-1}.$$

So now $M_F^R C = I$ implies that $M_F^R C(h) = h$, for all $h \in \mathcal{D}_\alpha$. If we set $C(h) = \mathbf{U}_h$, then we get $M_F^R(\mathbf{U}_h) = h$, for all $h \in \mathcal{D}_\alpha$. Since $\|\mathbf{U}_h\|_{\mathcal{D}_\alpha}^2 = \|Ch\|_{\mathcal{D}_\alpha}^2 \leq \delta^{-2}\|h\|_{\mathcal{D}_\alpha}^2$, the proof is finished. □

Now we are ready to prove Theorem A′. We will use a sequence of lemmas. Let's start by making the following observation. It suffices to prove Theorem A′ for any dense set of functions in $\mathcal{D}_\alpha$. Then a compactness argument establishes Theorem A′ in the general case.

The general plan is as follows. Assume $F$ is analytic on $\mathbf{D}_{1+\varepsilon}(0)$, given $h \in \mathcal{D}_\alpha$ and $h$ analytic on $\mathbf{D}_{1+\varepsilon}(0)$; write the most general solution of the pointwise problem, $F(z)\mathbf{U}_h(z) = h(z)$ for $z \in D$, by

$$\mathbf{U}_h(z) = F(z)^*[F(z)F(z)^*]^{-1}h(z) - Q(z)\underline{\mathcal{X}}(z) \quad \text{on } \mathbf{D},$$

where $\operatorname{Range} Q(z) = \operatorname{Kernel} F(z)$, $Q(z)$ is analytic, and $\underline{\mathcal{X}}(z) \in l^2$ for $z \in \overline{\mathbf{D}}$.

We will need the Cauchy transform of $\mathcal{K}$. (See Lemma 10 for the explicit definition of the Cauchy transform.) Now we choose $\underline{\mathcal{X}}(z)$ to make $\mathbf{U}_h(z)$ analytic in $\mathbf{D}$. So let $\underline{\mathcal{K}} = [Q(z)^*\, F'(z)^* h(z)]/(F(z)F(z)^*)^2$ and take $\mathcal{X} = \widehat{\mathcal{K}}$. It follows that $\overline{\partial_z}\underline{\widehat{\mathcal{K}}} = \underline{\mathcal{K}}$ in $\mathbf{D}$. (For the proof see [**1**].) A short computation shows that $\mathbf{U}_h$ is analytic. Thus, we are done in the case where $F$ is smooth if we show that:

$$\|\mathbf{U}_h\|_{\oplus_1^\infty \mathcal{D}_\alpha} \leq \frac{C_\alpha}{\varepsilon^4}\, \|h\|_{\mathcal{D}_\alpha}.$$

Our procedure is to show that

(i) $\left\|F^*(FF^*)^{-1}h\right\|_{\oplus_1^\infty \mathcal{H}\mathcal{D}_\alpha} \leq C_1\|h\|_{\mathcal{D}_\alpha}$,

(ii) $\|Q\underline{\widehat{\mathcal{K}}}\|_{\oplus_1^\infty \mathcal{D}_\alpha H D_\alpha} \leq C_2\|\underline{\widehat{\mathcal{K}}}\|_{\oplus_1^\infty \mathcal{D}_\alpha}$

and the main estimate

(iii) $\|\widehat{\mathcal{K}}\|_{\oplus_1^\infty \mathcal{HD}_\alpha} \leq C_2 \|h\|_{\mathcal{D}_\alpha}$.

We proceed to prove (i), (ii) and (iii) for the smooth case. Lemmas 4–9 prove (i) and (ii), while the Cauchy transform and Schur's lemma help us establish (iii). Note that Lemma 4–Lemma 7 extend multipliers on $\mathcal{D}_\alpha$ to multipliers on $\mathcal{HD}_\alpha$ with estimates.

**Lemma 4.** (a) *Let* $M_\varphi \in M(\mathcal{D}_\alpha)$. *Then* $M_\varphi \in M(\mathcal{HD}_\alpha)$ *and* $\|M_\varphi\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \leq \sqrt{20}\|M_\varphi\|_{\mathcal{B}(\mathcal{D}_\alpha)}$.

(b) *Let* $\{f_i\}_{i=1}^\infty \subseteq \mathcal{M}(\mathcal{D}_\alpha)$.

*Then*

$$\left\|M_F^C\right\|_{\mathcal{B}(\mathcal{HD}_\alpha, \oplus_1^\infty \mathcal{HD}_\alpha)} \leq \sqrt{20}\left\|M_F^C\right\|_{\mathcal{B}(\mathcal{D}_\alpha, \oplus_1^\infty \mathcal{D}_\alpha)}.$$

*Proof.* We need only prove (a), since (b) follows by summing the result of (a). Let $M_\varphi \in M(\mathcal{D}_\alpha)$ and $\|M_\varphi\| = 1$. We prove the result for trigonometric polynomials, i.e., polynomials of the type $f(t) = \sum_{n=-N}^N C_n e^{\mathrm{int}}$. To this end, we consider analytic polynomials $p$ and $q_0$ with $q_0(0) = 0$, and we need only estimate $\|M_\varphi(p+\overline{q}_0)\|_{\mathcal{HD}_\alpha}$, since $\{p+\overline{q}_0\}$ is dense in $\mathcal{HD}_\alpha \subseteq L^2(\mathbf{D})$.

