ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 1, 2012

TOPOLOGICAL APPROACH TO UNBOUNDED OPERATORS ON HILBERT C^*-MODULES

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ABSTRACT. Suppose V, W are Hilbert C^*-modules and $R(V,W)$ is the set of all unbounded regular operators between V, W. Suppose $B_1(V,W)$ denotes the set of all bounded adjointable operators $T : V \to W$ of norm $\|T\| \le 1$. We equip the sets with gap topology and extended gap topology, respectively. Then the adjoint-preserving bijection $R(V,W) \to \{F \in B(V,W) : \|F\| \le 1$ and Range $(1-F^*F)$ is dense in $V\}$ which associates to an unbounded regular operator t its bounded transform $F_t = t(1+t^*t)^{-1/2}$ will be bicontinuous.

1. Introduction and preliminaries. Suppose t is a densely defined closed operator on an arbitrary Hilbert space H, by a result due to von Neumann [**10**] the operators $(1+t^*t)^{-1}$ and $t(1+t^*t)^{-1}$ are everywhere defined and bounded. Moreover, $(1+t^*t)^{-1}$ is symmetric and positive (see also the book by Riesz and Sz.-Nagy [**13**, Section 118]). In 1991, Woronowicz [**15**] obtained a C^*-algebra version of these facts. Indeed, he identified a densely defined operator t on an arbitrary C^*-algebra $\mathcal{A}$ whose graph is orthogonally complemented in $\mathcal{A} \oplus \mathcal{A}$ by the domain Dom $(t) = (1-z^*z)^{1/2}(\mathcal{A})$ and $t(1-z^*z)^{1/2}(a) = za$ for all $a \in \mathcal{A}$ and some element z in multiplier algebra $M(\mathcal{A})$ of norm $\|z\| \le 1$. A little later when the theory of Hilbert C^*-modules was investigated in more detail, Lance in his book [**8**] generalized the identification to Hilbert C^*-modules as follows.

Let V, W be Hilbert C^*-modules and $R(V,W)$ the set of all 'unbounded adjointable operators,' or what are now known as regular operators. Let $B(V,W)$ be the set of bounded adjointable operators between V, W; then the map $t \to F_t = t(1+t^*t)^{-1/2}$ defines a bijection

2010 AMS *Mathematics subject classification.* Primary 46L08, Secondary 46C05, 47L60, 47B50.
Keywords and phrases. Hilbert C^*-module, unbounded operators, gap topology.
Received by the editors on March 3, 2009, and in revised form on June 17, 2009.

DOI:10.1216/RMJ-2012-42-1-285

$$(1.1) \quad R(V,W) \longrightarrow \{F \in B(V,W) : \|F\| \leq 1 \text{ and } \\ \text{Range}\,(1-F^*F) \text{ is dense in } V\},$$

whose map is adjoint-preserving, i.e., $F_t^* = F_{t^*}$.

There is a natural metric on the set of unbounded regular operators, the so-called *gap metric*. The gap topology is induced by the metric $\text{gap}(t,s) = \|P_{G(t)} - P_{G(s)}\|$ where $P_{G(t)}$ and $P_{G(s)}$ are projections onto the graphs of unbounded regular operators t, s, respectively. Suppose that $R(V,W)$ is equipped with the gap topology; then the map $t \to F_t$ will not be continuous if we equip the closed unit ball of $B(V,W)$ with the norm topology (see [**14**, Example 2.9]). The main goal of our paper is to introduce a suitable weaker topology on the closed unit ball of $B(V,W)$ such that the map (1.1) becomes a homeomorphism. We can use this fact to occasionally think of $R(V,W)$ as being a subspace of the closed unit ball of $B(V,W)$. The result is essentially new even in the case of Hilbert spaces.

Throughout the present paper $\mathcal{A}$ is an arbitrary C^*-algebra. Since we deal with bounded and unbounded operators at the same time we simply denote bounded operators by capital letters and unbounded operators by lower case letters. We use the notations $\text{Dom}\,(\cdot)$ and $\text{Ran}\,(\cdot)$ for domain and range of operators.

A (left) *pre-Hilbert C^*-module* over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module V equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathcal{A}$, $(x,y) \mapsto \langle x, y \rangle$, which is $\mathcal{A}$-linear in the first variable x and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \;\; \langle x, x \rangle \geq 0 \text{ with equality only when } x = 0.$$

A pre-Hilbert $\mathcal{A}$-module V is called a *Hilbert $\mathcal{A}$-module* if V is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. A Hilbert $\mathcal{A}$-submodule V of a Hilbert $\mathcal{A}$-module W is an orthogonal summand if $V \oplus V^\perp = W$, where $V^\perp$ denotes the orthogonal complement of V in W. If V, W are two Hilbert $\mathcal{A}$-modules, then the set of all ordered pairs of elements $V \oplus W$ from V and W is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_V + \langle y_1, y_2 \rangle_W$. It is called the direct *orthogonal sum* of V and W. We denote by $B(V,W)$ the Banach space of all bounded adjointable operators from a Hilbert $\mathcal{A}$-module V to another Hilbert $\mathcal{A}$-module W. Papers [**2, 3, 9, 12**] and the book by Lance [**8**] can be used as standard sources of reference.

