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NUMERICAL RANGES OF COMPOSITION OPERATORS WITH INNER SYMBOLS

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ABSTRACT. Operators on function spaces acting by composition to the right with a fixed self-map φ of some set are called composition operators with the symbol φ. In this paper, composition operators on the Hilbert Hardy space over the unit disk are considered. The numerical ranges of composition operators with inner symbol of parabolic automorphic type or hyperbolic type are shown to be circular.

1. Introduction. Let H^2 denote the Hilbert Hardy space on the open unit disk $\mathbf{U}$, that is, the space of all functions f analytic in $\mathbf{U}$ satisfying the condition

$$\|f\|_2 := \sup_{0<r<1} \left(\int_{\partial \mathbf{U}} |f(r\zeta)|^2 \, dm(\zeta) \right)^{1/2} < +\infty, \tag{1}$$

where m is the normalized Lebesgue measure. Actually, the integrals in formula (1) tend increasingly to $\|f\|_2$ as $r \to 1^-$.

It is well known that $\| \, \|_2$ can also be calculated with the formula

$$\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |c_n|^2}, \tag{2}$$

where $\{c_n\}$ is the sequence of Maclaurin coefficients of f.

The space H^∞ is the space of all bounded analytic functions on $\mathbf{U}$ endowed with the supremum norm $\| \, \|_\infty$. Obviously, $H^\infty \subseteq H^2$ since $\|f\|_2 \leq \|f\|_\infty$. A well-known fact about H^2-functions is the fact that, by a classical result of Fatou [**10**, Theorem 1.3], eventually extended by F. and M. Riesz, those functions have nontangential limits almost

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everywhere on $\partial\mathbf{U}$. The nontangential limit function of any f in H^2 will be denoted by the same symbol as the function itself. It is known that it is an $L^2_{\partial\mathbf{U}}$-function and

$$\|f\|_2 = \left(\int_{\partial\mathbf{U}} |f(\zeta)|^2 \, dm(\zeta) \right)^{1/2}, \quad f \in H^2. \tag{3}$$

A bounded analytic function on $\mathbf{U}$ is called an *inner function* if it has unimodular nontangential limits almost everywhere on $\partial\mathbf{U}$.

For each analytic self-map φ of $\mathbf{U}$ *the composition operator with symbol* φ is the following operator

$$C_\varphi f = f \circ \varphi, \quad f \in H^2. \tag{4}$$

Such operators are bounded, as a consequence of Littlewood's subordination principle, [**10**, Theorem 1.7], saying that composition operators whose symbol fixes the origin are contractions. If φ is a conformal automorphism of $\mathbf{U}$, we call C_φ an automorphic composition operator.

The numerical range of a Hilbert space operator T is the set $W(T) = \{\langle Tf, f\rangle : \|f\| = 1\}$. It is well known that the numerical range of a bounded operator is a convex subset of the complex plane whose closure contains the spectrum of the given operator, [**12**, Chapter 22]. The quantity $w(T) = \sup\{|\langle Tf, f\rangle| : \|f\| = 1\}$ is called the numerical radius of the operator T.

Our main goal in this paper is proving that composition operators with inner symbol of parabolic automorphic type or hyperbolic type have circular numerical ranges. Some explanations are in order here.

Analytic self-maps of $\mathbf{U}$ are classified as *symbols of hyperbolic*, respectively *parabolic type*, based on a noted theorem. We state it in the following and use in its text the notation $\varphi^{[n]}$ to designate the nth iterate of a self-map φ of $\mathbf{U}$, that is, $\varphi^{[n]} = \varphi \circ \cdots \circ \varphi$, n-times, $n = 1, 2, 3, \dots$.

Theorem 1 (Denjoy-Wolff). *Let φ be an analytic self-map of $\mathbf{U}$ other than the identity or an elliptic disk automorphism. Then the sequence of iterates $\{\varphi^{[n]}\}$ converges uniformly on compacts to a constant $\omega \in \overline{\mathbf{U}}$ called the Denjoy-Wolff point of φ.*

For any self-map φ and any $z \in \mathbf{U}$ the set $O_\varphi(z) = \{z, \varphi(z), \ldots, \varphi^{[n]}(z), \ldots\}$ is called the *orbit* of z under φ.