Computing

$$\begin{aligned}
&\|\varphi(p+\overline{q}_0)\|^2_{\mathcal{HD}_\alpha}\\
&= \int_{-\pi}^{\pi} |\varphi(p+\overline{q}_0)|^2 \, d\sigma\\
&\quad + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(\varphi(p+\overline{q}_0))(e^{it}) - (\varphi(p+\overline{q}_0))(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} \, d\sigma \, d\sigma\\
&\leq 2\int_{-\pi}^{\pi} |\varphi p|^2 \, d\sigma\\
&\quad + 2\int_{-\pi}^{\pi} |\varphi\overline{q}_0|^2 \, d\sigma + 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(\varphi p)(e^{it}) - (\varphi p)(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} \, d\sigma \, d\sigma\\
&\quad + 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(\varphi\overline{q}_0)(e^{it}) - (\varphi\overline{q}_0)(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} \, d\sigma \, d\sigma\\
&\leq 2\,\|\varphi p\|^2_{\mathcal{D}_\alpha} + 2\int_{-\pi}^{\pi} |\overline{q}_0|^2 \, d\sigma
\end{aligned}$$

$$
\begin{aligned}
&+2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|(\varphi\overline{q}_0)(e^{it})-(\varphi\overline{q}_0)(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}\,d\sigma\,d\sigma\\
\le{}& 2\left\|p\right\|_{\mathcal{D}_\alpha}^2+2\left\|q\right\|_\sigma^2\\
&+2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|(\varphi\overline{q}_0)(e^{it})-\varphi(e^{it})\overline{q}_0(e^{i\theta})+\varphi(e^{it})\overline{q}_0(e^{i\theta})-(\varphi\overline{q}_0)(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}\,d\sigma\,d\sigma\\
\le{}& 2\left\|p\right\|_{\mathcal{D}_\alpha}^2+2\left\|q_0\right\|_\sigma^2\\
&+4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|q_0(e^{it})-q_0(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}|\varphi(e^{it})|^2 d\sigma\,d\sigma\\
&+4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|\varphi(e^{it})-\varphi(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}|\,q_0(e^{i\theta})\,|^2 d\sigma\,d\sigma.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\|\varphi(p+\overline{q}_0)\right\|_{\mathcal{HD}_\alpha}^2 \le{}& 2\left\|p\right\|_\sigma^2+4\left\|\overline{q}_0\right\|_{\mathcal{HD}_\alpha}^2\\
&+4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|\varphi(e^{it})-\varphi(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}|q_0(e^{i\theta})|^2 d\sigma\,d\sigma.
\end{aligned}
$$

But $\|\overline{q}_0\|_{\mathcal{HD}_\alpha}=\|q_0\|_{\mathcal{D}_\alpha}$ and

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|\varphi(e^{it})-\varphi(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}|q_0(e^{i\theta})|^2 d\sigma\,d\sigma\\
&\quad\le 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|\varphi(e^{it})\;q_0(e^{it})-\varphi(e^{i\theta})q_0(e^{i\theta})|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}\,d\sigma\,d\sigma\\
&\qquad+2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|\varphi(e^{it})\;q_0(e^{i\theta})-\varphi(e^{it})q_0(e^{it})|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}\,d\sigma\,d\sigma\\
&\quad\le 2\left\|M_\varphi(q_0)\right\|_{\mathcal{D}_\alpha}^2+2\left\|q_0\right\|_{\mathcal{D}_\alpha}^2.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\left\|\varphi(p+\overline{q}_0)\right\|_{\mathcal{HD}_\alpha}^2 &\le 2\left\|p\right\|_{\mathcal{D}_\alpha}^2+4\left\|q_0\right\|_{\mathcal{D}_\alpha}^2+16\left\|q_0\right\|_{\mathcal{D}_\alpha}^2\\
&\le 20(\left\|p\right\|_{\mathcal{D}_\alpha}^2+\left\|q_0\right\|_{\mathcal{D}_\alpha}^2)\\
&\le 20(\left\|p+\overline{q}_0\right\|_{\mathcal{HD}_\alpha}^2.
\end{aligned}
$$

Thus,

$$\|M_\varphi\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \leq \sqrt{20}\,\|M_\varphi\|_{\mathcal{B}(\mathcal{D}_\alpha)}. \qquad \square$$

**Lemma 5.** *Assume* $\|M_F^C\| \leq 1$. *Then* $M_{FF^*} \in \mathcal{B}(\mathcal{HD}_\alpha)$ *and* $\|M_{FF^*}\| \leq 20$.

*Proof.* For $h \in \mathcal{D}_\alpha$, $M_{FF^*}(h) = M_F^R M_{F^*}^C(h) = M_F^R M_F^C(\overline{h})$. So $\|M_{FF^*}\| \leq \|M_F^C\|^2_{\mathcal{B}(\mathcal{HD}_\alpha, \oplus_1^\infty \mathcal{HD}_\alpha)}$ and by Lemma 4, $\|M_{FF^*}\| \leq 20$. $\square$

The next lemma proves that $M_{1/H}$ is in $M(\mathcal{D}_\alpha)$, provided $M_H \in M(\mathcal{D}_\alpha)$ and $H$ is bounded from below on $\mathbf{D}$.

**Lemma 6.** *Let* $H \in \mathcal{M}(\mathcal{HD}_\alpha)$ *with* $1 \geq |H(e^{it})| \geq \varepsilon > 0$, *for* $\sigma$ *almost everywhere* $t \in [-\pi, \pi]$. *Then* $M_{1/H} \in M(\mathcal{HD}_\alpha)$ *and* $\|M_{1/H}\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \leq (\sqrt{10}/\varepsilon^2)\|M_H\|_{\mathcal{B}(\mathcal{D}_\alpha)}$.

