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V-GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

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ABSTRACT. In this paper, we study some properties of V-Gorenstein projective, injective and flat modules. By analogy with the projective, injective and flat modules, we consider some properties and connections of V-Gorenstein projective, injective and flat modules.

1. Introduction. We use R-Mod (respectively, R^{op}-Mod) to denote the category of left (respectively, right) R-modules. For any R-module M, $\mathrm{pd}\,(M)$ (respectively, $\mathrm{id}\,(M)$, $\mathrm{fd}\,(M)$) denotes the projective (respectively, injective, flat) dimension. The character module $\mathrm{Hom}_Z(M, Q/Z)$ is denoted by M^+.

Since Auslander and Bridger [**1**] introduced the G-dimension of a finitely generated module in the study of Gorenstein dimensions of modules, it has been the subject of numerous publications. The use of equivalence introduced by Foxby has shown to be of great utility in this study. Enochs, Jenda and López-Ramos [**7**] studied V-Gorenstein modules relative to a dualizing module. These modules constitute a generalization of the well-known Gorenstein modules and at the same time an extension to the noncommutative case of Ω-Gorenstein modules [**6**]. They proved that, under certain conditions on the finiteness of projective dimension for flat modules, V-Gorenstein injectives and projectives form part of perfect cotorsion theories. In [**15**], we introduced the definition of a V-Gorenstein flat module and gave some characterizations of V-Gorenstein flat modules. In this paper, we continue the study of V-Gorenstein projective, injective and flat modules.

Keywords and phrases. V-Gorenstein projective module, V-Gorenstein injective module, V-Gorenstein flat module, dualizing module.

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Let R be a left and right Noetherian ring, and let V be an (R,R)-bimodule such that $\mathrm{End}\,({}_RV)=R$ and $\mathrm{End}\,(V_R)=R$. Then V is said to be a dualizing module if it satisfies the following three conditions:

(i) $\mathrm{id}\,({}_RV)\le r$ and $\mathrm{id}\,(V_R)\le r$ for some integer r;

(ii) $\mathrm{Ext}^i_R(V,V)=0$ and $\mathrm{Ext}^i_{R^{\mathrm{op}}}(V,V)=0$ for all $i\ge 1$;

(iii) ${}_RV$ and V_R are finitely generated.

The preceding definition is given in [**7**] for a bimodule ${}_SV_R$, where S and R are left and right Noetherian rings, respectively, but throughout this paper, we will consider the case $S=R$.

It is immediate that, if R is a Cohen-Macaulay local ring admitting a dualizing module Ω or R is an n-Gorenstein ring, then Ω and R are dualizing modules in this sense.

Let ${}_RV_R$ be a dualizing module of R. Enochs, Jenda and López-Ramos [**9**] introduced the left, right Auslander class $\mathcal{A}^l(R)$, $\mathcal{A}^r(R)$ and the left, right Bass class $\mathcal{B}^l(R)$, $\mathcal{B}^r(R)$:

$$\begin{aligned}
\mathcal{A}^l(R) &= \big\{M\in R\text{-Mod}\,|M\cong \mathrm{Hom}_R(V,V\otimes_R M),\mathrm{Tor}^R_i(V,M)\\
&\qquad = \mathrm{Ext}^i_R(V,V\otimes_R M)=0 \text{ for all } i\ge 1\big\},\\
\mathcal{A}^r(R) &= \big\{M\in \text{Mod-}R|M\cong \mathrm{Hom}_{R^{\mathrm{op}}}(V,M\otimes_R V),\mathrm{Tor}^R_i(M,V)\\
&\qquad = \mathrm{Ext}^i_{R^{\mathrm{op}}}(V,M\otimes_R V)=0 \text{ for all } i\ge 1\big\},\\
\mathcal{B}^l(R) &= \big\{N\in R\text{-Mod}\,|V\otimes_R \mathrm{Hom}_R(V,N)\cong N,\mathrm{Ext}^i_R(V,N)\\
&\qquad = \mathrm{Tor}^R_i(V,\mathrm{Hom}_R(V,N))=0 \text{ for all } i\ge 1\big\},\\
\mathcal{B}^r(R) &= \big\{N\in \text{Mod-}R|\mathrm{Hom}_{R^{\mathrm{op}}}(V,N)\otimes_R V\cong N,\mathrm{Ext}^i_{R^{\mathrm{op}}}(V,N)\\
&\qquad = \mathrm{Tor}^R_i(\mathrm{Hom}_{R^{\mathrm{op}}}(V,N),V)=0 \text{ for all } i\ge 1\big\}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
V\otimes_R - : \mathcal{A}^l(R) &\leftrightarrows \mathcal{B}^l(R) : \mathrm{Hom}_R(V,-)\\
-\otimes_R V : \mathcal{A}^r(R) &\leftrightarrows \mathcal{B}^r(R) : \mathrm{Hom}_{R^{\mathrm{op}}}(V,-)
\end{aligned}$$

give equivalences between the two subcategories.

Now let

$$\begin{aligned}
\mathcal{W} &= \big\{W\in R\text{-Mod}\mid W\cong V\otimes_R P,\ \text{where } P\in R\text{-Mod is projective}\},\\
\mathcal{X} &= \big\{X\in R\text{-Mod}\mid X\cong V\otimes_R F,\ \text{where } F\in R\text{-Mod is flat}\},\\
\mathcal{U} &= \big\{U\in R^{\mathrm{op}}\text{-Mod}\mid U\cong \mathrm{Hom}_{R^{\mathrm{op}}}(V,E),\\
&\qquad \text{where } E\in R^{\mathrm{op}}\text{-Mod is injective}\}.
\end{aligned}$$

Then clearly $\mathcal{W} \subseteq \mathcal{X} \subseteq \mathcal{B}^l(R)$ and $\mathcal{U} \subseteq \mathcal{A}^r(R)$. Every right R-module has a $\mathcal{U}$-preenvelope and every left R-module has a $\mathcal{W}$-precover and an $\mathcal{X}$-precover. The right $\mathcal{U}$-dimension of a right R-module, left $\mathcal{W}$-dimension and left $\mathcal{X}$-dimension of a left R-module are defined as usual.

We recall from [**7**] that a right R-module M is V-Gorenstein injective if there is an exact sequence

$$\cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow U^0 \longrightarrow U^1 \longrightarrow \cdots$$

of modules in $\mathcal{U}$ with $M = \operatorname{Ker}(U^0 \to U^1)$ such that $\operatorname{Hom}_{R^{\mathrm{op}}}(U, -)$ and $\operatorname{Hom}_{R^{\mathrm{op}}}(-, U)$ leave the sequence exact whenever $U \in \mathcal{U}$. A left R-module M is V-Gorenstein projective if there is an exact sequence

$$\cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

of modules in $\mathcal{W}$ with $M = \operatorname{Ker}(W^0 \to W^1)$ such that $\operatorname{Hom}_R(W, -)$ and $\operatorname{Hom}_R(-, W)$ leave the sequence exact whenever $W \in \mathcal{W}$. A left R-module M is V-Gorenstein flat if there exists an exact sequence

$$\cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

of modules in $\mathcal{X}$ with $M = \operatorname{Ker}(X^0 \to X^1)$ such that $\operatorname{Hom}_R(W, -)$ and $U \otimes_R -$ leave the sequence exact whenever $U \in \mathcal{U}$ and $W \in \mathcal{W}$.

It is clear that each module in $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{X}$ is V-Gorenstein injective, projective and flat, respectively. Moreover, if R is Gorenstein, then in this case V-Gorenstein injective, projective and flat modules are simply the usual Gorenstein injective, projective and flat modules respectively.

Let $\mathbf{A}$ be an abelian category and $\mathcal{F}$ a class of objects of $\mathbf{A}$. A left $\mathcal{F}$-resolution of an object M of $\mathbf{A}$ is a $\operatorname{Hom}(\mathcal{F}, -)$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ (not necessarily exact) with each $F_i \in \mathcal{F}$. A right $\mathcal{F}$-resolution of an object M of $\mathbf{A}$ is a $\operatorname{Hom}(-, \mathcal{F})$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots$ (not necessarily exact) with each $F^i \in \mathcal{F}$.

