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REAL ZEROS OF THREE DIFFERENT CASES OF POLYNOMIALS WITH RANDOM COEFFICIENTS

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ABSTRACT. There are many known asymptotic estimates for the expected number of real zeros of a trigonometric polynomial $V(\theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \cdots + a_n \cos n\theta$ with independent identically distributed random coefficients. However, recent study of random matrix theory as well as self-reciprocal random *algebraic* polynomials has led to development of a different class of random trigonometric polynomials. These applications, as well as, of course, mathematical interests, motivate us in this paper to provide asymptotic estimates for the expected number of zeros and the level-crossings of three different, albeit closely related, random trigonometric polynomials. The first two forms of random trigonometric polynomials have the self-reciprocal property of $a_j = a_{n-j}$. The third case is a random trigonometric polynomial which is in exact form arising from a transformation of an algebraic polynomial with the above self-reciprocal properties. It is shown that the expected number of real zeros of the classical trigonometric polynomial has been reduced by half, while for the other two cases this expected number remains the same as the previous cases.

1. Introduction. In this paper we study three different cases of random trigonometric polynomials. The first case is of classical form $V(\theta) = \sum_{j=0}^{n} a_j \cos j\theta$, in which the jth coefficient is equal to the $(n-j)$th term, that is, $a_j = a_{n-j}$. As we noted in Farahmand [**6**], this assumption on the coefficient naturally arises in the study of self-reciprocal random algebraic polynomials, see also [**7**]. Therefore, it is of interest to number theorists as well as, of course, those interested in the mathematical behavior of random polynomials, to study $V(\theta)$. The second case is the random trigonometric polynomial $R(\theta) = \sum_{j=0}^{n}(a_j \cos j\theta + b_j \sin j\theta)$, where $a_j = a_{n-j}$ and $b_j = b_{n-j}$. $T(\theta) = \sum_{j=0}^{n-1}\{\alpha_{n-j}\cos(j+1/2)\theta + \beta_{n-j}\sin(j+1/2)\theta\}$ is the third case we study, which is produced as seen in [**6**] from a self-reciprocal random algebraic polynomial. These three cases have some similarity on means

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of the distribution of the coefficients. In the study of all these cases it is interesting to see how the expected number of real zeros is affected by the distributions of the random coefficients.

The first work of random trigonometric polynomials was a study by Dunnage [**2**] in 1966. He found the number of real zeros, all save an exceptional set of measure zero, to be $2n/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\}$ for the classical random trigonometric polynomial $V(\theta)$ when n is large. He assumed that the coefficients are independent and normally-distributed with mean zero and variance one. Later, Sambandham and Renganathan [**9**] studied the case for $\mu \neq 0$. More recently, Farahmand [**4**] studied a more general case of K-level crossing with $K \neq 0$ and $\mu \neq 0$. Also, Wilkins [**11**] proved a more precise estimate for the expected number by significantly reducing the error terms involved. Earlier results on the subject are reviewed in the comprehensive work of Bharucha-Reid and Sambandham [**1**]. Here we prove:

Theorem 1. *For $V(\theta) = \sum_{j=0}^{n} a_j \cos j\theta$, where a_j are independent normally distributed random variables with means $\mu = 0$ and variances $\sigma^2 = 1$, $j = 1, \ldots, n$ and $a_j = a_{n-j}$, we let $N_{K,V}$ be the number of real roots of the equation $V(\theta) = K$ and $EN_{K,V}$ its expected value. Then, for all sufficient large n, $EN_{K,V}(0, 2\pi)$ is asymptotic to $n/\sqrt{3}$.*

This is half of the case when the assumption of self reciprocal is relaxed as can be seen in Farahmand [**5**, pages 75–79].

Theorem 2. *For $R(\theta) = \sum_{j=0}^{n}(a_j \cos j\theta + b_j \sin j\theta)$, where a_j and b_j are independent normally distributed random variables with means $\mu = 0$ and variances $\sigma^2 = 1$, $j = 1, \ldots, n$, and $a_j = a_{n-j}$, $b_j = b_{n-j}$, we get $EN_{K,R}(0, 2\pi) \sim 2n/\sqrt{3}$.*

In the following theorem, we obtain a result for $T(\theta)$. This polynomial is produced as seen in [**6**] from a self-reciprocal random algebraic polynomial $P_N(z) = \sum_{j=0}^{N} \eta_j z^j$, where $[n = (N - 1/2)]$ and $\{\alpha_j\}_{j=1}^{N-1}$, $\{\beta_j\}_{j=1}^{N-1}$ are sequences of independently normally distributed random variables with the same means μ and same variances σ^2, and $\eta_j = \alpha_j + i\beta_j$, $j = 1, 2, \ldots, N-1$ is a sequence of complex numbers with $\eta_N \equiv \eta_0 \equiv 1$.