*Proof.* Let $r \in \mathcal{HD}_\alpha$ be a trigonometric polynomial on $\partial\mathbf{D}$, and let $\|M_H\|_{\mathcal{B}(\mathcal{HD}_\alpha)} = 1$. Then

$$\begin{aligned}
\left\|M_{1/H} r\right\|^2_{\mathcal{HD}_\alpha} &= \int_{-\pi}^{\pi} |r/H|^2\, d\sigma \\
&\quad + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|r/H(e^{it}) - rH(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^{\alpha+1}}\, d\sigma\, d\sigma \\
&\leq \frac{1}{\varepsilon^2}\int_{-\pi}^{\pi} |r^2|\, d\sigma \\
&\quad + \frac{1}{\varepsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|H(e^{i\theta})r(e^{it}) - H(e^{it})r(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^{\alpha+1}}\, d\sigma\, d\sigma \\
&\leq \frac{1}{\varepsilon^2}\int_{-\pi}^{\pi} |r^2|\, d\sigma \\
&\quad + \frac{2}{\varepsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |H(e^{i\theta})|^2 \frac{|r(e^{it}) - r(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^{\alpha+1}}\, d\sigma\, d\sigma \\
&\quad + \frac{2}{\varepsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|H(e^{i\theta}) - H(e^{it})|^2}{|e^{it} - e^{i\theta}|^{\alpha+1}} |r(e^{i\theta})|^2 d\sigma\, d\sigma
\end{aligned}$$

$$\begin{aligned}
&\le \frac{2}{\varepsilon^4} \|r\|^2_{\mathcal{HD}_\alpha} \\
&\quad + \frac{2}{\varepsilon^4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|H(e^{i\theta}) - H(e^{it})|^2}{|e^{it} - e^{i\theta}|^{\alpha+1}} |r(e^{i\theta})|^2 d\sigma\, d\sigma \\
&\le \frac{2}{\varepsilon^4} \|r\|^2_{\mathcal{HD}_\alpha} + \frac{2}{\varepsilon^4} (4 \|r\|^2_{H\mathcal{D}_\alpha}) \\
&= \frac{10}{\varepsilon^4} \|r\|^2_{\mathcal{HD}_\alpha} .
\end{aligned}$$

We conclude that

$$\left\|M_{1/H}\right\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \le \frac{\sqrt{10}}{\varepsilon^2} \|M_H\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \le \frac{\sqrt{10}\sqrt{20}}{\varepsilon^2} \|M_H\|_{\mathcal{B}(\mathcal{D}_\alpha)} . \qquad \square$$

**Lemma 7.** *Let* $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$*. Assume that* $\|M_F^C\| \le 1$ *and* $0 < \varepsilon^2 \le F(z)F(z)^*$ *for all* $z \in \mathbf{D}$*. Then, for* $h \in \mathcal{D}_\alpha$*, we have*

$$\left\| \frac{F^*}{FF^*} h \right\|^2_{H\mathcal{D}_\alpha} \le \frac{10 \cdot 20^4}{\varepsilon^8} \|h\|^2_{\mathcal{D}_\alpha} .$$

*Proof.* Let $r$ be a trigonometric polynomial in $\partial\mathbf{D}$. Then, by Lemma 4,

$$\begin{aligned}
\|F^* r\|^2_{\mathcal{HD}_\alpha} &= \left\|M_F^C(\overline{r})\right\|^2_{\mathcal{HD}_\alpha} \\
&\le \left\|M_F^C\right\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)} \|\overline{r}\|^2_{\mathcal{HD}_\alpha} \\
&\le 20 \left\|M_F^C\right\|^2_{\mathcal{B}(\mathcal{D}_\alpha)} \|r\|^2_{\mathcal{HD}_\alpha} .
\end{aligned}$$

By Lemmas 5 and 6, we have

$$\left\|M_{(FF^*)^{-1}}\right\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)} \le \frac{10 \cdot 20}{\varepsilon^8} \|M_{FF^*}\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)} \le \frac{10 \cdot 20^3}{\varepsilon^8}.$$

Now, for h in $\mathcal{D}_\alpha$,

$$\begin{aligned}
\left\|(FF^*)^{-1} F^* h\right\|^2 &= \left\|M_{(FF^*)^{-1}} (F^* h)\right\|^2 \\
&\le \frac{10 \cdot 20^2}{\varepsilon^8} \|F^* h\|^2 \\
&\le \frac{10 \cdot 20^4}{\varepsilon^8} \|h\|^2_{\mathcal{D}_\alpha} . \qquad \square
\end{aligned}$$

The next lemma will enable us to write down the most general solution of $F\underline{U}_h(z) = h(z)$. For completeness, we include a proof.

**Lemma 8.** *Let* $\{c_j\}_{j=1}^{\infty} \in l^2$ *and* $C = (c_1, c_2, \dots) \in \mathcal{B}(l^2, \mathbf{C})$*. Then there exists a* $Q$ *such that entries of* $Q$ *are either* $0$ *or* $+c_j$ *or* $-c_j$ *for some* $j$ *and* $(CC^*)I - C^*C = QQ^*$*.*

*Proof.* For $k = 1, 2, \dots,$ let

$$A_k = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ c_{k+1} & c_{k+2} & c_{k+3} & \cdots \\ -c_k & 0 & 0 & \cdots \\ 0 & -c_k & 0 & \cdots \\ 0 & 0 & -c_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where the first $k$ rows of $A_k$ have only 0 entries. Then

$$A_k A_k^* = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \sum_{j=k+1}^{\infty} |c_j|^2 & -\overline{c}_k c_{k+2} & -\overline{c}_k c_{k+3} & \cdots \\ 0 & \cdots & 0 & -c_k \overline{c}_{k+2} & |c_k|^2 & 0 & \cdots \\ 0 & \cdots & 0 & -c_k \overline{c}_{k+3} & 0 & |c_k|^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} A_k A_k^* &= \begin{bmatrix} \sum_{k\neq 1}^{\infty} |c_k|^2 & -\overline{c}_1 c_2 & -\overline{c}_1 c_3 \cdots & \\ -\overline{c}_2 c_1 & \sum_{k\neq 2}^{\infty} |c_k|^2 & -\overline{c}_2 c_3 & \cdots \\ -\overline{c}_3 c_1 & -\overline{c}_3 c_2 & \sum_{k\neq 3}^{\infty} |c_k|^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= (CC^*)I - C^*C. \end{aligned}$$

This completes the proof, if we let

$$Q = [A_1, A_2, \dots] \in B\left(\bigoplus_1^{\infty} l^2, l^2\right). \qquad \square$$

Since $CC^*/C^*C$ is the projection onto the range of $C$, we see that $QQ^*/C^*C$ is the projection onto the kernel of $C$.