2. V-Gorenstein projective modules. In this section, we give a detailed treatment of V-Gorenstein projective left R-modules.

Lemma 2.1. *Let $N = \oplus_{i \in I} N_i$. Then $N \in \mathcal{B}^l(R)$ if and only if $N_i \in \mathcal{B}^l(R)$ for all $i \in I$.*

Proof. Since R is left Noetherian and $\mathrm{Hom}_R(V, \oplus_{i\in I} N_i) \cong \oplus_{i\in I}\mathrm{Hom}_R(V, N_i)$, it can be seen that the conclusion is true. $\square$

Lemma 2.2. *Let $0 \to M' \to M \to M'' \to 0$ be exact in R^{op}-Mod (respectively, R-Mod). Then if any two of M', M, M'' are in $\mathcal{A}^r(R)$ (respectively, $\mathcal{B}^l(R)$), then so is the third.*

Proof. By analogy with the proof of [**8**, Proposition 3.13]. $\square$

Theorem 2.3. *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be exact in R-Mod with M'' V-Gorenstein projective. Then M' is V-Gorenstein projective if and only if M is V-Gorenstein projective.*

Proof. "$\Rightarrow$." Let M' and M'' be V-Gorenstein projective. There exist exact sequences

$$\mathbf{W}' : \cdots \longrightarrow W_1' \longrightarrow W_0' \longrightarrow W'^0 \longrightarrow W'^1 \longrightarrow \cdots$$

of modules in $\mathcal{W}$ with $M' = \mathrm{Ker}\,(W'^0 \to W'^1)$ and

$$\mathbf{W}'' : \cdots \longrightarrow W_1'' \longrightarrow W_0'' \longrightarrow W''^0 \longrightarrow W''^1 \longrightarrow \cdots$$

of modules in $\mathcal{W}$ with $M'' = \mathrm{Ker}\,(W''^0 \to W''^1)$. Let $W_i \cong V \otimes_R P_i \in \mathcal{W}$ for $i = 1, 2$. Then

$$\begin{aligned}\mathrm{Ext}_R^i(W_1, W_2) &\cong \mathrm{Hom}_R(P_1, \mathrm{Ext}_R^i(V, V \otimes_R P_2)) \\ &\cong \mathrm{Hom}_R(P_1, \mathrm{Ext}_R^i(V, V) \otimes_R P_2) = 0\end{aligned}$$

by [**13**, page 258, 9.21] and [**5**, Theorem 3.2.15] for all $i \geq 1$, and so $\mathrm{Ext}_R^1(W_0'', M') = 0 = \mathrm{Ext}_R^1(M'', W'^0)$, which means that there is a homomorphism $h : W_0'' \to M$ such that $gh = d''s_0$. Consider the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_0' \arrow[r, "\varepsilon_0"] \arrow[d, "d_0'"'] & W_0' \oplus W_0'' \arrow[r, "\pi_0"] \arrow[d, "d_0"'] & W_0'' \arrow[r] \arrow[d, "d_0''"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $d_0 : W_0' \oplus W_0'' \to M$ is defined via $d_0(x,y) = fd_0'(x) + h(y)$ for $x \in W_0'$, $y \in W_0''$. Then d_0 is an epimorphism such that $fd_0' = d_0\varepsilon_0$, $gd_0 = d_0''\pi_0$ and K', K'' are V-Gorenstein projective. Continuing this procedure yields that $\cdots \to W_1' \oplus W_1'' \to W_0' \oplus W_0'' \to M \to 0$ is exact. Since $\mathrm{Ext}_R^1(M'', W'^0) = 0$, there exists a homomorphism $k : M \to W'^0$ such that $kf = d'^0$. Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "d'^0"'] & M \arrow[r, "g"] \arrow[d, "d^0"'] & M'' \arrow[r] \arrow[d, "d''^0"'] & 0 \\
0 \arrow[r] & W'^0 \arrow[r, "\varepsilon^0"] \arrow[d] & W'^0 \oplus W''^0 \arrow[r, "\pi^0"] \arrow[d] & W''^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $d^0 : M \to W'^0 \oplus W''^0$ is defined via $d^0(m) = (k(m), d''^0 g(m))$ for $m \in M$. Then d^0 is a monomorphism such that $d^0 f = \varepsilon^0 d'^0$, $\pi^0 d^0 = d''^0 g$ and C', C'' are V-Gorenstein projective. Continuing this procedure yields that $0 \to M \to W'^0 \oplus W''^0 \to W'^1 \oplus W''^1 \to \cdots$ is exact. Therefore, an exact sequence

$$\mathbf{W} : \cdots \to W'_1 \oplus W''_1 \to W'_0 \oplus W''_0 \to W'^0 \oplus W''^0 \to W'^1 \oplus W''^1 \to \cdots$$

of modules in $\mathcal{W}$ exists such that $M = \mathrm{Ker}\,(W'^0 \oplus W''^0 \to W'^1 \oplus W''^1)$. Let $W \in \mathcal{W}$. Then $\mathrm{Hom}_R(\mathbf{W}, W) \cong \mathrm{Hom}_R(\mathbf{W}', W) \oplus \mathrm{Hom}_R(\mathbf{W}'', W)$ is exact. Since $M', M'' \in \mathcal{B}^l(R)$ by [**7**, Theorem 3.4], $M \in \mathcal{B}^l(R)$, and so M is V-Gorenstein projective.

"$\Leftarrow$." Let M and M'' be V-Gorenstein projective. There exist exact sequences

$$\mathbf{W} : \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

of modules in $\mathcal{W}$ with $M = \mathrm{Ker}\,(W^0 \to W^1)$ and

$$\mathbf{W}'' : \cdots \longrightarrow W''_1 \longrightarrow W''_0 \longrightarrow W''^0 \longrightarrow W''^1 \longrightarrow \cdots$$

of modules in $\mathcal{W}$ with $M'' = \mathrm{Ker}\,(W''^0 \to W''^1)$. Consider the pushout of $M \to W^0$ and $M \to M''$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & W^0 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since M'' and C are V-Gorenstein projective, we have L is V-Gorenstein projective and $0 \to L \to W''^0 \oplus W^1 \to W''^1 \oplus W^2 \to \cdots$ is exact by the preceding proof, and so $\mathbf{W}'_r : 0 \to M' \to W^0 \to W''^0 \oplus W^1 \to \cdots$ is exact. Let $W \in \mathcal{W}$. Then $0 \to \operatorname{Hom}_R(L, W) \to \operatorname{Hom}_R(W^0, W) \to \operatorname{Hom}_R(M', W) \to 0$ is exact, and hence $\operatorname{Hom}_R(\mathbf{W}'_r, W)$ is exact. Let $0 \to K \to P \to \operatorname{Hom}_R(V, M') \to 0$ be exact with P projective. Then $0 \to V \otimes_R K \to V \otimes_R P \to M' \to 0$ is exact and $V \otimes_R K \in \mathcal{B}^l(R)$ since $M, M'' \in \mathcal{B}^l(R)$ by [**7**, Theorem 3.4]. Continuing this procedure yields that M' has an exact left $\mathcal{W}$-resolution $\mathbf{W}'_l : \cdots \to W'_1 \to W'_0 \to M' \to 0$ and $\operatorname{Hom}_R(\mathbf{W}'_l, W)$ is exact since $\operatorname{Ext}^i_R(M', W) = 0$ for all $i \geq 1$, and so

$$\mathbf{W}' : \cdots \longrightarrow W'_1 \longrightarrow W'_0 \longrightarrow W^0 \longrightarrow W''^0 \oplus W^1 \longrightarrow \cdots$$

is exact in $\mathcal{W}$ with $M' = \operatorname{Ker}(W^0 \to W''^0 \oplus W^1)$ and $\operatorname{Hom}_R(\mathbf{W}', W)$ is exact for any $W \in \mathcal{W}$. It follows that M' is V-Gorenstein projective by [**7**, Theorem 3.4]. $\square$