Theorem 3. *For $T(\theta) = \sum_{j=0}^{n-1}\{\alpha_{n-j}\cos(j+1/2)\theta + \beta_{n-j}\sin(j+1/2)\theta\}$ and the above assumptions of the distributions of the coefficients, we have $EN_{K,T}(0,2\pi) \sim 2n/\sqrt{3}$.*

To obtain the above results, we use an important formula known as the Kac-Rice formula, and its generalized form shown in Farahmand [**5**, page 36] as:

$$EN_K(a,b) = I_1(a,b) + I_2(a,b), \tag{1.1}$$

where

$$I_1(a,b) = \int_a^b \frac{\Delta}{\pi A^2}\exp\left(-\frac{(\alpha-K)^2B^2 + \beta^2A^2 - 2(\alpha-K)\beta C}{2\Delta^2}\right)d\theta, \tag{1.2}$$

and

$$\begin{aligned} I_2(a,b) = \int_a^b \frac{\sqrt{2}|\beta A^2 - C(\alpha-K)|}{\pi A^3}\exp\left(-\frac{(\alpha-K)^2}{2A^2}\right)& \\ \times \operatorname{erf}\left(\frac{|\beta A^2 - C(\alpha-K)|}{\sqrt{2}A\Delta}\right)d\theta.& \end{aligned} \tag{1.3}$$

In the above,

$$\begin{aligned} A^2 &= \operatorname{var}\{P(x)\}, & B^2 &= \operatorname{var}\{P'(x)\}, \\ C &= \operatorname{cov}\{P(x), P'(x)\}, & \Delta^2 &= A^2B^2 - C^2, \\ \alpha &= \mathrm{E}\{P(x)\}, & \beta &= \mathrm{E}\{P'(x)\}, \end{aligned}$$

and, as usual, the erf function is defined as

$$\operatorname{erf}(x) = \int_0^x \exp(-t^2)\,dt = \sqrt{\pi}\Phi(x\sqrt{2}) - \frac{\sqrt{\pi}}{2}.$$

2. Error terms. We define $S(\theta) = \sin(2n+1)\theta/\sin\theta$, see [**5**, page 74], which is continuous at $\theta = j\pi$. Since, for $\theta \in (\varepsilon, \pi-\varepsilon)$ and $(\pi+\varepsilon, 2\pi-\varepsilon)$, we have $|S(\theta)| < 1/\sin\varepsilon$, we can obtain

$$S(\theta) = O(1/\varepsilon).$$

Further,

$$S'(\theta) = \frac{(2n+1)\cos(2n+1)\theta}{\sin\theta} - \cot\theta S(\theta) = O\left(\frac{n}{\varepsilon}\right),$$

and also

$$\begin{aligned} S''(\theta) = &\frac{-(2n+1)^2\sin(2n+1)\theta}{\sin\theta} \\ &- \frac{(2n+1)\cos\theta\cos(2n+1)\theta}{\sin^2\theta} \\ &- \cot\theta S'(\theta) + \csc^2\theta S(\theta) \\ = &\, O\left(\frac{n^2}{\varepsilon}\right). \end{aligned}$$

Now, by expanding $\sin\theta(1+2\sum_{j=1}^{n}\cos 2j\theta)$, we can show

$$\sum_{j=1}^{n}\cos 2j\theta = \frac{\sin(2n+1)\theta}{2\sin\theta} - \frac{1}{2} = \frac{S(\theta)-1}{2} = O(1/\varepsilon), \tag{2.1}$$

$$\sum_{j=1}^{n} j\sin 2j\theta = -\frac{1}{2}\left\{\sum_{j=1}^{n}\cos 2j\theta\right\}' = -\frac{1}{4}S'(\theta) = O\left(\frac{n}{\varepsilon}\right), \tag{2.2}$$

and

$$\sum_{j=1}^{n} j^2\cos 2j\theta = -\frac{1}{4}\left\{\sum_{j=1}^{n}\cos 2j\theta\right\}'' = -\frac{1}{8}S''(\theta) = O\left(\frac{n^2}{\varepsilon}\right). \tag{2.3}$$

In a similar way to [**5**], we define $Q(\theta) = \cos\theta - \cos(2n+1)\theta/2\sin\theta$; then, for $\theta \in (\varepsilon, \pi-\varepsilon), (\pi+\varepsilon, 2\pi-\varepsilon)$, since $\cos\theta - \cos(2n+1)\theta =$

$2\sin(n+1)\theta\sin n\theta$, we also have $|Q(\theta)| < 1/\sin\varepsilon$. Hence, we can also obtain

$$Q(\theta) = O\left(1/\varepsilon\right).$$

Further, we have

$$\begin{aligned} Q'(\theta) &= \frac{-\sin\theta + (2n+1)\sin(2n+1)\theta}{2\sin\theta} - \cot\theta Q(\theta) \\ &= O\left(\frac{n}{\varepsilon}\right), \end{aligned}$$

and

$$\begin{aligned} Q''(\theta) = & \frac{-\cos\theta + (2n+1)^2\cos(2n+1)\theta}{2\sin\theta} \\ & + \frac{\cos\theta\left[\sin\theta - (2n+1)\sin(2n+1)\theta\right]}{2\sin^2\theta} \\ & - \cot\theta Q'(\theta) + \csc^2\theta Q(\theta) \\ = & O\left(\frac{n^2}{\varepsilon}\right). \end{aligned}$$