**Lemma 9.** *Let* $\{f_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$. *Assume that, for each* $j$, $f_j$ *is analytic on* $\mathbf{D}_{1+\varepsilon}(0)$ *and* $\|M_F^C\|_{\mathcal{B}(\mathcal{D}_\alpha)} \leq 1$. *Associate* $Q(z)$ *to* $F(z)$ *for each* $|z| = 1$ *as in the previous lemma. Then*

$$\|M_Q\|_{\mathcal{B}(\oplus_1^\infty \mathcal{H}\mathcal{D}_\alpha)} \leq \sqrt{86}.$$

*Proof.* $F(z) = (f_1(z), f_2(z), \dots)$, $z \in \mathbf{D}_{1+\varepsilon}(0)$. Recall that $\|M_F^C\|_{\mathcal{B}(\mathcal{D}_\alpha)} \leq 1$ implies $\|F(z)\|_{l^2}^2 \leq 1$. By Lemma 4, for each $z \in \overline{\mathbf{D}}$ there exists a $Q(z)$ such that

$$F(z)F(z)^*I - F(z)^*F(z) = Q(z)Q(z)^*.$$

Thus, $Q(z)Q(z)^* \leq (F(z)F(z)^*)I_{l^2}$, for all $z \in \overline{\mathbf{D}}$, so

$$\|Q(z)\|_{\mathcal{B}(l^2)} \leq 1. \tag{1}$$

Now let $\underline{r} \in \oplus_1^\infty \mathcal{H}\mathcal{D}_\alpha$ be a trigonometric polynomial in $z$; then

$$\begin{aligned}\|Q\underline{r}\|_{\oplus_1^\infty \mathcal{H}\mathcal{D}_\alpha} &= \int_{\partial\mathbf{D}} \left\|(Q\underline{r})(e^{it})\right\|^2 d\sigma \\ &\quad + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|(Q\underline{r})(e^{it}) - (Q\underline{r})(e^{i\theta})\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} d\sigma\, d\sigma.\end{aligned}$$

First, let's consider

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|F(e^{it}) - F(e^{i\theta})\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \|\underline{r}(e^{it})\|^2 d\sigma\, d\sigma.$$

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|F(e^{it}) - F(e^{i\theta})\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \|\underline{r}(e^{it})\|^2 d\sigma\, d\sigma \\ &\quad = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|(F\underline{r})(e^{it}) - F(e^{i\theta})\underline{r}(e^{it})\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma\end{aligned}$$

$$\begin{aligned}
&\leq 2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|F(e^{it})\right\|^2 \|\underline{r}(e^{it})-\underline{r}(e^{i\theta})\|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}\, d\sigma d\sigma \\
&\quad + 2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|(F\underline{r})(e^{it})-(F\underline{r})(e^{i\theta})\right\|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}\, d\sigma d\sigma \\
&\leq 2 \left\|\underline{r}\right\|^2_{\mathcal{HD}_\alpha} + 2 \left\|F\underline{r}\right\|^2_{\mathcal{B}(\mathcal{HD}_\alpha,\oplus\mathcal{HD}_\alpha)} \\
&\leq 2 \left\|\underline{r}\right\|^2_{\mathcal{HD}_\alpha} + 2 \left\|M_F^C\right\|^2_{\mathcal{B}(\mathcal{HD}_\alpha,\oplus\mathcal{HD}_\alpha)} \left\|\underline{r}\right\|^2_{\mathcal{HD}_\alpha} \\
&\leq 2 \left\|\underline{r}\right\|^2_{\mathcal{HD}_\alpha} + 2\cdot 20 \left\|\underline{r}\right\|^2_{\mathcal{HD}_\alpha}.
\end{aligned}$$

So

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|F(e^{it})-F(e^{i\theta})\right\|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} \|\underline{r}(e^{it})\|^2 d\sigma\, d\sigma \leq 42 \left\|\underline{r}\right\|^2_{\mathcal{HD}_\alpha}.$$

Now let $\underline{r} \in \oplus_1^\infty \mathcal{HD}_\alpha$; then

$$\begin{aligned}
\|Q\underline{r}\|^2_{\mathcal{HD}_\alpha} &= \int_{-\pi}^{\pi} \left\|(Q\underline{r})(e^{it})\right\|^2_{l^2} d\sigma \\
&\quad + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|(Q\underline{r})(e^{it})-(Q\underline{r})(e^{i\theta})\right\|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}\, d\sigma\, d\sigma.
\end{aligned}$$

Estimating, we get

$$\begin{aligned}
\|Q\underline{r}\|^2_{\mathcal{HD}_\alpha} \leq & \int_{-\pi}^{\pi} \left\|\underline{r}(e^{it})\right\|^2_{l^2} d\sigma \\
& + 2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|Q(e^{it})-Q(e^{i\theta})\right\|^2_{\mathcal{B}(l^2)}}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} \left\|\underline{r}(e^{it})\right\|^2_{l^2}\, d\sigma\, d\sigma \\
& + 2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|\underline{r}(e^{it})-\underline{r}(e^{i\theta})\right\|^2_{l^2}}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} \left\|Q(e^{i\theta})\right\|^2_{\mathcal{B}(l^2)}\, d\sigma\, d\sigma.
\end{aligned}$$