If the Denjoy-Wolff point ω of an analytic self-map φ is on $\partial\mathbf{U}$, then the angular derivative $\varphi'(\omega)$ is known to exist and satisfy the condition $0 < \varphi'(\omega) \leq 1$. If $\varphi'(\omega) < 1$, then φ is called a self-map *of hyperbolic type*. If $\varphi'(\omega) = 1$, then φ is called a self-map *of parabolic type*. Analytic self-maps of parabolic type are classified into two categories. The first is self-maps of *parabolic automorphic type*. This means that the self-map φ of parabolic type has *hyperbolically separated orbits*, that is,

$$\lim_{n\to+\infty} \rho(\varphi^{[n+1]}(z), \varphi^{[n]}(z)) > 0 \quad z \in \mathbf{U}, \tag{5}$$

where ρ is the pseudohyperbolic distance $\rho(z, w) = |(w - z)/(1 - \overline{w}z)|$, $z, w \in \mathbf{U}$. Either all the orbits of an analytic self-map of parabolic type are hyperbolically separated or all of them are hyperbolically non-separated, that is,

$$\lim_{n\to+\infty} \rho(\varphi^{[n+1]}(z), \varphi^{[n]}(z)) = 0, \quad z \in \mathbf{U}. \tag{6}$$

In case (6) holds, φ is called a self-map *of parabolic non-automorphic type*. The limits in (5) or (6) necessarily exist because the sequence under scrutiny is decreasing, by the Schwarz-Pick lemma [**22**, Section 4.3], saying that analytic self-maps of $\mathbf{U}$ are contractive under the pseudohyperbolic distance, that is, if φ is such a map, then

$$\rho(\varphi(z), \varphi(w)) \leq \rho(z, w), \quad z, w \in \mathbf{U}.$$

The terminology *parabolic automorphic type*, respectively *hyperbolic type* is also related to linear fractional model-theory for analytic self-maps of $\mathbf{U}$, a body of knowledge that contains Theorem 2 in the next section and similar additional results. We refer the reader to [**22**] for more details on it.

If the Denjoy-Wolff point is in $\mathbf{U}$, that is, if φ is an analytic self-map of $\mathbf{U}$ with a fixed point, then the numerical range of C_φ can exhibit quite a variety of shapes, [**16**]. If φ is inner, not the identity or a rotation, and the fixed point is the origin, then $W(C_\varphi) = \mathbf{U} \cup \{1\}$, [**16**]. The case when $\varphi(z) = \lambda z$, $|\lambda| = 1$, is very easy to handle, leading to a regular polygon inscribed in $\partial\mathbf{U}$ if λ is a root of 1, respectively to $\mathbf{U} \cup \{\lambda^n : n \geq 0\}$, when λ is not a root of 1, [**16**]. There are only two kinds of quadratic composition operators: those of constant symbol and those having symbol of the form $\alpha_p(z) = (p - z)/(1 - \overline{p}z)$, where $p \in \mathbf{U}$. They have elliptical numerical ranges, (possibly degenerate, that is, the elliptical disk can be reduced to the focal axis). The elliptical disk is closed in the case of constant symbols and open in the other case, [**17**]. Composition operators of symbol α_p are automorphic composition operators whose symbol fixes a point, (also known as elliptic automorphisms). For any elliptic automorphic symbol, that symbol is conformally conjugated to a rotation. If the fixed point is not the origin and the aforementioned rotation is not by a root of unity, then $\overline{W(C_\varphi)}$ is a disk about the origin, [**6**, Theorem 4.1], but the exact description of $W(C_\varphi)$ is currently unknown. Composition operators whose symbols are monomials fixing the origin can have polygonal or cone-like numerical ranges, [**16**]. Besides the few facts summarized in this section, very little is known about numerical ranges of composition operators on H^2.

According to [**6**], if φ is a parabolic or hyperbolic disk automorphism, then $W(C_\varphi)$ is a circular disk centered at the origin. Our main results in this paper are extensions of those facts. We show that $W(C_\varphi)$ is a circular disk centered at the origin if φ is an arbitrary inner function of parabolic automorphic type, respectively an arbitrary inner function of hyperbolic type (not just a parabolic or hyperbolic disk automorphism). Those results are obtained in Section 2, (Theorem 3), respectively Section 3 (Theorem 4).

2. Inner symbols of parabolic automorphic type. The following theorem, [**20**, Theorem 1], is the main tool in showing that composition operators with inner symbols of parabolic automorphic type have circular numerical ranges.