Remark 1.1. Let V, W be Hilbert $\mathcal{A}$-modules and $A \in B(V,V)$, $B \in B(W,V)$, $C \in B(V,W)$ and $D \in B(W,W)$ bounded adjointable operators. Suppose

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \in V \oplus W \quad \text{and} \quad P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B(V \oplus W, V \oplus W).$$

Then we have

$$\begin{aligned} \|Pz\| &= \|\langle Ax + By, Ax + By\rangle_V \\ &\qquad + \langle Cx + Dy, Cx + Dy\rangle_W\|^{1/2} \\ &\le \|\langle Ax + By, Ax + By\rangle_V\|^{1/2} \\ &\qquad + \|\langle Cx + Dy, Cx + Dy\rangle_W\|^{1/2} \\ &= \|Ax + By\| + \|Cx + Dy\| \\ &\le \|A\|\|x\| + \|B\|\|y\| + \|C\|\|x\| + \|D\|\|y\| \\ &\le (\|A\| + \|B\| + \|C\| + \|D\|)\ \|z\|. \end{aligned}$$

Furthermore, for every $x \in V$ of norm $\|x\| \le 1$ we find

$$\|Ax\| = \| \begin{pmatrix} Ax \\ 0 \end{pmatrix} \| \le \|P \begin{pmatrix} x \\ 0 \end{pmatrix} \| \le \|P\|.$$

Thus, $\|A\| \le \|P\|$. A similar argument shows that $\|B\|$, $\|C\|$, $\|D\| \le \|P\|$. We therefore have

$$\sup\{\|A\|, \|B\|, \|C\|, \|D\|\} \le \|P\| \le \|A\| + \|B\| + \|C\| + \|D\|.$$

An unbounded regular operator on a Hilbert C^*-module is an analogue of a closed operator on a Hilbert space that naturally arises in the study of locally compact quantum groups and in noncommutative geometry. Let us quickly recall the definition. A densely defined closed $\mathcal{A}$-linear map $t : \mathrm{Dom}\,(t) \subseteq V \to W$ is called *regular* if it is adjointable and the operator $1 + t^*t$ has a dense range. We denote the set of all regular operators from V to W by $R(V,W)$. It is well known that a densely defined operator $t : \mathrm{Dom}\,(t) \subseteq V \to W$ which has a densely defined adjoint operator t^* is regular if and only if its graph is orthogonally complemented in $V \oplus W$ (see, e.g., [**4**]). If t is regular, then t^* is regular and $t = t^{**}$; moreover, t^*t is regular and self-adjoint. Define $Q_t = (1+t^*t)^{-1/2}$ and $F_t = tQ_t$; then $\mathrm{Ran}\,(Q_t) = \mathrm{Dom}\,(t)$, $0 \leq Q_t \leq 1$ in $B(V,V)$ and $F_t \in B(V,W)$ [**8**, (10.4)]. The bounded operator F_t is called the bounded transform of the regular operator t. According to [**8**, Theorem 10.4], the map $t \to F_t$ defines an adjoint-preserving bijection

$$R(V,W) \longrightarrow \{F \in B(V,W) : \|F\| \leq 1 \text{ and } \mathrm{Ran}\,(1 - F^*F) \text{ is dense in } V\}.$$

We refer to the papers [**4, 5, 11**] and [**8**, Chapters 9, 10] for more detailed information.

2. Gap topologies. Let $t \in R(V,W)$ be a regular operator and $G(t) = \{(x, tx) : x \in \mathrm{Dom}\,(t)\}$ its graph; then $V \oplus W = G(t) \oplus \mathcal{V}(G(t^*))$ in which $\mathcal{V} \in B(V \oplus W, W \oplus V)$ is defined by $\mathcal{V}(x,y) = (y, -x)$. In view of [**8**, (9.7)] the orthogonal projection $P_{G(t)} : V \oplus W \to V \oplus W$ can be described through the following matrix

$$P_{G(t)} = \begin{pmatrix} Q_t^2 & Q_tF_t^* \\ F_tQ_t & F_tF_t^* \end{pmatrix}. \tag{2.1}$$

Definition 2.1. Let $t, s \in R(V,W)$. Then the gap metric on the space of all unbounded regular operators is defined by $\mathrm{gap}(t,s) = \|P_{G(t)} - P_{G(s)}\|$ where $P_{G(t)}$ and $P_{G(s)}$ are orthogonal projections onto $G(t)$ and $G(s)$, respectively. The topology induced by this metric is called gap topology.