If we set $C_j = f_j(e^{it}) - f_j(e^{i\theta})$, then we have $\{f_j(e^{it}) - f_j(e^{i\theta})\}_{j=1}^\infty$ is in $l^2$. Let

$$F(e^{it}) - F(e^{i\theta}) = (f_1(e^{it}) - f_1(e^{i\theta}),\, f_2(e^{it}) - f_2(e^{i\theta}), \dots).$$

It is important to observe that, by linearity, $Q(e^{it}) - Q(e^{i\theta})$ is the $Q$ associated with $F(e^{it}) - F(e^{i\theta})$ . Thus, by Lemma 8,

$$(Q(e^{it}) - Q(e^{i\theta}))(Q(e^{it}) - Q(e^{i\theta}))^* \leq \left\| F(e^{it}) - F(e^{i\theta}) \right\|_{l^2} I_{l^2},$$

which implies that

$$\left\| Q(e^{it}) - Q(e^{i\theta}) \right\|_{\mathcal{B}(l^2)}^2 \leq \left\| F(e^{it}) - F(e^{i\theta}) \right\|_{l^2}^2 .$$

We now have, by (1),

$$\begin{aligned} \|Q\underline{r}\|^2_{\oplus_1^\infty \mathcal{HD}_\alpha} &\leq \int_{-\pi}^{\pi} \left\| \underline{r}(e^{it}) \right\|^2 d\sigma \\ &\quad + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left\| \underline{r}(e^{it}) - \underline{r}(e^{i\theta}) \right\|^2}{\left| e^{it} - e^{i\theta} \right|^{\alpha+1}} \left\| Q(e^{i\theta}) \right\|^2 d\sigma \, d\sigma \\ &\quad + 2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left\| Q(e^{it}) - Q(e^{i\theta}) \right\|^2}{\left| e^{it} - e^{i\theta} \right|^{\alpha+1}} \left\| \underline{r}(e^{it}) \right\|^2 d\sigma \, d\sigma \\ &\leq 2 \left\| \underline{r} \right\|^2_{\mathcal{HD}_\alpha} \\ &\quad + 2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left\| F(e^{it}) - F(e^{i\theta}) \right\|^2}{\left| e^{it} - e^{i\theta} \right|^{\alpha+1}} \left\| \underline{r}(e^{it}) \right\| \, d\sigma \, d\sigma \\ &\leq 2 \left\| \underline{r} \right\|^2_{\mathcal{HD}_\alpha} + 2(42) \left\| \underline{r} \right\|^2_{\mathcal{HD}_\alpha}, \text{ using (2).} \end{aligned}$$

Thus,

$$\|M_Q\|_{\mathcal{B}(\oplus_1^\infty \mathcal{HD}_\alpha, \oplus_1^\infty \mathcal{HD}_\alpha)} \leq \sqrt{86}. \qquad \square$$

Let's summarize the results we have obtained so far.

First, set $\widehat{\underline{\mathcal{K}}} = (Q^* \widehat{F'^* h)/(} F F^*)^2$. Then, by Lemma 9, we have

$$\left\| M_Q \widehat{\underline{\mathcal{K}}} \right\|_{\mathcal{B}(\oplus_1^\infty \mathcal{HD}_\alpha)} \leq \sqrt{86} \left\| \widehat{\underline{\mathcal{K}}} \right\|_{\oplus_1^\infty \mathcal{HD}_\alpha} .$$

So we are left with showing that

$$\left\| \widehat{\underline{\mathcal{K}}} \right\|_{\oplus_1^\infty \mathcal{HD}_\alpha} \leq \frac{C_\alpha}{\varepsilon^3} \, \|h\|_{\mathcal{D}_\alpha} .$$

Now by Lemma 8, $\|Q(z)^*/\sqrt{F(z)F(z)^*}\|_{\mathcal{B}(l^2)} \leq 1$, so we have

$$\begin{aligned}
&\int_{\mathbf{D}} \|\mathcal{K}(z)\|^2 (1-|z|^2)^{1-\alpha}\, dA(z) \\
&\qquad = \int_{\mathbf{D}} \left\| \frac{Q^{*\prime} F^{\prime *} h}{(FF^*)^2}(z) \right\|^2 (1-|z|^2)^{1-\alpha} dA(z) \\
&\qquad \leq \frac{1}{\varepsilon^6} \int_{\mathbf{D}} \left\| \frac{Q^* F^{\prime *} h}{\sqrt{FF^*}}(z) \right\|^2 (1-|z|^2)^{1-\alpha} dA(z) \\
&\qquad \leq \frac{1}{\varepsilon^6} \int_{\mathbf{D}} \int_{\mathbf{D}} \|F^{\prime *} h(z)\|^2 (1-|z|^2)^{1-\alpha} dA(z) \\
&\qquad \leq \frac{1}{\varepsilon^6} \left\| M_F^C(\overline{h}) \right\|^2_{\oplus_1^\infty \mathcal{H}\mathcal{D}_\alpha} \\
&\qquad \leq \frac{20}{\varepsilon^6} \|h\|^2_{\mathcal{D}_\alpha}.
\end{aligned}$$

Thus we are done if we can show that

$$\left\| \underline{\widehat{\mathcal{K}}} \right\|^2_{\oplus_1^\infty \mathcal{H}\mathcal{D}_\alpha} \leq C_\alpha \int_{\mathbf{D}} \|\mathcal{K}(z)\|^2 (1-|z|^2)^{1-\alpha}\, dA(z).$$

In the following we find an estimate for $\|\widehat{k}\|_{\mathcal{D}_\alpha}$, where $\widehat{k}$ is the Cauchy transform of a function, $k$, which is smooth on $\mathbf{D}_{1+\varepsilon}$.