Theorem 2. *Let φ be an analytic self-map of $\mathbf{U}$ with Denjoy-Wolff point on $\partial\mathbf{U}$. If φ is of parabolic automorphic type, then there is some analytic σ from $\mathbf{U}$ into the right half-plane satisfying*

$$\sigma \circ \varphi = \sigma + ib \tag{7}$$

for some nonzero real constant b. If φ is of hyperbolic type, then

$$\sigma \circ \varphi = K\sigma \tag{8}$$

for some $K > 1$.

It should be added that the hyperbolic case was originally proved by Valiron [**24**], and Pommerenke's new contribution in [**20**] was to treat the case of symbols of parabolic type. He wrote a unified proof of both (7) and (8) which will be used in one of the following technical lemmas to show that, if φ is inner, then the maps σ satisfying (7), respectively (8), can be chosen so that they have purely imaginary nontangential limits almost everywhere.

In order to prove that, we record first a Hilbert-space principle that appears in [**4**, Proof of Theorem 3.1] and is very easy to prove.

Lemma 1. *If a sequence* $\{x_n\}$ *in the unit ball of a Hilbert space* H *converges weakly to a norm-one vector* x*, then* $\{x_n\}$ *is norm-convergent to* x.

Based on this principle we record a second technical lemma whose proof is straightforward.

Lemma 2. *A sequence of inner functions tends uniformly on compacts to an inner function if and only if it is* $\| \ \|_2$*-convergent to that function.*

Relative to weak convergence in H^2, we wish to note that H^2 is a reproducing kernel Hilbert space, (a Hilbert space where point evaluations are bounded functionals), and for that reason, a sequence in H^2 is weakly convergent if and only if it is norm-bounded and pointwise convergent. Using Montel's theorem in classical complex analysis, it can be shown that, actually, weakly convergent sequences in H^2 tend to their limit not just pointwise, but even uniformly on compacts.

Lemma 3. *If* φ *is an inner function of parabolic automorphic type, or of hyperbolic type, then the map* σ *satisfying* (7)*, respectively* (8)*, can* *be chosen so that its radial limit function has purely imaginary values almost everywhere.*

Proof. Without loss of generality, one can assume that the Denjoy-Wolff point of φ is 1. The existence of such σ for an arbitrary analytic self-map φ of parabolic automorphic type or hyperbolic type, having Denjoy-Wolff point 1, was proved by Pommerenke [**20**] by constructing σ as the uniform limit on compacts of the sequence

$$g_n(z) = \frac{(1+\varphi^{[n]}(z))/(1-\varphi^{[n]}(z)) - i\Im(1+\varphi^{[n]}(0))/(1-\varphi^{[n]}(0))}{\Re(1+\varphi^{[n]}(0))/(1-\varphi^{[n]}(0))}, \; z \in \mathbf{U}.$$

For all n, the radial limits of g_n are purely imaginary almost everywhere, (since φ is inner). Thus, the sequence

$$f_n = \frac{g_n - 1}{g_n + 1}, \quad n = 1, 2, 3, \dots ,$$

consists of inner functions and tends uniformly on compacts to the function $(\sigma - 1)/(\sigma + 1)$. In order to establish that σ has purely imaginary radial limits almost everywhere, it is necessary (and sufficient) to prove that $(\sigma - 1)/(\sigma + 1)$ is inner. By Lemma 2, this happens if and only if the sequence $\{f_n\}$ is $\| \, \|_2$-Cauchy. Note that $\{f_n\}$ is a sequence of inner functions, and $|f_m/f_n| \leq 1$ if $m \geq n$, which is a consequence of the fact that, by the Schwarz-Pick lemma, $|f_{n+1}| \leq |f_n|$ for all n, (see also [**20**, (3.6)]). Thus, the family $\{f_m/f_n\}_{m \geq n}$ consists of inner functions, and hence $\|f_m - f_n\|_2 = \|f_m/f_n - 1\|_2$, $m \geq n$. Therefore, in order to prove $\|f_m - f_n\|_2 \to 0$ as $m, n \to +\infty$, it will suffice to show that $f_m/f_n \to 1$ pointwise on $\mathbf{U}$ as $m, n \to +\infty$, (by Lemma 1). Indeed, $f_n \to f = (\sigma - 1)/(\sigma + 1)$ uniformly on compacts and hence pointwise. The function f is a nonzero analytic self-map of $\mathbf{U}$, so for each $z \in \mathbf{U}$ with property $f(z) \neq 0$, $f_m(z)/f_n(z) \to f(z)/f(z) = 1$ if $m, n \to +\infty$. If, arguing by contradiction, one assumes that there is a point $p \in \mathbf{U}$ and some $\varepsilon_0 > 0$ so that $|f_m(p)/f_n(p) - 1| \geq \varepsilon_0$ for infinitely many $m > n$, one can use Montel's theorem to get a sequence $\{f_{m_k}/f_{n_k}\}$ that satisfies $|f_{m_k}(p)/f_{n_k}(p) - 1| \geq \varepsilon_0$ and tends uniformly on compacts to a, necessarily holomorphic map h. By what we noted above, h coincides to 1 on the subset of $\mathbf{U}$ where f is nonzero. By the analytic continuation principle, it follows that h is identically 1, hence