Proposition 2.4. *The class V-$\mathcal{GP}$ of V-Gorenstein projective left R-modules is closed under arbitrary direct sums and arbitrary direct summands.*

Proof. Let $M = \oplus_{i \in I} M_i$ with $M_i \in V$-$\mathcal{GP}$ for all $i \in I$. Then $M_i \in \mathcal{B}^l(R)$ by [**7**, Theorem 3.4] for all $i \in I$, and hence $M \in \mathcal{B}^l(R)$ by Lemma 2.1. For each $i \in I$, an exact sequence

$$\mathbf{W}_i : \cdots \longrightarrow W_{i1} \longrightarrow W_{i0} \longrightarrow W_i^0 \longrightarrow W_i^1 \longrightarrow \cdots$$

of modules exists in $\mathcal{W}$ with $M_i = \operatorname{Ker}(W_i^0 \to W_i^1)$. Then

$$\bigoplus_{i \in I} \mathbf{W}_i : \cdots \longrightarrow \bigoplus_{i \in I} W_{i1} \longrightarrow \bigoplus_{i \in I} W_{i0} \longrightarrow \bigoplus_{i \in I} W_i^0 \longrightarrow \bigoplus_{i \in I} W_i^1 \longrightarrow \cdots$$

is exact in $\mathcal{W}$ such that $M = \operatorname{Ker}(\oplus_{i \in I} W_i^0 \to \oplus_{i \in I} W_i^1)$. Let $W \in \mathcal{W}$. Then $\operatorname{Hom}_R(\oplus_{i \in I} \mathbf{W}_i, W) \cong \prod_{i \in I} \operatorname{Hom}_R(\mathbf{W}_i, W)$ is exact, and hence $M \in V$-$\mathcal{GP}$ by [**7**, Theorem 3.4].

Let $M = M_1 \oplus M_2$ with $M \in V$-$\mathcal{GP}$. Then $M_i \in \mathcal{B}^l(R)$ by [**7**, Theorem 3.4] and Lemma 2.1. Let $L = M_1 \oplus M_2 \oplus M_1 \oplus M_2 \oplus \cdots$. Then

$L \in V\text{-}\mathcal{GP}$. Consider the exact sequence $0 \to M_1 \to M \to M_2 \to 0$. Then $0 \to M_1 \to L \to L \to 0$ is exact, and so $M_1 \in V\text{-}\mathcal{GP}$ by Theorem 2.3. Similarly, $M_2 \in V\text{-}\mathcal{GP}$. $\square$

Proposition 2.5. *Let M be any left R-module. Consider the following exact sequences of left R-modules:*

$$0 \longrightarrow K_m \longrightarrow G_{m-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow K'_m \longrightarrow G'_{m-1} \longrightarrow \cdots \longrightarrow G'_0 \longrightarrow M \longrightarrow 0,$$

where $G_{m-1}, \ldots, G_0$ and $G'_{m-1}, \ldots, G'_0$ are V-Gorenstein projective modules. Then K_m is V-Gorenstein projective if and only if K'_m is V-Gorenstein projective.

Proof. Let K_m be V-Gorenstein projective. Then $M \in \mathcal{B}^l(R)$ by [**7**, Theorem 3.4] and Lemma 2.2, and so M has an exact left $\mathcal{W}$-resolution $0 \to L_m \to W_{m-1} \to \cdots \to W_0 \to M \to 0$. Since $\mathrm{Ext}^i_R(W, K_m) = 0$ for all $i \geq 1$ and any $W \in \mathcal{W}$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & L_m \arrow[r] \arrow[d] & W_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & W_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_m \arrow[r] & G_{m-1} \arrow[r] & \cdots \arrow[r] & G_0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Thus, $0 \to L_m \to W_{m-1} \oplus K_m \to \cdots \to G_0 \to 0$ is exact, and hence L_m is V-Gorenstein projective by Theorem 2.3. Since each $\mathrm{Hom}_R(V, W_i)$ is projective and $K'_m \in \mathcal{B}^l(R)$ by Lemma 2.2 we have the following commutative diagram with exact rows: Therefore, $0 \to \mathrm{Hom}_R(V, L_m) \to \mathrm{Hom}_R(V, W_{m-1}) \oplus \mathrm{Hom}_R(V, K'_m) \to \cdots \to \mathrm{Hom}_R(V, G'_0) \to 0$ is exact, and so $0 \to L_m \to W_{m-1} \oplus K'_m \to \cdots \to G'_0 \to 0$ is exact. Thus, $W_{m-1} \oplus K'_m$ is V-Gorenstein projective by Theorem 2.3, which implies that K'_m is V-Gorenstein projective. Similarly, if K'_m is V-Gorenstein projective, then K_m is V-Gorenstein projective. $\square$

It is well known that R is a perfect ring if and only if any direct limit of projective R-modules is projective by [**14**, Theorem 1.2.13].

Theorem 2.6. *Let R be left perfect. If $M_0 \to M_1 \to M_2 \to \cdots$ is a sequence of V-Gorenstein projective left R-modules, then the direct limit $\lim\limits_{\to} M_n$ is again V-Gorenstein projective.*

Proof. For each n, an exact sequence

$$\mathbf{W}_n : \cdots \longrightarrow V \otimes_R P_n^{-2} \longrightarrow V \otimes_R P_n^{-1} \longrightarrow V \otimes_R P_n^{0} \longrightarrow V \otimes_R P_n^{1} \longrightarrow \cdots$$

of modules in $\mathcal{W}$ exists with $M_n = \mathrm{Ker}\,(V \otimes_R P_n^0 \to V \otimes_R P_n^1)$. Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & V \otimes_R P_0^{-2} \arrow[r] \arrow[d, "\varphi_{1,0}^{-2}"'] & V \otimes_R P_0^{-1} \arrow[r] \arrow[d, "\varphi_{1,0}^{-1}"'] & M_0 \arrow[r] \arrow[d, "\varphi_{1,0}"'] & 0 \\
\cdots \arrow[r] & V \otimes_R P_1^{-2} \arrow[r] \arrow[d] & V \otimes_R P_1^{-1} \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & M_0 \arrow[r] \arrow[d, "\varphi_{1,0}"'] & V \otimes_R P_0^{0} \arrow[r] \arrow[d, "\varphi_{1,0}^{0}"'] & V \otimes_R P_0^{1} \arrow[r] \arrow[d, "\varphi_{1,0}^{1}"'] & \cdots \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & V \otimes_R P_1^{0} \arrow[r] \arrow[d] & V \otimes_R P_1^{1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

So $\varphi_{n+1,n}^k = 1_V \otimes \psi_{n+1,n}^k$ for some morphism; namely, $\psi_{n+1,n}^k = \mathrm{Hom}_R(V, \varphi_{n+1,n}^k)$ since $\mathrm{Hom}_R(V, V \otimes_R P_n^k) \cong P_n^k$, and hence (P_n^k) and $(V \otimes_R P_n^k)$ are direct systems for $k = \ldots, -1, 0, 1, \ldots$. Thus,

$$\begin{aligned}\varinjlim \mathbf{W}_i : \cdots \longrightarrow V \otimes_R \varinjlim P_i^{-2} &\longrightarrow V \otimes_R \varinjlim P_i^{-1} \\ &\longrightarrow V \otimes_R \varinjlim P_i^{0} \longrightarrow V \otimes_R \varinjlim P_i^{1} \longrightarrow \cdots\end{aligned}$$

is exact in $\mathcal{W}$. Let $W \cong V \otimes_R Q \in \mathcal{W}$. Then Q is pure-injective by [**14**, Lemma 3.1.6]. Thus, Q is isomorphic to a summand of Q^{++}, and so W is isomorphic to a summand of $V \otimes_R Q^{++} \cong \mathrm{Hom}_R(V, Q^+)^+ \cong (V \otimes_R Q)^{++}$. Hence, $\mathrm{Hom}_R(\varinjlim \mathbf{W}_i, W)$ is exact by analogy with the proof of [**12**, Theorem 2.1], which implies that $\varinjlim M_n$ is V-Gorenstein projective. $\square$

In [**5**, Exercise 9] it is proved that an R-module M is Gorenstein projective if and only if $M \in \mathcal{G}_0(R)$ and $\Omega \otimes_R M$ is Ω-Gorenstein projective.