**Lemma 10.** *Let $k \in C^2(\mathbf{D}_{1+\varepsilon})$ and $\widehat{k}(z) = \int_{\mathbf{D}} \frac{k(w)}{z-w}\, dA(w)$, where $z \in \mathbf{D}$ and $dA(w) = (1/\pi)dm$. Then $\|\widehat{k}\|^2_\alpha \sim \int_{\mathbf{D}} \int_{\mathbf{D}} k(w)\overline{k(z)}[1/(1-w\overline{z})^{1+\alpha}]\, dA(w)\, dA(z)$.*

*Proof.* $\widehat{k}(e^{i\theta}) = \sum_{n=1}^\infty \widehat{k}_{-n}\, e^{-in\theta}$, where $\widehat{k}_{-n} = \langle \widehat{k}, e^{-int}\rangle = \int_{-\pi}^{\pi} \widehat{k}(e^{it})e^{int}\, d\sigma(t)$.

$$\begin{aligned}
\int_{-\pi}^{\pi} \widehat{k}(e^{it})e^{int} d\sigma(t) &= \int_{-\pi}^{\pi} \left( -\int_{\mathbf{D}} \frac{k(w)}{w-e^{it}}\, dA(w) \right) e^{int} d\sigma(t) \\
&= \int_{\mathbf{D}} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} e^{it(n-1-k)} d\sigma(t) w^k k(w)\, dA(w) \\
&= \int_{\mathbf{D}} k(w) w^{n-1}\, dA(w).
\end{aligned}$$

On the other hand, since $\widehat{k}(z) = \sum_{n=1}^{\infty} \widehat{k}_{-n}\overline{z}^n$ for $z \in \partial\mathbf{D}$, the weighted Dirichlet norm for $\widehat{k}$ is given by

$$\begin{aligned}\left\|\widehat{k}\right\|_{\alpha}^{2} &= \sum_{n=1}^{\infty}(|-n|+1)^{\alpha}\left|\widehat{k}_{-n}\right|^{2}\\ &= \sum_{n=1}^{\infty}(n+1)^{\alpha}\left|\widehat{k}_{-n}\right|^{2}\\ &= \sum_{n=1}^{\infty}(n+1)^{\alpha}\left|\int_{\mathbf{D}} k(w)w^{n-1}dA(w)\right|^{2}.\end{aligned}$$

By Stirling's formula there is a constant, $C_\alpha$, so that for all $n = 1, 2, \ldots,$

$$\frac{1}{C_\alpha}\frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)\Gamma(n)} \le (n+1)^{\alpha} \le C_\alpha \frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)\Gamma(n)}.$$

Thus,

$$\left\|\widehat{k}\right\|_{\alpha}^{2} \le C_\alpha \sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)\Gamma(n)}\int_{\mathbf{D}}\int_{\mathbf{D}} k(w)\overline{k(z)}w^{n-1}\overline{z}^{n-1}dA(w)\,dA(z)$$

and

$$\begin{aligned}\int_{\mathbf{D}}\int_{\mathbf{D}} k(w)\overline{k(z)}\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)\Gamma(n+1)}(w\,\overline{z})^{n}dA(w)\,dA(z)\\ = \int_{\mathbf{D}}\int_{\mathbf{D}} k(w)\overline{k(z)}\frac{1}{(1-w\overline{z})^{1+\alpha}}dA(w)\,dA(z).\end{aligned}$$

This completes the proof. $\square$

**Lemma 11.** *There is a constant $C_\alpha < \infty$, so that if $k$ is smooth on $\mathbf{D}_{1+\varepsilon}$, then*

$$\|\widehat{k}\|_{\alpha}^{2} \le C_\alpha \int_{\mathbf{D}} |k(z)|^{2}(1-|z|^{2})^{(1-\alpha)}dA(z).$$

*Proof.* Set $L(z) = |k(z)|(1-|z|^2)^{(1-\alpha)/2}$. Then by the previous lemma,

$$\left\|\widehat{k}\right\|_\alpha^2 \le \int_{\mathbf{D}}\int_{\mathbf{D}} L(w)L(z)K(z,\, w)\, dA(z)\, dA(w),$$

for

$$K(z,w) = \frac{1}{(1-|w|^2)^{(1-\alpha)/2}(1-|z|^2)^{(1-\alpha)/2}|1-w\overline{z}|^{\alpha+1}}.$$

So we need only show that there exists a $C_\alpha < \infty$, so that

$$\int_{\mathbf{D}}\int_{\mathbf{D}} L(z)\, L(w)\, K(z,w)\, dA(z)\, dA(w) \le C_\alpha^2 \int_{\mathbf{D}} L(z)^2\, dA(z),$$

for all real valued $L \in L^2(A)$. By Schur's test (see Zhu [**13**, page 45]), we must find a positive function, $P$, on $\mathbf{D}$ satisfying

$$\int_{\mathbf{D}} \frac{P(z)}{(1-|w|^2)^{(1-\alpha)/2}(1-|z|^2)^{(1-\alpha)/2}|1-w\overline{z}|^{1+\alpha}}\, dA(z) \le C_\alpha P(w),$$

for all $w \in \mathbf{D}$. Using $P(z) = (1-|z|^2)^{(\alpha-2)/4}$, this reduces to showing that

$$\int_{\mathbf{D}} \frac{(1-|z|^2)^{(3/4)\alpha-1}}{|1-w\overline{z}|^{1+\alpha}}\, dA(z) \le C_\alpha \frac{1}{(1-|w|^2)^{\alpha/4}} \quad \text{for all } w \in \mathbf{D}.$$

But this estimate follows from the well-known result that, for $t > -1$ and $c > 0$, we have

$$\int_{\mathbf{D}} \frac{(1-|z|^2)^t}{|1-w\overline{z}|^{2+t+c}}\, dA(z) \approx \frac{1}{(1-|w|^2)^c} \quad \text{as } |w| \uparrow 1 [\mathbf{13}, \text{page } 27]. \qquad \square$$

We are now ready to give the proof of Theorem A$'$ in the smooth case.