Now, by expanding $2\sin\theta\sum_{j=1}^{n}\sin 2j\theta$ this time, we can get a series of the following results

(2.4)
$$\sum_{j=1}^{n}\sin 2j\theta = \frac{\cos\theta - \cos(2n+1)\theta}{2\sin\theta} = Q(\theta) = O\left(1/\varepsilon\right),$$

(2.5)
$$\sum_{j=1}^{n} j\cos 2j\theta = \frac{1}{2}\left\{\sum_{j=1}^{n}\sin 2j\theta\right\}' = \frac{1}{2}Q'(\theta) = O\left(\frac{n}{\varepsilon}\right),$$

and

(2.6)
$$\sum_{j=1}^{n} j^2\sin 2j\theta = -\frac{1}{4}\left\{\sum_{j=1}^{n}\sin 2j\theta\right\}'' = -\frac{1}{4}Q''(\theta) = O\left(\frac{n^2}{\varepsilon}\right).$$

Now, using the above identities, we are able to evaluate the characteristics required in using the Kac-Rice formula (1.1)–(1.3).

3. Preliminary analysis. In this section, we first discuss the roots in the intervals $(\varepsilon, \pi-\varepsilon)$ or $(\pi+\varepsilon, 2\pi-\varepsilon)$, which make the main contribution for the three different cases separately. For this part, we obtain our results by applying the Kac-Rice formula (1.1). Then we discuss the intervals that are not well-behaved around 0, π and 2π, that include the intervals $(0,\varepsilon)$, $(\pi-\varepsilon, \pi+\varepsilon)$ and $(2\pi-\varepsilon, 2\pi)$, where ε is any positive constant, smaller than π, which is chosen later. It should also be positive and small enough to facilitate handling the roots in the intervals $(\varepsilon, \pi-\varepsilon)$ and $(\pi+\varepsilon, 2\pi-\varepsilon)$. We will see for $\varepsilon = n^{-1/4}$ satisfies both requirements. We prove the number of real roots in intervals $(0,\varepsilon)$, $(\pi-\varepsilon, \pi+\varepsilon)$, $(2\pi-\varepsilon, 2\pi)$ are negligible.

Case 1. Let $V(\theta) = \sum_{j=0}^{n} a_j \cos j\theta$, where a_j are independent normally distributed random variables with means $\mu = 0$ and variances $\sigma^2 = 1$, $j = 1, \dots, n$ and $a_j = a_{n-j}$. We can write $V(\theta)$ in the following form, assuming n is odd

$$V(\theta) = \sum_{j=0}^{(n-1)/2} a_j \left[\cos j\theta + \cos(n-j)\theta\right]. \tag{3.1}$$

For the case where n is even, all terms in (3.1) will remain the same, but we have one additional term $a_{n/2}\cos((n\theta)/2)$. Here, without loss of generality, we only consider the case of n odd.

In order to use the Kac-Rice formula, we first evaluate the asymptotic value for each variable needed by using the error terms obtained at above estimators in (2.1)–(2.3) and (2.4)–(2.6). Since $E\{a_j\} = E\{b_j\} = \mu = 0$, we can easily get

$$\alpha_V \equiv E(V(\theta)) = 0 \quad \text{and} \quad \beta_V \equiv E(V'(\theta)) = 0. \tag{3.2}$$

Then, we obtain the variance of the polynomial by substituting (3.1), using the results of (2.1) and (2.4) and making some trigonometric identities

$$A_V^2 = \text{var }(T(\theta)) = \text{var}\left\{\sum_{j=0}^{(n-1)/2} a_j\left[2\cos\left(\frac{n\theta}{2}\right)\cos\left(\frac{n-2j}{2}\right)\theta\right]\right\}$$
$$= 2\cos^2\left(\frac{n\theta}{2}\right)\left\{\sum_{j=0}^{(n-1)/2}[1+\cos(n-2j)\theta]\right\}$$
$$= [1+\cos(n\theta)]\left\{\frac{n-1}{2}+\sum_{j=0}^{(n-1)/2}[\cos n\theta\cos 2j\theta+\sin n\theta\sin 2j\theta]\right\}$$
$$= [1+\cos(n\theta)]\left\{\frac{n}{2}+O\left(\frac{\cos n\theta+\sin n\theta}{\varepsilon}\right)\right\}. \tag{3.3}$$