Proposition 2.7. *M is a Gorenstein projective left R-module if and only if $M \in \mathcal{A}^l(R)$ and $V \otimes_R M$ is a V-Gorenstein projective left R-module.*

Proof. By analogy with the proof of [**15**, Proposition 2.5]. $\square$

Theorem 2.8. *The following are equivalent for a ring R:*

(1) *R is a* QF *ring*;

(2) *Every left R-module is Gorenstein projective*;

(3) *Every left R-module is V-Gorenstein projective.*

Proof. (1) $\Rightarrow$ (2). Let M be any left R-module, and let $\mathbf{E} : 0 \to M \to E^0 \to E^1 \to \cdots$ be an injective resolution of M. Then, each E^i is projective. Let Q be any projective left R-module. Then Q is injective, and so $\mathrm{Ext}^i_R(M, Q) = 0$ for all $i \geq 1$ and $\mathrm{Hom}_R(\mathbf{E}, Q)$ is exact. Thus, M is Gorenstein projective.

(2) $\Rightarrow$ (1). Let P be any projective left R-module. Consider the exact sequence $0 \to P \to E(P) \to C \to 0$. Since C is Gorenstein projective, we have $\mathrm{Ext}^1_R(C, P) = 0$. So P is injective.

(2) $\Rightarrow$ (3). Let M be any left R-module. Then $\mathrm{Hom}_R(V, M)$ is Gorenstein projective, and hence $V \otimes_R \mathrm{Hom}_R(V, M) \in \mathcal{B}^l(R)$. Let $0 \to M \to E^0 \to E^1 \to \cdots$ be an injective resolution of M. Consider the exact sequence $0 \to \mathrm{Hom}_R(V, M) \to \mathrm{Hom}_R(V, E^0) \to C \to 0$. Since $\mathrm{Hom}_R(V, E^0), \mathrm{Hom}_R(V, M) \in \mathcal{A}^l(R)$ by [**8**, Proposition 3.9], then $C \in \mathcal{A}^l(R)$, and so we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \otimes_R \mathrm{Hom}_R(V,M) \arrow[r] \arrow[d, "\sigma_M"'] & V \otimes_R \mathrm{Hom}_R(V,E^0) \arrow[r] \arrow[d, "\cong"'] & V \otimes_R C \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] & E^0. & &
\end{tikzcd}$$

Then σ_M is monic. Consider the exact sequence $0 \to V \otimes_R \mathrm{Hom}_R(V,M) \to M \to L \to 0$. Then

$$0 \to \mathrm{Hom}_R(V, V \otimes_R \mathrm{Hom}_R(V,M)) \to \mathrm{Hom}_R(V,M) \to \mathrm{Hom}_R(V,L) \to 0$$

is exact since $\mathrm{Ext}^1_R(V, V \otimes_R \mathrm{Hom}_R(V,M)) = 0$, and so $\mathrm{Hom}_R(V,L) = 0$ since $\mathrm{Hom}_R(V, V \otimes_R \mathrm{Hom}_R(V,M)) \cong \mathrm{Hom}_R(V,M)$, which implies that $L = 0$. Thus, $M \cong V \otimes_R \mathrm{Hom}_R(V,M) \in \mathcal{B}^l(R)$, and M is V-Gorenstein projective by Proposition 2.7.

(3) $\Rightarrow$ (2). The proof is dual to that of (2) $\Rightarrow$ (3). $\square$

A ring R is said to be left (respectively, right) n-perfect if every flat left (respectively, right) R-module has projective dimension less than or equal to n. An R-module M is called strongly Gorenstein projective if a projective resolution exists of the form:

$$\mathbf{P} : \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Ker}\, f$ and such that $\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact for any projective module Q. An R-module M is called strongly Gorenstein flat if a flat resolution exists of the form

$$\mathbf{F} : \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Ker}\, f$ and such that $I \otimes_R \mathbf{F}$ is exact for any injective module I (see [**2**]).

Proposition 2.9. *Let R be a left n-perfect ring. Then the following are equivalent*:

(1) *R is left perfect*;

(2) *Every strongly Gorenstein flat left R-module is strongly Gorenstein projective*;

(3) *Every Gorenstein flat left R-module is Gorenstein projective*;

(4) *Every V-Gorenstein flat left R-module is V-Gorenstein projective.*

Proof. (1) $\Rightarrow$ (2). Let M be a strongly Gorenstein flat left R-module. Then an exact sequence $0 \to M \to F \to M \to 0$ exists with F flat and $M \in \mathcal{A}^l(R)$ by [**15**, Proposition 2.5]. Let Q be any projective left R-module. By [**8**, Corollary 3.19], $\mathrm{Gpd}\,({}_RM) \leq r$, and hence $\mathrm{Ext}_R^{r+i}(M, Q) = 0$ for all $i \geq 1$. Since F is projective,

$$\mathrm{Ext}_R^1(M, Q) \cong \mathrm{Ext}_R^2(M, Q) \cong \cdots \cong \mathrm{Ext}_R^r(M, Q) \cong \mathrm{Ext}_R^{r+1}(M, Q) = 0,$$

and so $\mathrm{Ext}_R^i(M, Q) = 0$ for all $i \geq 1$. Thus M is strongly Gorenstein projective by [**2**, Proposition 2.9].

(2) $\Rightarrow$ (3). This is a simple consequence of [**2**, Theorem 2.7, Theorem 3.5].

(3) $\Rightarrow$ (4). Since M is V-Gorenstein flat if and only if $M \in \mathcal{B}^l(R)$ and $\mathrm{Hom}_R(V, M)$ is Gorenstein flat by [**15**, Theorem 2.3], then $M \in \mathcal{B}^l(R)$ and $\mathrm{Hom}_R(V, M)$ is Gorenstein projective by (3), and hence $V \otimes_R \mathrm{Hom}_R(V, M) \cong M$ is V-Gorenstein projective by Proposition 2.7.

(4) $\Rightarrow$ (3). By analogy with the proof of (3) $\Rightarrow$ (4).

(3) $\Rightarrow$ (1). Let F be any flat left R-module. Then F is Gorenstein projective and $\mathrm{pd}\,({}_RF) \leq n$. Thus, F is projective by [**11**, Proposition 2.27]. $\square$

Proposition 2.10. *Every V-Gorenstein projective left R-module is V-Gorenstein flat.*

Proof. Let M be a V-Gorenstein projective left R-module. An exact sequence

$$\mathbf{W} : \cdots \longrightarrow V \otimes_R P_1 \longrightarrow V \otimes_R P_0 \longrightarrow V \otimes_R P^0 \longrightarrow V \otimes_R P^1 \longrightarrow \cdots$$

of modules in $\mathcal{W}$ exists with $M = \mathrm{Ker}\,(V \otimes_R P^0 \to V \otimes_R P^1)$. Let $U \in \mathcal{U}$. Then $\mathrm{fd}\,(U_R) \leq r$, and so there is an exact sequence of right R-modules $0 \to F_r \to \cdots \to F_0 \to U \to 0$, where each F_i is flat. Consider the exact sequence $0 \to F_r \to F_{r-1} \to U_{r-1} \to 0$. Then $0 \to F_r \otimes_R V \to F_{r-1} \otimes_R V \to U_{r-1} \otimes_R V \to 0$ is exact, and so

$0 \to F_r \otimes_R \mathbf{W} \to F_{r-1} \otimes_R \mathbf{W} \to U_{r-1} \otimes_R \mathbf{W} \to 0$ is exact. Since $F_r \otimes_R \mathbf{W}$ and $F_{r-1} \otimes_R \mathbf{W}$ are exact, we see that $U_{r-1} \otimes_R \mathbf{W}$ is exact. Continuing this procedure yields that $U \otimes_R \mathbf{W}$ is exact. Thus, M is V-Gorenstein flat. □

3. V-Gorenstein injective and flat modules. In this section, we give a detailed treatment of V-Gorenstein injective right R-modules and consider some connections of V-Gorenstein injective and V-Gorenstein flat modules.