*Proof of Theorem* A$'$. Assume that $\{f_j\}_{j=1}^{\infty}$ are analytic in $|z| < 1+\delta$ for all $j = 1, 2, \dots$ and $\|M_F^C\| \le 1$ and $0 < \varepsilon^2 \le F(z)F(z)^*$ for all

$z \in \mathbf{D}$, where $F(z) = (f_1(z), f_2(z), \dots)$. Let $h$ be analytic in $|z| < 1+\delta$. Define

$$(*) \qquad \mathbf{U}_h = F^*(FF^*)^{-1}h - Q\left(\frac{\widehat{Q^*F'^*h}}{(FF^*)^2}\right)$$

pointwise on $\overline{\mathbf{D}}$.

Since $\mathbf{U}_h$ is analytic in $\mathbf{D}$ and $M_F^R(\mathbf{U_h}) = h$, we need only show that

$$\|\mathbf{U}_h\|_{\oplus_1^\infty \mathcal{D}_\alpha} \leq \frac{C_\alpha}{\varepsilon^4} \|h\|_{\mathcal{D}_\alpha}.$$

Taking norms in $(*)$ above and letting $E_\alpha$ denote the best constant (depending only upon $\alpha$) from Lemma 11, we get

$$\begin{aligned}
\|\mathbf{U_h}\|_{\oplus_1^\infty \mathcal{D}_\alpha} &= \left\| F^*(FF^*)^{-1}h + Q\frac{\widehat{Q^*F'^*h}}{(FF^*)^2}\right\|_{\oplus_1^\infty \mathcal{HD}_\alpha} \\
&\leq \left\|F^*(FF^*)^{-1}h\right\| + \left\| Q\frac{\widehat{Q^*F'^*h}}{(FF^*)^2}\right\|_{\oplus_1^\infty \mathcal{HD}_\alpha} \\
&\leq \frac{20^2\cdot 10}{\varepsilon^4}\|h\|_{\mathcal{D}_\alpha} + \sqrt{86}\left\| \frac{\widehat{Q^*F'^*h}}{(FF^*)^2}\right\|_{\oplus_1^\infty \mathcal{HD}_\alpha} \\
&\leq \frac{20^2\cdot 10}{\varepsilon^4}\|h\| + \sqrt{86}\left\|\widehat{k}\right\|_{\oplus_1^\infty \mathcal{HD}_\alpha} \\
&\leq \frac{20^2\cdot 10}{\varepsilon^4}\|h\|_{\mathcal{D}_\alpha} + \sqrt{86}\,\frac{E_\alpha}{\varepsilon^3}\|h\|_{\mathcal{D}_\alpha} \\
&\leq \left(\frac{20^2\cdot 10}{\varepsilon^4} + \sqrt{86}\frac{E_\alpha}{\varepsilon^3}\right)\|h\|_{\mathcal{D}_\alpha}
\end{aligned}$$

where $C_\alpha = 20^2 \cdot 10 + \sqrt{86}\, E_\alpha$. ◻

Note that a better estimate for solutions of order "$1/\varepsilon^3$" would follow if an estimate of order "$1/\varepsilon^3$" could be obtained in Lemma 7.

We will show that the same estimates hold without our smoothness assumptions. The following two lemmas actually hold for any complete Nevanlinna-Pick reproducing kernel which is analytic on the unit ball or the unit polydisk in $\mathbf{C}^n$.

**Lemma 12.** *Let* $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D}_\alpha)$ *with* $\|M_F^C\| = 1$. *For* $0 \leq r \leq 1$, *let* $F_r(z) = F(rz)$. *Then* $\|M_{F_r}^C\| \leq \|M_F^C\|$, *and thus* $F_r \in M(\mathcal{D}_\alpha, \oplus_1^\infty \mathcal{D}_\alpha)$.

*Proof.* We claim that

$$I - M_{F_r}^C (M_{F_r}^C)^* \geq 0.$$

That is, for any $\{\underline{c}_j\}_{j=1}^n \subset l^2$ and $\{z_j\}_{j=1}^n \subset D$,

$$0 \leq \sum_{j=1}^n \sum_{k=1}^n \langle (I - F(rz_k)F(rz_j)^*)\underline{c}_j, \underline{c}_k \rangle k_{z_j}(rz_k). \tag{3}$$

But

$$(3) = \sum_{j=1}^n \sum_{k=1}^n \langle (I - F(rz_k)F(rz_j)^*)\underline{c}_j, \underline{c}_k \rangle k_{rz_j}(rz_k) \cdot \left[ \frac{k_{z_j}(z_k)}{k_{rz_j}(rz_k)} \right]. \tag{4}$$

The expression (4) without the "boxed terms" is positive since $I - M_F^C M_F^{C*} \geq 0$. We need only note that the matrix whose $(k, j)$th entry is the boxed term is positive. Then the Schur product theorem gives us that (4) is positive.