Next, we calculate the variance of the derivative of $T(\theta)$ with respect to θ in a similar way to A^2. Using the results from (2.2), (2.3), (2.5) and (2.6), we obtain

$$B_V^2 = \text{var }(T'(\theta)) = \sum_{j=0}^{n} j^2\sin^2 j\theta + \sum_{j=0}^{(n-1)/2} 2j(n-j)\sin(j\theta)\sin(n-j)\theta$$
$$= \frac{1}{2}\sum_{j=0}^{n} j^2 - n\cos n\theta\sum_{j=0}^{(n-1)/2} j + \cos n\theta\sum_{j=0}^{(n-1)/2} j^2 - \frac{1}{2}\sum_{j=0}^{n} j^2\cos 2j\theta + \sum_{j=0}^{(n-1)/2} j(n-j)\cos(n-2j)\theta$$
$$= \frac{n^3}{12}(2-\cos n\theta)+O\left(\frac{n^2(\cos n\theta+\sin n\theta)}{\varepsilon}\right). \tag{3.4}$$

At last, we turn to the covariance between the polynomial and its derivative. Here $E\{T(\theta)\}E\{T'(\theta)\}=0$ as the mean of the coefficients

$\mu = 0$. Therefore,

$$\begin{aligned}C_V &= E\left\{ \sum_{j=0}^{(n-1)/2} a_j\left[\cos j\theta + \cos(n-j)\theta\right] \right.\\ &\qquad\qquad \left. \times\, a_j\left[-j\sin j\theta - (n-j)\sin(n-j)\theta\right] \right\}\\ &= -\sum_{j=0}^{n} j\sin j\theta\cos j\theta - \sum_{j=0}^{n} j\sin j\theta\cos(n-j)\theta \\ &= \sum_{j=0}^{n} -j\sin j\theta\cos j\theta + \cos n\theta\sum_{j=0}^{n} -j\sin j\theta\cos j\theta \\ &\quad - \frac{1}{2}\sin n\theta\sum_{j=0}^{n} j(1-\cos 2j\theta) \\ &= -\frac{n^2\sin(n\theta)}{4} - O\left(\frac{n(1+\cos n\theta)}{\varepsilon}\right).\end{aligned} \tag{3.5}$$

Then, finally from (3.3), (3.4) and (3.5), we can get

$$\Delta_V^2 = A_V^2 B_V^2 - C_V^2 = \frac{n^4(\cos n\theta + 1)^2}{48} + O\left(\frac{n^3(1+\cos n\theta)}{\varepsilon}\right). \tag{3.6}$$

From (1.2) and (1.3), substituting $K = 0$ and the results of (3.2)–(3.6) into the Kac-Rice formula, we can obtain

$$EN_{K,V}(\varepsilon, \pi-\varepsilon) = EN_{K,V}(\pi+\varepsilon, 2\pi-\varepsilon) \sim \frac{n}{2\sqrt{3}}. \tag{3.7}$$

This is the result for Theorem 1.

Case 2. Let $R(\theta) = \sum_{j=0}^{n}(a_j\cos j\theta + b_j\sin j\theta)$, where a_j and b_j are independent normally distributed random variables with means $\mu = 0$ and variances $\sigma^2 = 1$, $j = 1, \ldots, n$ and $a_j = a_{n-j}$, $b_j = b_{n-j}$. We can also write $T(\theta)$ in the following form by assuming n is odd. The polynomial will have one additional term for n even, and we will not discuss this case as it is explained above. With our assumptions we can evaluate $R(\theta)$ as

$$\begin{aligned}R(\theta) = \sum_{j=0}^{(n-1)/2} &\{a_j\left[\cos j\theta + \cos(n-j)\theta\right] \\ &+ b_j\left[\sin j\theta + \sin(n-j)\theta\right]\}.\end{aligned} \tag{3.8}$$

In a similar way, we evaluate each term in the Kac-Rice formula in (1.2) and (1.3) as follows. First, we note

$$\alpha_R \equiv E(R(\theta)) = 0 \quad \text{and} \quad \beta_R \equiv E(R'(\theta)) = 0.$$

Then, from (3.8), we can obtain the variance of the polynomial

(3.9)
$$\begin{aligned}
A_R^2 &= \sum_{j=0}^{(n-1)/2} \text{var}\,\{a_j\}\,\{\cos j\theta + \cos(n-j)\theta\}^2 \\
&\quad + \sum_{j=0}^{(n-1)/2} \text{var}\,\{b_j\}\,\{\sin j\theta + \sin(n-j)\theta\}^2 \\
&= 4\cos^2\left(\frac{n\theta}{2}\right) + 4\sin^2\left(\frac{n\theta}{2}\right)\left\{\sum_{j=0}^{(n-1)/2}\left[\cos\left(\frac{n-2j}{2}\right)\theta\right]^2\right\} \\
&= (n-1) + O\bigg(\frac{\cos n\theta + \sin n\theta}{\varepsilon}\bigg).
\end{aligned}$$

The variance of the derivative of the polynomial with respect to θ can be obtained:

(3.10)
$$\begin{aligned}
B_R^2 &= \sum_{j=0}^{(n-1)/2} \text{var}\,\{a_j\}\,[-j\sin j\theta - (n-j)\sin(n-j)\theta]^2 \\
&\quad + \sum_{j=0}^{(n-1)/2} \text{var}\,\{b_j\}\,[j\cos j\theta + (n-j)\cos(n-j)\theta]^2 \\
&= \sum_{j=0}^{(n-1)/2} \Big\{ j^2 \sin^{\,2} j\theta + 2j(n-j)\sin(j\theta)\sin(n-j)\theta \\
&\quad + (n-j)^2\sin^{\,2}(n-j)\theta + j^2\cos^{\,2} j\theta \\
&\quad + 2j(n-j)\cos(j\theta)\cos(n-j)\theta + (n-j)^2\cos^2(n-j)\theta \Big\} \\
&= \frac{n^3}{3} + O\bigg(\frac{n^2(\cos n\theta + \sin n\theta)}{\varepsilon}\bigg).
\end{aligned}$$

At last, we get the covariance between the polynomial and its derivative which is similar to the first case:

(3.11)
$$\begin{aligned}
C_R = &\sum_{j=0}^{(n-1)/2} E\left\{a_j^2\right\}\Big\{[\cos j\theta + \cos(n-j)\theta] \\
&\qquad\qquad \times [-j\sin j\theta - (n-j)\sin(n-j)\theta]\Big\} \\
&+ \sum_{j=0}^{(n-1)/2} E\left\{b_j^2\right\}\Big\{[\sin j\theta + \sin(n-j)\theta] \\
&\qquad\qquad \times [j\cos j\theta + (n-j)\cos(n-j)\theta]\Big\} \\
= &\sum_{j=0}^{(n-1)/2} -(n-j)\sin(n-j)\theta\cos j\theta + j\sin(n-j)\theta\cos j\theta \\
&+ \sum_{j=0}^{(n-1)/2} (n-j)\sin j\theta\cos(n-j)\theta - j\sin j\theta\cos(n-j)\theta \\
= &\,O\bigg(\frac{n(\sin n\theta - \cos n\theta)}{\varepsilon}\bigg).
\end{aligned}$$

Then, from (3.9), (3.10) and (3.11), we can get

$$\Delta_R^2 = A_R^2 B_R^2 - C_R^2 = \frac{n^4}{3} + O\bigg(\frac{n^2}{\varepsilon}\bigg). \tag{3.12}$$

Finally, with the results above, for $K = 0$, we therefore have

$$EN_{K,R}(\varepsilon, \pi - \varepsilon) = EN_{K,R}(\pi + \varepsilon, 2\pi - \varepsilon) \sim \frac{n}{\sqrt{3}}. \tag{3.13}$$

This is the result of Theorem 2.

Case 3. Here we consider a random trigonometric polynomial which is produced from a random algebraic polynomial with complex variables and complex coefficients, with a self-reciprocal property. Let $\{\alpha_j\}_{j=1}^{N-1}$ and $\{\beta_j\}_{j=1}^{N-1}$ be sequences of independently normally distributed random variables with the same means μ and same variances σ^2, and let

$\eta_j = \alpha_j + i\beta_j$, $j = 1, 2, \dots, N-1$ be a sequence of complex numbers with $\eta_N \equiv \eta_0 \equiv 1$. We define a (complex) random algebraic polynomial as

$$P_N(z) = \sum_{j=0}^{N} \eta_j z^j.$$

And we define $P_N(z)$ as a self-reciprocal random algebraic polynomial, that is, for all N and z, the polynomial satisfies the relation $P_N(z) = z^N P_N(1/z)$. This self-reciprocal polynomial has the property of $\eta_N \equiv \eta_0 \equiv 1$, and η_{N-j} is the complex conjugate of η_j, $j = 1, 2, \dots, N-1$. Therefore, we get

$$P_N(z) = 1 + \eta_1 z + \eta_2 z^2 + \cdots + \eta_{N-1} z^{N-1} + z^N. \tag{3.14}$$

For $z = r\exp(i\theta)$, we can simply transform $P_N(z)$ in (3.14) as

$$\begin{aligned} e^{iN\theta/2} P_N(e^{i\theta}) = 2\cos\left(\frac{N\theta}{2}\right) &+ 2\alpha_1 \cos\left(\frac{N-2}{2}\right)\theta + 2\beta_1 \sin\left(\frac{N-2}{2}\right)\theta \\ &+ 2\alpha_2 \cos\left(\frac{N-4}{2}\right)\theta + 2\beta_2 \sin\left(\frac{N-4}{2}\right)\theta \\ &\vdots \\ &+ F_N(\theta), \end{aligned}$$

where

$$F_N(\theta) = \begin{cases} 2\alpha_{(N/2-1)}\cos\theta + 2\beta_{(N/2-1)}\sin\theta + \alpha_{N/2} + i\beta_{N/2} & \text{for } n \text{ even} \\ 2\alpha_{(N-1)/2}\cos\theta + 2\beta_{(N-1)/2}\sin\theta & \text{for } n \text{ odd.} \end{cases}$$