$f_{m_k}(p)/f_{n_k}(p) \to 1$, a contradiction. We proved by contradiction that $\{f_n\}$ is $\| \, \|_2$-Cauchy. □

The following principle is used in the particular setting of hyperbolic automorphic composition operators to establish one of the main results in [**6**].

Proposition 1. *A composition operator C_φ on H^2 which has inner eigenfunctions associated to each $\lambda \in \partial\mathbf{U}$ must have circular numerical range with center at* 0.

Proof. If $w \in W(C_\varphi)$, that is, $w = \langle C_\varphi f, f\rangle$, for some $f \in H^2$, $\|f\|_2 = 1$, then choose any $\lambda \in \partial\mathbf{U}$ and any inner eigenfunction u of C_φ corresponding to the eigenvalue λ and note that $\|uf\|_2 = 1$ and $\langle C_\varphi uf, uf\rangle = \lambda\langle uf \circ \varphi, uf\rangle = \lambda\langle C_\varphi f, f\rangle = \lambda w$. This shows that $W(C_\varphi)$ has circular symmetry, (that is, if it contains a point, then it contains the whole circle about the origin passing through that point). Since numerical ranges are convex, $W(C_\varphi)$ must be a circular disk, (open or closed) about the origin. □

The principle above is satisfied in the following situation:

Proposition 2. *The numerical range of a composition operator which has a singular inner eigenfunction associated to some unimodular eigenvalue $\lambda \neq 1$ is a disk centered at the origin.*

Proof. Recall that singular inner functions are of type

$$S_\nu(z) = \exp\left(- \int_{\partial\mathbf{U}} \frac{u+z}{u-z}\, d\nu(u)\right), \quad z \in \mathbf{U},$$

where ν is a finite, nonnegative, Borel measure which is singular with respect to Lebesgue measure. If S_ν is an eigenfunction of some composition operator C_φ associated to some $\lambda = e^{i\theta_0} \neq 1$, then

$$\left(-\int_{\partial\mathbf{U}} \frac{u+\varphi(z)}{u-\varphi(z)}\, d\nu(u)\right) - \left(-\int_{\partial\mathbf{U}} \frac{u+z}{u-z}\, d\nu(u)\right) = (\theta_0 + 2k\pi)i, \quad z \in \mathbf{U},$$

for some integer k. Therefore, for all $a > 0$, $S_{a\nu}$ is an eigenfunction of C_φ corresponding to the eigenvalue $\lambda_a = e^{ia(\theta_0 + 2k\pi)}$. On the other hand, the eigenvalues λ_a, $a > 0$, exhaust the unit circle. By Proposition 1, the numerical range of C_φ is a disk centered at 0. $\square$

Based on the technical lemmas and propositions above, we can treat the case of symbols of parabolic automorphic type. This is the main result of the current section of this paper.

Theorem 3. *The numerical range of a composition operator whose symbol φ is an inner function of parabolic automorphic type is a circular disk centered at the origin having radius larger than* 1.

Proof. Let φ be an analytic self-map of $\mathbf{U}$ of parabolic automorphic type, and let σ be associated to φ, as in Theorem 2. It is easy to check that, for each $0 < \theta < 2\pi$, the function $\exp(-(\theta/b)\sigma)$, if $b > 0$ respectively, $\exp((\theta/b)\sigma)$, if $b < 0$ is an eigenfunction of C_φ corresponding to the eigenvalue $e^{-i\theta}$, respectively $e^{i\theta}$. Since σ is valued in the right half-plane and purely imaginary almost everywhere on $\partial\mathbf{U}$, the eigenfunctions above are zero-free inner functions, that is, singular inner functions. By Proposition 2, it follows that $W(C_\varphi)$ is a disk centered at 0. That disk has radius larger than 1. Indeed 1 is an interior point of $W(C_\varphi)$, whenever $\varphi(0) \neq 0$, [**16**, Proposition 3.3]. $\square$

A consequence of the proof above is the following.