By the dual proof of Theorem 2.3 and Proposition 2.4, we have the following result.

Theorem 3.1. *Let $0 \to M' \to M \to M'' \to 0$ be exact in R^{op}-Mod with M' V-Gorenstein injective. Then M is V-Gorenstein injective if and only if M'' is V-Gorenstein injective. Furthermore, the class V-$\mathcal{GI}$ of V-Gorenstein injective right R-modules is closed under arbitrary direct products and arbitrary direct summands.*

By the dual proof of Proposition 2.7, we have the following result.

Proposition 3.2. *M is a Gorenstein injective right R-module if and only if $M \in \mathcal{B}^r(R)$ and $Hom_{R^{\mathrm{op}}}(V, M)$ is a V-Gorenstein injective right R-module.*

It is well known that, if R is right coherent, then M is a Gorenstein flat left R-module if and only if M^+ is a Gorenstein injective right R-module by [**11**, Theorem 3.6]. Here we have the following result.

Theorem 3.3. *The following are equivalent*:

(1) *M is a V-Gorenstein flat left R-module*;

(2) *M^+ is a V-Gorenstein injective right R-module*;

(3) *$M \in \mathcal{B}^l(R)$, $Tor_i^R(U, M) = 0$ for all $i \geq 1$ and any $U \in \mathcal{U}$, and there is an exact right $\mathcal{X}$-resolution $0 \to M \to X^0 \to X^1 \to \cdots$.*

Proof. Since $\mathrm{Tor}_i^R(V, \mathrm{Hom}_R(V, M))^+ \cong \mathrm{Ext}^i_{R^{\mathrm{op}}}(V, \mathrm{Hom}_R(V, M)^+) \cong \mathrm{Ext}^i_{R^{\mathrm{op}}}(V, M^+ \otimes_R V)$ by [**5**, Theorem 3.2.11], then $\mathrm{Tor}_i^R(V, \mathrm{Hom}_R(V, M)) = 0$ if and only if $\mathrm{Ext}^i_{R^{\mathrm{op}}}(V, M^+ \otimes_R V) = 0$ for all $i \geq 1$. Since

$\mathrm{Ext}^i_R(V,M)^+ \cong \mathrm{Tor}^R_i(M^+,V)$ by [**5**, Theorem 3.2.13], $\mathrm{Ext}^i_R(V,M)=0$ if and only if $\mathrm{Tor}^R_i(M^+,V) = 0$ for all $i \geq 1$. Since $(V \otimes_R \mathrm{Hom}_R(V,M))^+ \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, \mathrm{Hom}_R(V,M)^+) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, M^+ \otimes_R V)$, we have $M \in \mathcal{B}^l(R)$ if and only if $M^+ \in \mathcal{A}^r(R)$.

(1) $\Rightarrow$ (2). Since M is V-Gorenstein flat, we have $M \in \mathcal{B}^l(R)$ by [**7**, Proposition 3.2], and an exact sequence

$$\mathbf{X} : \cdots \longrightarrow V \otimes_R F_1 \longrightarrow V \otimes_R F_0 \longrightarrow V \otimes_R F^0 \longrightarrow V \otimes_R F^1 \longrightarrow \cdots$$

of modules in $\mathcal{X}$ exists with $M = \mathrm{Ker}\,(V \otimes_R F^0 \to V \otimes_R F^1)$. Then

$$\begin{aligned}\mathbf{X}^+ : \cdots \longrightarrow \mathrm{Hom}_{R^{\mathrm{op}}}(V, F^{1+}) \longrightarrow \mathrm{Hom}_{R^{\mathrm{op}}}(V, F^{0+})\\ \longrightarrow \mathrm{Hom}_{R^{\mathrm{op}}}(V, F_0^+) \longrightarrow \mathrm{Hom}_{R^{\mathrm{op}}}(V, F_1^+) \longrightarrow \cdots\end{aligned}$$

is an exact sequence of modules in $\mathcal{U}$ with $M^+ \cong$ Coker $(\mathrm{Hom}_R(V, F^{1+}) \to \mathrm{Hom}_R(V, F^{0+}))$. Let $U \in \mathcal{U}$. Then $\mathrm{Hom}_{R^{\mathrm{op}}}(U, \mathbf{X}^+) \cong (U \otimes_R \mathbf{X})^+$ is exact, and so M^+ is V-Gorenstein injective by [**7**, Theorem 2.4].

(2) $\Rightarrow$ (3). Since M^+ is V-Gorenstein injective, then $M^+ \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, M^+ \otimes_R V)$ by [**7**, Theorem 2.4], and so $\mathrm{Hom}_R(V,M)^+ \cong M^+ \otimes_R V$ is Gorenstein injective by Proposition 3.2. Thus, $\mathrm{Hom}_R(V,M)$ is Gorenstein flat, and there is an exact right flat resolution

$$\mathbf{F} : 0 \longrightarrow \mathrm{Hom}_R(V,M) \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots .$$

Therefore, $V \otimes_R \mathbf{F} : 0 \to M \to V \otimes_R F^0 \to V \otimes_R F^1 \to \cdots$ is exact. Let $X \cong V \otimes_R F \in \mathcal{X}$. Then $\mathrm{Hom}_R(V \otimes_R \mathbf{F}, X) \cong \mathrm{Hom}_R(\mathbf{F}, F)$ is exact.

(3) $\Rightarrow$ (1). Let $\mathbf{F}_l : \cdots \to F_1 \to F_0 \to \mathrm{Hom}_R(V,M) \to 0$ be a flat resolution of $\mathrm{Hom}_R(V,M)$. Then

$$V \otimes_R \mathbf{F}_l : \cdots \longrightarrow V \otimes_R F_1 \longrightarrow V \otimes_R F_0 \longrightarrow M \longrightarrow 0$$

is exact and $U \otimes_R (V \otimes_R \mathbf{F}_l)$ is exact for any $U \in \mathcal{U}$ since $\mathrm{Tor}^R_i(U,M) = 0$ for all $i \geq 1$. Let $\mathbf{X}_r : 0 \to M \to X^0 \to X^1 \to \cdots$ be an exact right $\mathcal{X}$-resolution of M. Then $(U \otimes_R \mathbf{X}_r)^+ \cong \mathrm{Hom}_R(\mathbf{X}_r, U^+)$ is exact since $U^+ \in \mathcal{X}$, and so $U \otimes_R \mathbf{X}_r$ is exact. Thus,

$$\mathbf{X} : \cdots \longrightarrow V \otimes_R F_1 \longrightarrow V \otimes_R F_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

is exact with $M = \mathrm{Ker}\,(X^0 \to X^1)$ and $U \otimes_R \mathbf{X}$ is exact for any $U \in \mathcal{U}$, and hence M is V-Gorenstein flat by [**15**, Theorem 2.3]. $\square$

Corollary 3.4. *The following are equivalent for an (R,S)-bimodule M:*

(1) *M is a V-Gorenstein flat left R-module*;

(2) $\mathrm{Hom}_S(M,E)$ *is a V-Gorenstein injective right R-module for any $E \in$* Mod-S *injective*;

(3) $\mathrm{Hom}_S(M,E)$ *is a V-Gorenstein injective right R-module for any injective cogenerator E for* Mod-S.