As seen from the results above, not all the coefficients η_j in (3.14) can have a matching conjugate for N even; therefore, our main interest is for the case of N odd. However, we still present $P_N(z)$ for both n odd and even to complete the problem. Then, for N odd, we have

$$\begin{aligned} P_N(\theta) = 2 &\sum_{j=1}^{(N-1)/2} \left\{ \alpha_j \cos\left(\frac{N-2j}{2}\right)\theta + \beta_j \sin\left(\frac{N-2j}{2}\right)\theta \right\} \\ &+ 2\cos\left(\frac{N\theta}{2}\right), \end{aligned} \tag{3.15}$$

and, for N even,

$$P_N(\theta) = 2\sum_{j=1}^{N/2-1}\left\{\alpha_j \cos\left(\frac{N-2j}{2}\right)\theta + \beta_j \sin\left(\frac{N-2j}{2}\right)\theta\right\}$$
$$+ 2\cos\left(\frac{N\theta}{2}\right) + \alpha_{N/2} + i\beta_{N/2}.$$

Now, from (3.15), that is, for N odd, we have the polynomial of the form

$$P_N(\theta) = T(\theta) + \cos(n+\theta/2),$$

where $n = (N-1)/2$ and

$$T(\theta) = \sum_{j=0}^{n-1}\{\alpha_{n-j}\cos(j+1/2)\theta + \beta_{n-j}\sin(j+1/2)\theta\}. \tag{3.16}$$

Therefore, the study of random algebraic polynomials with complex coefficients with self-reciprocal properties will lead to the trigonometric polynomial with real random coefficients. Here we change N to express the degree of the trigonometric polynomial in order to keep consistent with our previous cases. Now we need to consider the number of level crossings of $T(\theta)$ with $K = -\cos(n+\theta/2)$, which is the same problem to study as the real zeros of $P_N(\theta)$. From (2.1), (2.2), (2.4) and (2.5), we can get the means of the polynomial and its derivative as

(3.17)
$$\alpha_T = \mu\sum_{j=0}^{n-1}\left\{\cos\left(j+\frac{1}{2}\right)\theta + \sin\left(j+\frac{1}{2}\right)\theta\right\} = O\left(\frac{\mu}{\varepsilon}\right),$$

$$\beta_T = -\mu\sum_{j=0}^{n-1}\left(j+\frac{1}{2}\right)\left\{\sin\left(j+\frac{1}{2}\right)\theta - \cos\left(j+\frac{1}{2}\right)\theta\right\}$$

(3.18)
$$= O\left(\frac{\mu n}{\varepsilon}\right).$$

Next, we calculate A^2 and B^2, which are the variances of the polynomial and its derivative, as

(3.19)
$$A_T^2 = \operatorname{var}\left\{\sum_{j=0}^{n-1}\left[\alpha_{n-j}\cos\left(j+\frac{1}{2}\right)\theta + \beta_{n-j}\sin\left(j+\frac{1}{2}\right)\theta\right]\right\}$$

$$
\begin{aligned}
&= \sigma^2 \sum_{j=0}^{n-1} \cos^2 \left(j + \frac{1}{2}\right)\theta + \sigma^2 \sum_{j=0}^{n-1} \sin^2 \left(j + \frac{1}{2}\right)\theta \\
&= n\sigma^2
\end{aligned}
$$

and

(3.20)

$$
\begin{aligned}
B_T^2 &= \sigma^2 \left\{ \sum_{j=0}^{n-1} \left[-\left(j+\frac{1}{2}\right)^2 \sin^2\left(j+\frac{1}{2}\right)\theta + \left(j+\frac{1}{2}\right)^2 \cos^2\left(j+\frac{1}{2}\right)\theta \right] \right\} \\
&= \sigma^2 \sum_{j=0}^{n-1} \left(j + \frac{1}{2}\right)^2 \\
&= \frac{n\sigma^2}{3}\left(n^2 - \frac{1}{4}\right).
\end{aligned}
$$

At last, we turn to the covariance between the polynomial and its derivative. In this calculation, most of the terms are canceled with each other as the assumptions of $\{\alpha_j\}_{j=1}^{N-1}$ and $\{\beta_j\}_{j=1}^{N-1}$ are sequences of independent normally distributed random variables. We will see whether the mean is zero or non-zero; it will not affect our results. Therefore, we get