Remark 1. The composition operators that satisfy the assumptions in Proposition 1 and, hence, all composition operators induced by inner functions of parabolic automorphic type have point spectrum with circular symmetry.

Indeed, if $\lambda \neq 0$ is an eigenvalue of C_φ and f an eigenfunction corresponding to it, then for each μ with property $|\mu| = |\lambda|$ one can choose an inner eigenfunction corresponding to the eigenvalue $\mu\lambda^{-1}$ and consider the function $uf \neq 0$ which is visibly an eigenfunction of C_φ corresponding to the eigenvalue μ.

Can relation (8) be used to produce inner eigenfunctions in the case of symbols of hyperbolic type? It is known that it can be used to produce outer eigenfunctions (see [**10**] or [**13**] for the notion of *outer function*). This has been first noted by Nordgren [**18**]in the case of hyperbolic disk automorphisms and eventually extended to symbols of hyperbolic type, using Theorem 2 in the process. More exactly, if φ is such a symbol, then based on Theorem 2, one can consider any unimodular $\lambda = e^{i\theta}$ and, by using the notation in Theorem 2, construct the function $f_\lambda(z) := \exp(i\theta \log \sigma(z)/ \log K)$. Based on (8), one readily obtains $C_\varphi f_\lambda = \lambda f_\lambda$. The fact that, for all $\lambda \in \partial\mathbf{U}$, one has that $f_\lambda \in H^2$ is proved in [**9**, Theorem 7.21], where it is actually shown that $f_\lambda,\ 1/f_\lambda \in H^\infty$; thus, f_λ is a bounded outer eigenfunction. The interesting consequence of all these considerations is:

Remark 2. If φ is a symbol of hyperbolic type, then

$$\overline{\mathbf{U}} \subseteq W(C_\varphi). \tag{9}$$

Relative to Remark 1, the circular symmetry of the whole spectrum, not just the point spectrum, is established for arbitrary composition operators with symbol of hyperbolic type in [**9**, Theorem 7.21], (by using the eigenfunctions f_λ). This makes it very tempting to conjecture that, at least when the symbol is inner, the numerical range is a disk about 0. We are able to prove that fact in the next section.

3. Inner symbols of hyperbolic type. The well-known Herglotz theorem [**21**, Theorem 11.19] says that each nonnegative harmonic function on $\mathbf{U}$ is representable as the Poisson integral of a finite positive Borel measure.

For each $\alpha \in \partial\mathbf{U}$, τ_α, the Aleksandrov measure of index α of φ, is the measure whose Poisson integral equals $P(\varphi(z), \alpha)$, where $P(w, \alpha)$, $w \in \mathbf{U}$, $\alpha \in \partial\mathbf{U}$ is the usual Poisson kernel. These measures were introduced and studied by Aleksandrov [**2**] and used by Cima and Matheson [**7**] to understand compact composition operators on Hardy spaces. This author used them to prove essential norm formulas for composition operators with inner symbols [**15**].

Related to that, we consider the operator A_φ:

$$P_{A_\varphi(\mu)} = P_\mu \circ \varphi, \quad \mu \in \mathcal{M}.$$

Above, $\mathcal{M}$ is the space of complex Borel measures on $\partial\mathbf{U}$ and, for all measures ν, P_ν denotes the Poisson integral of ν. In other words, $A_\varphi(\mu)$ is, by definition, the measure whose Poisson integral equals $P_\mu \circ \varphi$. The existence of such a measure is a consequence of representing each complex Borel measure as a finite linear combination with complex coefficients of nonnegative Borel measures and Herglotz's theorem.

It is elementary to see that

$$V\mu = P_\mu, \quad \mu \in \mathcal{M}$$

is an isometry of $\mathcal{M}$ onto h^1, the space of complex-valued harmonic functions f on $\mathbf{U}$ satisfying the mean growth condition

$$\|f\|_1 = \sup_{0<r<1} \int_{\partial\mathbf{U}} |f(ru)|\, dm(u) < +\infty,$$

(see also [**21**, Theorem 11.19]).