Gap topology has been studied systematically in the book [**6**] and papers [**1, 14**] and references therein. Following [**14**], the space of all bounded adjointable operators between Hilbert C^*-modules is an open dense subset of the space of all unbounded regular operators with respect to the gap topology. The restriction of gap topology on the space of all bounded adjointable operators is equivalent with the topology which is generated by the usual operator norm. However, the uniform structures induced by the gap metric and by the operator norm on the space of bounded adjointable operators are different. This follows from the fact that the metric which is given by the usual norm of bounded operator is complete while the gap metric on the set of bounded adjointable operators is not complete.

Remark 2.2. Let F be a bounded adjointable operator in $B(V, W)$ of norm $\|F\| \leq 1$; then $\mathcal{F}(F) := 1 - F^*F$ is a positive operator. In particular, for a regular operator $t \in R(V, W)$ and its bounded transform F_t we have

$$Q_t^2 = 1 - F_t^* F_t = \mathcal{F}(F_t), \quad 1 - F_t F_t^* = \mathcal{F}(F_t^*), \quad F_t Q_t = F_t \mathcal{F}(F_t)^{1/2}.$$

An equivalent picture of the gap metric is now definable via the bounded transforms of regular operators t and s. Indeed, the following metric is uniformly equivalent to the gap metric

$$\begin{aligned} d(t,s) = \sup \{ &\|\mathcal{F}(F_t) - \mathcal{F}(F_s)\|, \|\mathcal{F}(F_t^*) - \mathcal{F}(F_s^*)\|, \\ &\|F_t\mathcal{F}(F_t)^{1/2} - F_s\mathcal{F}(F_s)^{1/2}\|\}. \end{aligned} \tag{2.2}$$

To see this, we use the fact that $(F_t Q_t)^* = Q_t F_t^*$ and get

$$\begin{aligned} &P_{G(t)} - P_{G(s)} \\ &\quad = \begin{pmatrix} \mathcal{F}(F_t) - \mathcal{F}(F_s) & (F_t\mathcal{F}(F_t)^{1/2} - F_s\mathcal{F}(F_s)^{1/2})^* \\ F_t\mathcal{F}(F_t)^{1/2} - F_s\mathcal{F}(F_s)^{1/2} & \mathcal{F}(F_t^*) - \mathcal{F}(F_s^*) \end{pmatrix}. \end{aligned}$$

Now the inequalities $d(t,s) \leq \|P_{G(t)} - P_{G(s)}\| \leq 4\, d(t,s)$ follow from Remark 1.1.

Definition 2.3. Let V, W be two Hilbert $\mathcal{A}$-modules and $B_1(V, W)$ the set of bounded adjointable operators $T : V \to W$ of norm $\|T\| \leq 1$. We define the pseudo metric

$$\begin{aligned} \sigma(T,S) = \sup \{ &\|\mathcal{F}(T) - \mathcal{F}(S)\|, \|\mathcal{F}(T^*) - \mathcal{F}(S^*)\|, \\ &\|T\mathcal{F}(T)^{1/2} - S\mathcal{F}(S)^{1/2}\|\} \end{aligned} \tag{2.3}$$

on the set $B_1(V, W)$. The topology induced by this pseudo metric is called extended gap topology.

Corollary 2.4. *The extended gap topology is weaker than the topology which is generated by the usual operator norm on $B_1(V,W)$.*

Proof. Let $\{T_n\}$ be a sequence in $B_1(V,W)$ which converges to an element T in $B_1(V,W)$ with respect to the norm topology, i.e., $\|T_n - T\| \to 0$. By elementary methods and continuity of the function $f(x) = x^{1/2}$ on $[0,+\infty)$, we find

$$\begin{aligned}
&\|\mathcal{F}(T_n) - \mathcal{F}(T)\| = \|T_n^* T_n - T^*T\| \longrightarrow 0,\\
&\|\mathcal{F}(T_n^*) - \mathcal{F}(T^*)\| = \|T_n T_n^* - TT^*\| \longrightarrow 0,\\
&\|\mathcal{F}(T_n)^{1/2} - \mathcal{F}(T)^{1/2}\| \longrightarrow 0,\\
&\|T_n\mathcal{F}(T_n)^{1/2} - T\mathcal{F}(T)^{1/2}\| \longrightarrow 0.
\end{aligned}$$