Now $k_w(z)$ is a complete Nevanlinna-Pick kernel, as previously noted, so

$$1 - \frac{1}{k_w(z)} = \sum_{n=1}^\infty c_n z^n \overline{w}^n \quad \text{and } c_n > 0 \text{ for all } n.$$

Thus,

$$\begin{aligned} \frac{k_{z_j}(z_k)}{k_{z_j r}(z_k r)} &= \left(1 - \sum_{n=1}^\infty c_n r^{2n} \overline{z}_j^n z_k^n \right) k_{z_j}(z_k) \\ &= \left(1 - \sum_{n=1}^\infty c_n \overline{z}_j^n z_k^n + \sum_{n=1}^\infty (1 - r^{2n}) c_n \overline{z}_j^n z_k^n \right) k_{z_j}(z_k) \\ &= 1 + \sum_{n=1}^\infty c_n (1 - r^{2n}) z_k^n \overline{z}_j^n k_{z_j}(z_k). \end{aligned}$$

Hence, $[k_{z_j}(z_k)/k_{rz_j}(rz_k)]_{j,k=1}^n$ is positive, and we are done. □

**Lemma 13.** *Let $\mathcal{F} \in M(\oplus_1^\infty \mathcal{D}_\alpha)$. Then $s - \lim_{r\to 1^-} M^*_{\mathcal{F}_r} = M^*_{\mathcal{F}}$.*

*Proof.* Assume that $\|M_{\mathcal{F}}\|_{B(\oplus_1^\infty \mathcal{D}_\alpha)} \leq 1$. By Lemma 12, $\|M_{\mathcal{F}_r}\|_{B(\oplus_1^\infty \mathcal{D}_\alpha)} \leq 1$ for all $0 \leq r \leq 1$. Thus, we need only show that $\lim_{r\to 1^-} \|(M^*_{\mathcal{F}_r} - M^*_{\mathcal{F}})\underline{x}\| = 0$ for $\underline{x}$ in a dense subset of $\oplus_1^\infty \mathcal{D}_\alpha$. By considering finite sums of the form $\sum_{j=1}^N \underline{c}_j k_{z_j}$, with $\{\underline{c}_j\}_{j=1}^N \subset l^2$ and $\{z_j\}_{j=1}^N \subset D$, we need only show that for $\underline{e} \in l^2$ and $z \in D$, $\lim_{r\to 1^-} \|(M^*_{\mathcal{F}_r} - M^*_{\mathcal{F}})\underline{e}k_z\|_{\mathcal{D}_\alpha} = 0$.

Now

$$\begin{aligned}(M^*_{\mathcal{F}_r} - M^*_{\mathcal{F}})(\underline{e}k_z) &= \mathcal{F}(rz)^*\underline{e}k_z - \mathcal{F}(z)^*\underline{e}k_z \\ &= \mathcal{F}(rz)^*\underline{e}k_{rz}\frac{k_z}{k_{rz}} - \mathcal{F}(z)^*\underline{e}k_z \\ &= M^*_{\mathcal{F}}(\underline{e}k_{rz})\frac{k_z}{k_{rz}} - M^*_{\mathcal{F}}(\underline{e}k_z).\end{aligned}$$

Thus,

$$\begin{aligned}(5) \quad \|(M^*_{\mathcal{F}_r} - M^*_{\mathcal{F}})(\underline{e}k_z)\|_{\mathcal{D}_\alpha} &\leq \|k_{rz} - k_z\|_{\mathcal{D}_\alpha} \\ &+ \|k_z\|_{\mathcal{D}_\alpha}\left(\sup_{w\in D}\left|\frac{k_z(w)}{k_{rz}(w)} - 1\right| + \sup_{w\in D}\left|\left(\frac{k_z(w)}{k_{rz}(w)}\right)'\right|\right).\end{aligned}$$

But, for fixed $z$, the righthand side of (5) goes to 0 as $r \uparrow 1$, by an explicit calculation using $k_z(w) = 1/(1-\overline{z}w)^{1-\alpha}$. □

Now we are in a position to prove Theorem A′ and hence the $\mathcal{M}(\mathcal{D}_\alpha)$ Corona theorem for the general case, without smoothness assumptions on $F(z)$.

*Proof.* Let $\{f_j\}_{j=1}^\infty \subseteq \mathcal{M}(\mathcal{D}_\alpha)$, $\|M_F^C\| \leq 1$ and $0 < \varepsilon^2 \leq F(z)F(z)^*$ for all $z < 1$. By Lemma 13 for $0 \leq r < 1$, we have $\|M_{F_r}^C\| \leq 1$ and $0 < \varepsilon^2 \leq F_r(z)F_r(z)^*$ for all $z < 1$.

By the proof of Theorem A′ in the smooth case, we have

$$\left(\frac{C_\alpha}{\varepsilon^4}\right)^{-2} I \leq M_{F_r}^R (M_{F_r}^R)^* \leq I.$$

By Theorem B, there exists a $G_r \in M(\mathcal{D}_\alpha, \oplus_1^\infty \mathcal{D}_\alpha)$ so that

$$M_{F_r}^R (M_{G_r}^C)^* = I \quad \text{and } \|M_{G_r}^C\| \le \frac{C_\alpha}{\varepsilon^4}.$$

By compactness, we may choose a net with $(M_{G_{r_n}}^C)^* \overset{WOT}{\to} (M_G^C)^*$ as $r_n \to 1^-$. Note that $M_G^C \in M(\mathcal{D}_\alpha, \oplus_1^\infty \mathcal{D}_\alpha)$, since the multiplier algebra (as operators) is weak operator closed (WOT).

Now by Lemma 13, $(M_{F_{r_n}}^R)^* \overset{s}{\to} (M_F^R)^*$. Thus, we get

$$I = (M_{G_{r_n}}^C)^* (M_{F_{r_n}}^R)^* \overset{WOT}{\longrightarrow} (M_G^C)^* (M_F^R)^*,$$

which implies $M_F^R M_G^C = I$ with entries of $G$ in $\mathcal{M}(\mathcal{D}_\alpha)$ and $\|M_G^C\| \le C_\alpha/\varepsilon^4$.

This ends the proof of the $\mathcal{M}(\mathcal{D}_\alpha)$ Corona theorem. $\square$

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