Proof. (1) $\Rightarrow$ (2). Let E be any injective right S-module. Then E is isomorphic to a summand of S^{+X} for some set X. Thus, $\mathrm{Hom}_S(M,E)$ is isomorphic to a summand of $\mathrm{Hom}_S(M,S^{+X}) \cong M^{+X}$, and hence $\mathrm{Hom}_S(M,E)$ is a V-Gorenstein injective right R-module by Theorem 3.1 and Theorem 3.3.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Since S^+ is an injective cogenerator for Mod-S, $M^+ \cong \mathrm{Hom}_S(M,S^+)$ is a V-Gorenstein injective right R-module, and so M is a V-Gorenstein flat left R-module by Theorem 3.3. $\square$

Theorem 3.5. *Let $0 \to M' \to M \to M'' \to 0$ be exact in R-Mod with M'' V-Gorenstein flat. Then M is V-Gorenstein flat if and only if M' is V-Gorenstein flat. The class V-$\mathcal{GF}$ of V-Gorenstein flat left R-modules is closed under arbitrary direct sums and arbitrary direct summands.*

Proof. Use Theorems 3.1 and 3.3. $\square$

Theorem 3.6. *Let R be right n-perfect, and let $M \in R^{\mathrm{op}}$*-Mod. *Then M is a V-Gorenstein injective right R-module if and only if M^+ is a V-Gorenstein flat left R-module.*

Proof. By analogy with the proof of Theorem 3.3, we have $M \in \mathcal{A}^r(R)$ if and only if $M^+ \in \mathcal{B}^l(R)$.

"$\Rightarrow$." Let M be V-Gorenstein injective. Then $M \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, M \otimes_R V)$ by [**7**, Theorem 2.4], and so $M \otimes_R V$ is Gorenstein injective by Proposition 3.2. Thus, there is an exact sequence of right R-modules

$$\cdots \longrightarrow E_r \longrightarrow E_{r-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \otimes_R V \longrightarrow 0,$$

where each E_i injective, and so $0 \to \mathrm{Hom}_R(V, M^+) \to E_0^+ \to \cdots \to E_{r-1}^+ \to C \to 0$ is exact, where $C = \mathrm{Im}\,(E_{r-1}^+ \to E_r^+)$. Since $M^+ \in \mathcal{B}^l(R)$, $\mathrm{Hom}_R(V, M^+) \in \mathcal{A}^l(R)$, and so $C \in \mathcal{A}^l(R)$. Thus, $\mathrm{Hom}_R(V, M^+)$ is Gorenstein flat by [**15**, Lemma 3.1], which implies that M^+ is V-Gorenstein flat.

"$\Leftarrow$." Let M^+ be V-Gorenstein flat. Then $(M \otimes_R V)^+ \cong \mathrm{Hom}_R(V, M^+)$ is Gorenstein flat by [**15**, Theorem 2.3], and an exact sequence of left R-modules

$$0 \longrightarrow (M \otimes_R V)^+ \longrightarrow F_0 \longrightarrow \cdots \longrightarrow F_{n+r-1} \longrightarrow F_{n+r} \longrightarrow \cdots$$

exists with each F_i flat, and hence

$$\cdots \longrightarrow F_{n+r}^+ \longrightarrow F_{n+r-1}^+ \longrightarrow \cdots \longrightarrow F_0^+ \longrightarrow (M \otimes_R V)^{++} \longrightarrow 0$$

is exact. We successively pick injective right R-modules $E_0, E_1, \ldots$ such that

$$F_0^+ \oplus E_0 \cong F_0^{+++}, \;\; F_i^+ \oplus E_{i-1} \oplus E_i \cong (F_i^+ \oplus E_{i-1})^{++}, \text{ for all } i = 1, 2, \ldots .$$

By adding $0 \to E_i \to E_i \to 0$ to the preceding sequence in degrees $i+2$ and $i+1$, we obtain the following exact sequence

$$\begin{aligned} \cdots \longrightarrow (F_{n+r}^+ \oplus E_{n+r-1})^{++} \longrightarrow (F_{n+r-1}^+ \oplus E_{n+r-2})^{++} \longrightarrow \cdots \\ \longrightarrow (F_1^+ \oplus E_0)^{++} \longrightarrow F_0^{+++} \longrightarrow (M \otimes_R V)^{++} \longrightarrow 0. \end{aligned}$$

Set $K = \mathrm{Ker}\,(F_{n+r}^+ \oplus E_{n+r-1} \to F_{n+r-1}^+ \oplus E_{n+r-2})$. Then

$$0 \to K \to F_{n+r}^+ \oplus E_{n+r-1} \to \cdots \to F_1^+ \oplus E_0 \to F_0^+ \to M \otimes_R V \to 0$$

is exact. Since $M \in \mathcal{A}^r(R)$, $M \otimes_R V \in \mathcal{B}^r(R)$, and so $K \in \mathcal{B}^r(R)$. Thus, $M \otimes_R V$ is Gorenstein injective by [**8**, Theorem 3.17] and $M \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, M \otimes_R V)$ is V-Gorenstein injective by Proposition 3.2. ◻

Lemma 3.7. *V-$\mathcal{GF}$ is closed under arbitrary direct products.*

Proof. Let $M = \prod_{i\in I} M_i$ with each M_i a V-Gorenstein flat left R-module. Then $\mathrm{Hom}_R(V, M_i)$ is Gorenstein flat by [**15**, Theorem 2.3] and there is an exact sequence of left R-modules

$$0 \longrightarrow \mathrm{Hom}_R(V, M_i) \longrightarrow F_i^0 \longrightarrow F_i^1 \longrightarrow \cdots \longrightarrow F_i^{r-1} \longrightarrow C_i \longrightarrow 0,$$

where $C_i \in \mathcal{A}^l(R)$ and F_i^j is flat for $j = 0, 1, \ldots, r-1$ by [**15**, Lemma 3.1], and so

$$0 \to \mathrm{Hom}_R(V, M) \to \prod_{i\in I} F_i^0 \to \prod_{i\in I} F_i^1 \to \cdots \to \prod_{i\in I} F_i^{r-1} \to \prod_{i\in I} C_i \to 0$$

is exact and $\prod_{i\in I} F_i^j$ is flat for $j = 0, 1, \ldots, r-1$. Since $\prod_{i\in I} C_i \in \mathcal{A}^l(R)$, we have $\mathrm{Hom}_R(V, M) \in \mathcal{A}^l(R)$ is Gorenstein flat. Thus, $M \cong V \otimes_R \mathrm{Hom}_R(V, M)$ is V-Gorenstein flat by [**15**, Proposition 2.5]. $\square$

Corollary 3.8. *Let R be right n-perfect. Then the following are equivalent for an (S, R)-bimodule M:*

(1) *M is a V-Gorenstein injective right R-module*;

(2) $\mathrm{Hom}_S(M, E)$ *is a V-Gorenstein flat left R-module for all injective left S-modules E*;

(3) $\mathrm{Hom}_S(M, E)$ *is a V-Gorenstein flat left R-module for any injective cogenerator E for S-*Mod;

(4) *$F \otimes_S M$ is a V-Gorenstein injective right R-module for all flat right S-modules F*;

(5) *$F \otimes_S M$ is a V-Gorenstein injective right R-module for any faithfully flat right S-module F.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). By analogy with the proof of Corollary 3.4.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (4). Let F be any flat right S-module. Then $(F \otimes_S M)^+ \cong \mathrm{Hom}_S(M, F^+)$ is a V-Gorenstein flat left R-module, and so $F \otimes_S M$ is a V-Gorenstein injective right R-module by Theorem 3.6. $\square$

Corollary 3.9. *If* $\mathrm{id}\,(R_R) \le n$*, then the following are equivalent for an* (S,R)*-bimodule* M:

(1) M *is a* V*-Gorenstein injective right* R*-module*;

(2) $\mathrm{Hom}_S(M,E)$ *is a* V*-Gorenstein flat left* R*-module for all injective left* S*-modules* E;