Therefore (2.3) implies that

$$\begin{aligned}
\sigma(T_n, T) = \sup\{\|\mathcal{F}(T_n) - \mathcal{F}(T)\|,\ &\|\mathcal{F}(T_n^*) - \mathcal{F}(T^*)\|,\\
&\|T_n\mathcal{F}(T_n)^{1/2} - T\mathcal{F}(T)^{1/2}\|\} \longrightarrow 0,
\end{aligned}$$

i.e., the sequence $\{T_n\}$ converges to the bounded adjointable operator T in the extended gap topology. $\square$

The following example shows that the extended gap topology is strictly weaker than the topology which is generated by the usual operator norm.

Example 2.5. Let $\mathcal{A}$ be unital C^*-algebra and $H_\mathcal{A}$ be the standard Hilbert $\mathcal{A}$-module which is countably generated by orthonormal basis $\xi_j = (0,\dots,0,1,0,\dots,0)$, $j \in \mathbf{N}$. For any integer n define the bounded self-adjoint operator $T_n : H_\mathcal{A} \to H_\mathcal{A}$ by

$$T_n(\xi_j) = \begin{cases} n/(n+1)\,\xi_j & \text{if } n \text{ is even},\\ -n/(n+1)\,\xi_j & \text{if } n \text{ is odd}.\end{cases}$$

Suppose $T = 1$ is the identity operator of $B(H_\mathcal{A}, H_\mathcal{A})$; then

$$\begin{aligned}
\|\mathcal{F}(T_n) - \mathcal{F}(T)\| = \|\mathcal{F}(T_n^*) - \mathcal{F}(T^*)\| = \|T_n^2 - T^2\| \longrightarrow 0,&\\
\|T_n\mathcal{F}(T_n)^{1/2} - T\mathcal{F}(T)^{1/2}\| = \|\frac{\sqrt{2n+1}}{n+1}\ T_n - 0\| \longrightarrow 0.&
\end{aligned}$$

Consequently, $\sigma(T_n, T) \to 0$ as $n \to +\infty$. Thus the sequence $\{T_n\}$ converges to the bounded operator T in the extended gap topology. However, the sequence T_n of bounded self-adjoint operators is not convergent to any operator in $B_1(H_{\mathcal{A}}, H_{\mathcal{A}})$ in the norm topology.

We define $(\mathcal{X}, \sigma)$ to be the metric space that we obtain when we equip the set $\mathcal{X} = \{F \in B(V,W) : \|F\| \leq 1 \text{ and } \mathrm{Ran}\,(1 - F^*F) \text{ is dense in } V\}$ with the extended gap topology. Indeed, the restriction of the pseudo metric σ to the set $\mathcal{X}$ (still denoted by σ) is a metric on $\mathcal{X}$. To see this, suppose $T, S \in \mathcal{X}$ and $\sigma(T,S) = 0$; then $T\mathcal{F}(T)^{1/2} = S\mathcal{F}(S)^{1/2} = S\mathcal{F}(T)^{1/2}$ which implies $T = S$ on V, since $\mathcal{F}(T)^{1/2} = (1 - T^*T)^{1/2}$ has dense range.

Theorem 2.6. *Let V, W be Hilbert $\mathcal{A}$-modules. Suppose that $R(V,W)$ is equipped with the gap topology. Then $\tau : R(V,W) \to (\mathcal{X}, \sigma)$, $\tau(t) = F_t$ is an isometric adjoint-preserving map of $R(V,W)$ onto the metric space $(\mathcal{X}, \sigma)$. In particular, τ is an adjoint-preserving homeomorphism.*

Proof. In view of (2.2) and (2.3), we obtain

$$\sigma(\tau(t), \tau(s)) = \sigma(F_t, F_s) = d(t, s).$$

That is, τ is an isometry and is therefore continuous and open. It also preserves the adjoint operation by Theorem 10.4 of [**8**]. $\square$

Beside the results of [**14**], we can use the above fact to occasionally think of $R(V,W)$ as being a subspace of $B_1(V,W)$ with respect to the extended gap topology.

Suppose $\mathcal{A}$ is an arbitrary C^*-algebra of compact operators and V, W are Hilbert $\mathcal{A}$-modules. Then every densely defined closed operator $t : \mathrm{Dom}\,(t) \subseteq V \to W$ is automatically regular (see [**4, 5**]). This fact enables us to reformulate Theorem 2.6 in terms of densely defined closed operators on Hilbert C^*-modules over C^*-algebras of compact operators, or in terms of densely defined closed operators on Hilbert spaces.

Acknowledgments. The authors would like to thank the referee for his/her useful comments.

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