(3.21)

$$
\begin{aligned}
C_T &= \text{cov}\ [T(\theta)T'(\theta)] \\
&= -(\mu^2 + \sigma^2) \sum_{j=0}^{n-1} \left(j + \frac{1}{2}\right) \sin\left(j + \frac{1}{2}\right)\theta \cos\left(j + \frac{1}{2}\right)\theta \\
&\quad + (\mu^2 + \sigma^2) \sum_{j=0}^{n-1} \left(j + \frac{1}{2}\right) \sin\left(j + \frac{1}{2}\right)\theta \cos\left(j + \frac{1}{2}\right)\theta \\
&\quad + \mu^2 \sum_{j=0}^{n-1} \left(j + \frac{1}{2}\right) \sin\left(j + \frac{1}{2}\right)\theta \cos\left(j + \frac{1}{2}\right)\theta \\
&\quad - \mu^2 \sum_{j=0}^{n-1} \left(j + \frac{1}{2}\right) \sin\left(j + \frac{1}{2}\right)\theta \cos\left(j + \frac{1}{2}\right)\theta \\
&= 0.
\end{aligned}
$$

From the Kac-Rice formula (1.1) and by substituting the results into (3.17)–(3.21), we therefore have

$$\Delta_T^2 = A_T^2 B_T^2 - C_T^2 = A_T^2 B_T^2 = \frac{n^2\sigma^4}{3}(n^2 - 1/4), \tag{3.22}$$

Hence, we can obtain

$$I_1(\varepsilon, \pi - \varepsilon) \sim \frac{n}{\sqrt{3}} \tag{3.23}$$

and

$$I_2(\varepsilon, \pi - \varepsilon) = O\left(n^{3/4}\right). \tag{3.24}$$

Hence, from (1.1), (3.23) and (3.24), denote the number of real zeros in the interval (a, b) by $N(a, b)$ and its expected value by $EN(a, b)$; then we can obtain

$$\begin{aligned} EN_{K,T}(\varepsilon, \pi - \varepsilon) &= EN_{K,T}(\pi + \varepsilon, 2\pi - \varepsilon) \\ &= \frac{n}{\sqrt{3}} + O(n^{3/4}). \end{aligned} \tag{3.25}$$

4. Proof of the negligible part. In this section, we are going to show the expected number of real zeros in the intervals $(0, \varepsilon)$, $(\pi - \varepsilon, \pi + \varepsilon)$ and $(2\pi - \varepsilon, 2\pi)$ is negligible. We will modify the method based upon Jensen's theorem [**8**, page 125] or [**10**, page 332], which has been used by Dunnage [**3**]. Here we just take the last case as an example, since the process for the other two cases is the same, and the final results of this part remain the same for all three cases.

We consider the function of the complex variable ζ,

$$T(\zeta) - K = \sum_{j=0}^{n-1} \{\alpha_{n-j} \cos[(j + 1/2)\zeta] + \beta_{n-j} \sin[(j + 1/2)\zeta]\} - K,$$

which is normally distributed with means $n\mu$ and variance $n\sigma^2$. The period of $T(\theta)$ is 2π, and the number of real zeros in the interval $(0, \varepsilon)$, $(2\pi - \epsilon, 2\pi)$ or $(\pi - \varepsilon, \pi + \varepsilon)$ is the same as the number in $(-\varepsilon, \varepsilon)$, and this certainly does not exceed the number of real zeros in the circle $|\zeta| < \varepsilon$. Therefore, we seek an upper bound to the number of real zeros in the segment of the real axis joining the points $\pm\varepsilon$. To this end, let $N(r) \equiv N(r, K)$ denote the number of real zeros of $T(\zeta) - K = 0$ in $|\zeta| < \varepsilon$. By Jensen's theorem,

$$\int_{\varepsilon}^{2\varepsilon} r^{-1} N(r)\, dr \leq \int_{0}^{2\varepsilon} r^{-1} N(r)\, dr = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \log \left| \frac{T(2\varepsilon e^{i\theta}) - K}{T(0) - K} \right| d\theta.$$

Assuming that $T(0) \neq K$, we obtain

$$N(\varepsilon) \log 2 \leq \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \log \left| \frac{T(2\varepsilon e^{i\theta}) - K}{T(0) - K} \right| d\theta. \tag{4.1}$$

Now we can see that, for any positive ν,

$$\begin{aligned} (4.2)\quad & Pr(-e^{-\nu} \leq T(0) - K \leq e^{-\nu}) \\ &= (2\pi n\sigma^2)^{-1/2} \int_{K-e^{-\nu}}^{K+e^{-\nu}} \exp\left\{ -\frac{(t-n\mu)^2}{2n\sigma^2} \right\} dt \\ &< \sqrt{\frac{2}{\pi n \sigma^2}} e^{-\nu}. \end{aligned}$$

The distribution function of $|\alpha_j|$ and $|\beta_j|$ is

$$F(x) = \begin{cases} 1/\sqrt{2\pi\sigma^2} \int_0^x \exp\left\{\frac{-(t-\mu)^2}{2\sigma^2}\right\} dt & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let M be defined as the maximum value among $|\alpha_{n-j}|$ and $|\beta_{n-j}|$ where $j = 1, 2, \ldots, n$, that is, $M = \operatorname{Max}\{\operatorname{Max}(\alpha_{n-j}), \operatorname{Max}(\beta_{n-j})\}$. Since $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$, we have