(3) $\mathrm{Hom}_S(M,E)$ *is a* V*-Gorenstein flat left* R*-module for any injective cogenerator* E *for* S-Mod;

(4) $F \otimes_S M$ *is a* V*-Gorenstein injective right* R*-module for all flat right* S*-modules* F;

(5) $F \otimes_S M$ *is a* V*-Gorenstein injective right* R*-module for any faithfully flat right* S*-module* F*.*

Proof. Since $\mathrm{id}\,(R_R) \le n$, R is right n-perfect by [5, Proposition 9.1.2]. The result holds. □

Corollary 3.10. *Let* R *be an* n*-Gorenstein ring* M *an* (S,R)*-bimodule. Then the following are equivalent*:

(1) M *is a Gorenstein injective right* R*-module*;

(2) $\mathrm{Hom}_S(M,E)$ *is a Gorenstein flat left* R*-module for all injective left* S*-modules* E;

(3) $\mathrm{Hom}_S(M,E)$ *is a Gorenstein flat left* R*-module for any injective cogenerator* E *for* S-Mod;

(4) $F \otimes_S M$ *is a Gorenstein injective right* R*-module for all flat right* S*-modules* F;

(5) $F \otimes_S M$ *is a Gorenstein injective right* R*-module for any faithfully flat right* S*-module* F*.*

Proof. Since R is n-Gorenstein, then $\mathrm{id}\,(R_R) \le n$. The result holds by Corollary 3.9. □

Let $\mathcal{F}$ be a class of R-modules. Then $\mathcal{F}$ is said to be a Kaplansky class if a cardinal $\mathcal{N}$ exists such that, for every $M \in \mathcal{F}$ and for each $x \in M$, a submodule F of M exists such that $x \in F \subseteq M$, F, $M/F \in \mathcal{F}$ and $\mathrm{Card}\,(F) \le \mathcal{N}$.

Lemma 3.11. *Let R be a right n-perfect ring. Then V-$\mathcal{GI}$ is a Kaplansky class.*

Proof. Let $F \in V$-$\mathcal{GI}$ and $G \subseteq F$ be a pure submodule of F. Then $0 \to (F/G)^+ \to F^+ \to G^+ \to 0$ is split, and so G^+, $(F/G)^+$ are V-Gorenstein flat by Theorem 3.5. Thus G, F/G are V-Gorenstein injective by Theorem 3.6. So V-$\mathcal{GI}$ is a Kaplansky class by [**5**, Lemma 5.3.12]. $\square$

Lemma 3.12. *Let R be a right n-perfect ring. Then every inductive limit of V-Gorenstein injective right R-modules is V-Gorenstein injective.*

Proof. Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be a representative set of indecomposable injective right R-modules. Let M be a right R-module, $S(M) = \oplus_{\alpha\in\Lambda} E_\alpha^{(\mathrm{Hom}_{R^{\mathrm{op}}}(E_\alpha, M))}$ and $C(M) : S(M) \to M$ the evaluation map. Then

$$\begin{tikzcd} & E_\alpha \arrow[dl, dashed] \arrow[d] \\ S(M) \arrow[r] & M \end{tikzcd}$$

can always be completed. So $S(M) \to M$ is an injective precover of M. Now let $S_0(M) = S(M)$ and $S_1(M) = S_0(\mathrm{Ker}\,(C(M)))$. This gives a complex $S_1(M) \to S_0(M) \to M \to 0$. Proceeding in this manner, we define $S_2, S_3, \ldots$ and have a left $\mathcal{I}$nj-resolution $\cdots \to S_1(M) \to S_0(M) \to M \to 0$.

If M is Gorenstein injective, then this resolution is exact. So if $((G_i), (\varphi_{ji}))$ is an inductive system with G_i Gorenstein injective, then $G_i \in \mathcal{B}^r(R)$ by [**8**, Proposition 3.8], and we get an inductive system $(S_m(G_i), S_m(\varphi_{ji}))$ for any $m \geq 1$. Thus,

$$0 \longrightarrow K \longrightarrow \varinjlim S_{n+r}(G_i) \longrightarrow \cdots \longrightarrow \varinjlim S_0(G_i) \longrightarrow \varinjlim G_i \longrightarrow 0$$

is exact, where $K = \mathrm{Ker}\,(\varinjlim S_{n+r}(G_i) \to \varinjlim S_{n+r-1}(G_i))$. Since $\varinjlim G_i \in \mathcal{B}^r(R)$ by the definition of $\mathcal{B}^r(R)$ and each $\varinjlim S_m(G_i)$ is injective, we have $K \in \mathcal{B}^r(R)$, and so $\varinjlim G_i$ is Gorenstein injective by [**8**, Theorem 3.17]. That is, the direct limit of Gorenstein injective modules is Gorenstein injective. Let $((M_i),(\psi_{ji}))$ be an inductive system with M_i V-Gorenstein injective. Then $M_i \otimes_R V$ is Gorenstein injective by [**7**, Theorem 2.4], and hence $\varinjlim M_i \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, \varinjlim M_i \otimes_R V) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, \varinjlim(M_i \otimes_R V))$ is V-Gorenstein injective by Proposition 3.2. $\square$

Theorem 3.13. *Every right R-module has a V-Gorenstein injective preenvelope.*

Proof. Use Theorem 3.1, Lemma 3.11, Lemma 3.12 and [**10**, Theorem 2.5]. $\square$

Theorem 3.14. *Let R be right n-perfect. Then every right R-module has a V-Gorenstein injective cover.*

Proof. Consider any homomorphism $G \to M$ with G a V-Gorenstein injective right R-module. We wish to prove that $G \to M$ can be factored through a V-Gorenstein injective right R-module G' with $\mathrm{Card}\,(G') < k$ for some cardinal k. If $\mathrm{Card}\,(G) < k$, let $G' = G$. So suppose $\mathrm{Card}\,(G) \geq k$. Consider a submodule $S \subset G$ maximal with respect to the two properties that S is pure in G and that $S \subset \mathrm{Ker}\,(G \to M)$. Let $G' = G/S$. Then G' is V-Gorenstein injective by the proof of Lemma 3.11. We want to argue that $\mathrm{Card}\,(G') < k$. Let $K = \mathrm{Ker}\,(G' \to M)$. Then $\mathrm{Card}\,(G'/K) \leq \mathrm{Card}\,(M)$. So if $\mathrm{Card}\,(G') \geq k$, there is a nonzero pure submodule T/S of G/S contained in K by [**3**, Theorem 5]. But then T is pure in G and is contained in the kernel of $G \to M$. This contradicts the choice of S. That is, $G \to M$ can be factored $G \to G' \to M$ with G' V-Gorenstein injective and $\mathrm{Card}\,(G') < k$. Thus, every right R-module has a V-Gorenstein injective cover by [**5**, Proposition 5.2.2] and Lemma 3.12. $\square$

A ring R has a Matlis dualizing module if there is an (R,R)-bimodule E such that ${}_RE$ and E_R are both injective cogenerators and such that the canonical maps $R \to \mathrm{Hom}_R(E,E)$ and $R \to \mathrm{Hom}_{R^{\mathrm{op}}}(E,E)$ are both biinjections. E will be called a Matlis dualizing module for R (see [**9**]).

Theorem 3.15. *Let R be a ring admitting a Matlis dualizing module E and M a finitely generated left R-module. Then M is a Gorenstein projective left R-module if and only if* $\mathrm{Hom}_R(M,E)$ *is a Gorenstein injective right R-module.*

Proof. By [**9**, Proposition 4], $M \in \mathcal{A}^l(R)$ if and only if $\mathrm{Hom}_R(M,E) \in \mathcal{B}^r(R)$.

"$\Rightarrow$." Since R is left Noetherian and M is Gorenstein projective, M has a monic flat preenvelope $f : M \to F$. Since M is finitely generated, f can be factored through a finitely generated free module R^{n_0}, and so we can assume $F = R^{n_0}$. Set $C = \mathrm{Coker}\, f$, and let P be any projective left R-module. Then

$$0 \longrightarrow \mathrm{Hom}_R(C,P) \longrightarrow \mathrm{Hom}_R(R^{n_0},P) \longrightarrow \mathrm{Hom}_R(M,P) \longrightarrow 0$$

is exact, and so $\mathrm{Ext}^1_R(C,P) = 0$, which implies that C is finitely generated Gorenstein projective by [**11**, Corollary 2.11]. Continuing this procedure yields that M has an exact right flat resolution of the form $0 \to M \to R^{n_0} \to R^{n_1} \to \cdots$. Then

$$\cdots \longrightarrow E^{n_1} \longrightarrow E^{n_0} \longrightarrow \mathrm{Hom}_R(M,E) \longrightarrow 0$$

is exact. Let I be any injective right R-module. Then I is isomorphic to a summand of E^X for some set X. Thus, $\mathrm{Ext}^i_{R^{\mathrm{op}}}(E^X, \mathrm{Hom}_R(M,E)) \cong \mathrm{Hom}_R(\mathrm{Tor}^R_i(E^X,M),E) = 0$ for all $i \geq 1$, and we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}_{R^{\mathrm{op}}}(E^X,E^{n_0}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{R^{\mathrm{op}}}(E^X,\mathrm{Hom}_R(M,E)) \arrow[r] \arrow[d, "\cong"'] & 0 \\
\cdots \arrow[r] & \mathrm{Hom}_R(E^X \otimes_R R^{n_0},E) \arrow[r] & \mathrm{Hom}_R(E^X \otimes_R M,E) \arrow[r] & 0
\end{tikzcd}$$

with the lower row exact by [**4**, Lemma 1]. So

$$\cdots \to \mathrm{Hom}_{R^{\mathrm{op}}}(I,E^{n_1}) \to \mathrm{Hom}_{R^{\mathrm{op}}}(I,E^{n_0}) \to \mathrm{Hom}_{R^{\mathrm{op}}}(I,\mathrm{Hom}_R(M,E)) \to 0$$

is exact and $\mathrm{Ext}^i_{R^{\mathrm{op}}}(I,\mathrm{Hom}_R(M,E)) = 0$ for all $i \geq 1$. Thus, $\mathrm{Hom}_R(M,E)$ is Gorenstein injective.

"$\Leftarrow$." Since M is finitely generated and $\mathrm{Hom}_R(M,E)$ is Gorenstein injective, we have $\mathrm{Hom}_R(M,E)$ has an epic injective precover g :

$E^{m_0} \to \mathrm{Hom}_R(M,E)$ by analogy with the proof of [**9**, Proposition 5]. Set $K = \mathrm{Ker}\, g$. Then $\mathrm{Hom}_{R^{\mathrm{op}}}(K,E)$ is finitely generated and $\mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{op}}}(K,E),E) \cong K$. Let I be any injective right R-module. Then

$$0 \to \mathrm{Hom}_{R^{\mathrm{op}}}(I,K) \to \mathrm{Hom}_{R^{\mathrm{op}}}(I,E^{m_0}) \to \mathrm{Hom}_{R^{\mathrm{op}}}(I,\mathrm{Hom}_R(M,E)) \to 0$$

is exact, and so $\mathrm{Ext}^1_{R^{\mathrm{op}}}(I,K) = 0$ and K is Gorenstein injective. Continuing this procedure yields that an exact left $\mathcal{I}$nj-resolution of the form $\cdots \to E^{m_1} \to E^{m_0} \to \mathrm{Hom}_R(M,E) \to 0$ and

$$0 \to \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(M,E),E) \to \mathrm{Hom}_{R^{\mathrm{op}}}(E^{m_0},E) \to \mathrm{Hom}_{R^{\mathrm{op}}}(E^{m_1},E) \to \cdots$$

is exact. That is, $0 \to M \to R^{m_0} \to R^{m_1} \to \cdots$ is exact. Since $\mathrm{Hom}_R(M,E)$ is Gorenstein injective and E is an injective cogenerator, then M is Gorenstein flat by analogy with the proof of [**12**, Corollary 2.6]. So $\mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Ext}^i_R(M,R^{(Y)}),E) \cong \mathrm{Tor}^R_i(\mathrm{Hom}_{R^{\mathrm{op}}}(R^{(Y)},E),M) = 0$ for all $i \geq 1$, which implies that $\mathrm{Ext}^i_R(M,R^{(Y)}) = 0$ for all $i \geq 1$. Set $C = \mathrm{Coker}\,(M \to R^{m_0})$. Then C is finitely generated and $\mathrm{Hom}_R(C,E) \cong K$ is Gorenstein injective. Thus,

$$0 \to \mathrm{Hom}_R(C,R^{(Y)}) \to \mathrm{Hom}_R(R^{m_0},R^{(Y)}) \to \mathrm{Hom}_R(M,R^{(Y)}) \to 0$$

is exact. Continuing this procedure yields that

$$\cdots \to \mathrm{Hom}_R(R^{m_1},R^{(Y)}) \to \mathrm{Hom}_R(R^{m_0},R^{(Y)}) \to \mathrm{Hom}_R(M,R^{(Y)}) \to 0$$

is exact. Therefore, for any projective left R-module P,

$$\cdots \longrightarrow \mathrm{Hom}_R(R^{m_1},P) \longrightarrow \mathrm{Hom}_R(R^{m_0},P) \longrightarrow \mathrm{Hom}_R(M,P) \longrightarrow 0$$

is exact and $\mathrm{Ext}^i_R(M,P) = 0$ for all $i \geq 1$. It follows that M is Gorenstein projective. □

Corollary 3.16. *Let R be a ring admitting a Matlis dualizing module E and M a finitely generated left R-module. Then M is a V-Gorenstein projective left R-module if and only if $Hom_R(M,E)$ is a V-Gorenstein injective right R-module.*

Proof. Let M be a V-Gorenstein projective left R-module. Then $V \otimes_R \mathrm{Hom}_R(V,M) \cong M$, and hence $\mathrm{Hom}_R(V,M) \in \mathcal{A}^l(R)$ is

Gorenstein projective by Proposition 2.7. So $\mathrm{Hom}_R(M,E)\otimes_R V \cong \mathrm{Hom}_R(\mathrm{Hom}_R(V,M),E) \in \mathcal{B}^r(R)$ and $\mathrm{Hom}_R(M,E)\otimes_R V$ is Gorenstein injective by Theorem 3.15. Thus, $\mathrm{Hom}_R(M,E)$ is a V-Gorenstein injective right R-module by Proposition 3.2. Similarly, if $\mathrm{Hom}_R(M,E)$ is a V-Gorenstein injective right R-module, then M is a V-Gorenstein projective left R-module. $\square$

Proposition 3.17. *Let R be a right n-perfect ring. Then $-\otimes-$ is right balanced by V-$\mathcal{GI}\times V$-$\mathcal{GF}$ on* Mod-$R\times R$-Mod.

Proof. Let $M \in R$-Mod. Then a right V-$\mathcal{GF}$-resolution $0\to M\to F^0\to F^1\to\cdots$ exists by [**15**, Theorem 3.10]. Let $G\in V$-$\mathcal{GI}$. Then $0\to G\otimes_R M\to G\otimes_R F^0\to G\otimes_R F^1\to\cdots$ is exact if and only if $\cdots\to \mathrm{Hom}_R(F^1,G^+)\to \mathrm{Hom}_R(F^0,G^+)\to \mathrm{Hom}_R(M,G^+)\to 0$ is exact. But the last sequence is exact by Theorem 3.6.

Let $N\in R^{\mathrm{op}}$-Mod. A right V-$\mathcal{GI}$-resolution $0\to M\to E^0\to E^1\to\cdots$ exists by Theorem 3.13. Let $G\in V$-$\mathcal{GF}$. By analogy with the preceding proof, we have $0\to G\otimes_R M\to G\otimes_R E^0\to G\otimes_R E^1\to\cdots$ is exact. $\square$

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