(4.3)

$$\begin{aligned} T(2\varepsilon e^{i\theta}) &= \sum_{j=0}^{n-1} \left\{ \alpha_{n-j} \cos\left(j + \frac{1}{2}\right) 2\varepsilon e^{i\theta} + \beta_{n-j} \sin\left(j + \frac{1}{2}\right) 2\varepsilon e^{i\theta} \right\} \\ &\leq M \sum_{j=0}^{n-1} \Bigg\{ \cosh\{(2j+1)\varepsilon \sin\theta\} + \sinh\{(2j+1)\varepsilon\sin\theta\} \\ &\qquad + \{\cosh(2j+1)\varepsilon\sin\theta\} + \sinh\{(2j+1)\varepsilon\sin\theta\} \Bigg\} \\ &\leq 4Mn \exp(2n+1)\varepsilon. \end{aligned}$$

The probability $M > ne^{\nu}$ for any positive ν and all sufficiently large n is

$$
\begin{aligned}
Pr(M > ne^{\nu}) &\le nPr(|g_1| > ne^{\nu}) \\
&= n\sqrt{\frac{1}{2\pi\sigma^2}} \int_{ne^{\nu}}^{\infty} \exp\left\{ -\frac{(t-\mu)^2}{2\sigma^2} \right\} dt \\
&= \frac{\sigma}{\sqrt{2\pi}} \exp\left\{ -\nu - \frac{(ne^{\nu}-\mu)^2}{2\sigma^2} \right\}.
\end{aligned} \tag{4.4}
$$

Then, we can get

$$
\begin{aligned}
\left|T(2\varepsilon e^{i\theta}) - K\right| &\le |4Mn \exp(2n+1)\varepsilon - \cos(n+\theta/2)| \\
&\le |4n^2 \exp(2n+1)\varepsilon - 1| \\
&\sim 4n^2 \exp(2n+1)\varepsilon.
\end{aligned} \tag{4.5}
$$

From (4.2)–(4.5), we obtain

$$
\begin{aligned}
\left|\frac{T(2\varepsilon e^{i\theta}) - K}{T(0) - K}\right| &\le e^{\nu} \left|4n^2 \exp((2n+1)\varepsilon + \nu)\right| \\
&\le 4n^2 \exp((2n+1)\varepsilon + 2\nu)
\end{aligned} \tag{4.6}
$$

except for the sample functions of measure not exceeding

$$
\sqrt{\frac{2}{\pi n\sigma^2}}\, e^{-\nu} + \frac{\sigma}{\sqrt{2\pi}} \exp\left\{ -\nu - \frac{(ne^{\nu}-\mu)^2}{2\sigma^2} \right\}.
$$

Therefore from (4.1) and (4.6),

$$
N(\varepsilon) \le \frac{\log 4 + 2\log n + (2n+1)\varepsilon + 2\nu}{\log 2}. \tag{4.7}
$$

Then, since $\varepsilon = n^{-1/4}$, it follows from (4.7) and for any sufficiently large n that

$$
\begin{aligned}
Pr\{N(\varepsilon) > 3n\varepsilon + 2\nu\} \le \sqrt{\frac{2}{\pi\sigma^2}}\, e^{-\nu} \\
+ \frac{\sigma}{\sqrt{2\pi}} \exp\left\{ -\nu - \frac{(ne^{\nu}-\mu)^2}{2\sigma^2} \right\}.
\end{aligned} \tag{4.8}
$$

Let $n' = [3n^{3/4}]$ be the greatest integer less than or equal to $3n^{3/4}$; then, from (4.4) and (4.8), for all sufficiently large n, we obtain

$$
\begin{aligned}
EN(\varepsilon) &= \sum_{j>0} jPr\{N(\varepsilon) = j\} \\
&= \sum_{j>0} Pr\{N(\varepsilon) \geq j\} \\
&= \sum_{1 \geq j \geq n'} Pr\{N(\varepsilon) > j\} \\
&\quad + \sum_{j>0} Pr\{N(\varepsilon) > n' + j\} \\
&\leq n' + \sqrt{\frac{2}{n\pi\sigma^2}} \sum_{j>1} e^{-j/2} \\
&\quad + \frac{\sigma}{\sqrt{2\pi}} \exp\left\{ - \frac{-j/2 - (ne^{-j/2} - \mu)^2}{2\sigma^2} \right\} \\
&= O(n^{3/4}).
\end{aligned}
\tag{4.9}
$$

Hence, from (3.25) and (4.9), we can finally obtain the expected number of level crossings of $T(\theta)$ with $-\cos(n + \theta/2)$ in the interval $(0, 2\pi)$ where $2n/\sqrt{3} + O(n^{3/4})$. Also, we can get the real zeros of $V(\theta)$ are $n/\sqrt{3} + O(n^{3/4})$, and $R(\theta)$ is $2n/\sqrt{3} + O(n^{3/4})$ in the same way.

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