If the composition operator of symbol φ acting on h^1

$$C_\varphi f = f \circ \varphi, \quad f \in h^1,$$

is also denoted C_φ, then $V^{-1}C_\varphi V = A_\varphi$. Thus, the operators A_φ are similar copies of composition operators. To avoid confusion we chose to denote them by A_φ (although some authors [**23**] call these operators too composition operators). The symbol C_φ will designate from now on only the composition operator on H^2 with symbol φ.

The Aleksandrov measures are related to the operators A_φ as follows: $\tau_\alpha = A_\varphi(\delta_\alpha)$, $\alpha \in \partial\mathbf{U}$, where as usual δ_α is the unit point mass measure at α. Thus, the Aleksandrov measure of index α of φ is the nonnegative, finite measure τ_α on $\partial\mathbf{U}$ whose Poisson integral equals the nonnegative harmonic function

$$F_\alpha(z) = \Re\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = P(\varphi(z), \alpha), \quad z \in \mathbf{U}. \tag{10}$$

A well-known fact about the Poisson integral of a measure on $\partial\mathbf{U}$ is that its radial limit function is the absolutely continuous part in its Lebesgue decomposition with respect to the normalized arc-length measure m on $\partial\mathbf{U}$. Therefore, if we denote by σ_α the singular part of τ_α, then (10) leads to the equality

$$d\tau_\alpha(\xi) = \frac{1-|\varphi(\xi)|^2}{|\alpha-\varphi(\xi)|^2}\, dm(\xi) + d\sigma_\alpha(\xi). \tag{11}$$

An immediate consequence of the above equality is the fact that φ is inner if and only if $\tau_\alpha = \sigma_\alpha$, $\alpha \in \partial\mathbf{U}$. A formula obtained by this author is the key to understanding the action of A_φ on singular measures, in the particular case of automorphic symbols.

Proposition 3. *If φ is a disk automorphism and $\mu \perp m$ a Borel measure on $\partial\mathbf{U}$, then*

$$A_\varphi(\mu)\varphi^{-1} = P(\varphi(0), u)\, d\mu(u). \tag{12}$$

Proof. It is elementary to see that $|S_\mu| = \exp(-P_\mu)$. On the other hand, $S_\mu \circ \varphi$ is inner and zero-free, hence a unimodular multiple of the singular inner function S_ν, where ν is given by

$$d\nu\varphi^{-1}(v) = P(\varphi(0), v)\, d\mu(v),$$

which is proved in [**14**, Lemma 3.1]. Thus $P_\nu = P_\mu \circ \varphi$. $\square$

A straightforward application of formula (12) is calculating the Aleksandrov measures of disk automorphisms. Indeed, combining (11) and (12) leads to the formula

$$A_\varphi(\delta_\alpha) = \tau_\alpha = \sigma_\alpha = P(\varphi(0), \alpha)\delta_{\varphi^{-1}(\alpha)}, \quad \alpha \in \partial\mathbf{U}, \tag{13}$$

valid for any disk automorphism φ. An application of this formula is proving the following.

Lemma 4. *Automorphic, hyperbolic composition operators have singular inner eigenfunctions associated to unimodular eigenvalues different from* 1.

Proof. Let the measure μ be defined as follows:

$$\mu := \sum_{n=-\infty}^{+\infty} w_n \delta_{\varphi^{[n]}(u)},$$

where $u \in \partial\mathbf{U}$ is any unimodular number which is not a fixed point of φ and, for a negative integer n, $\varphi^{[n]}$ designates the $|n|$-fold iterate of φ^{-1}, whereas $\varphi^{[0]}$ is the coordinate function.

The numbers w_n are defined as follows:

$$w_0 = 1, \qquad w_n P(\varphi(0), \varphi^{[n]}(u)) = w_{n-1}, \quad n = \pm 1, \pm 2, \pm 3, \dots .$$

The measure μ is a finite, nonnegative, singular measure because $|w_n/w_{n-1}| \to 1/P(\varphi(0), \omega)$ if $n \to +\infty$, where ω denotes the Denjoy-Wolff point of φ. On the other hand, $1/P(\varphi(0), \omega) \leq \sup\{P(z, \omega)/P(\varphi(z), \omega) : z \in \mathbf{U}\} = \varphi'(\omega) < 1$, [**1**].

A similar argument involving the hyperbolic automorphism φ^{-1} shows that $\lim_{n\to-\infty} |w_{n-1}/w_n| < 1$.

By formula (13), $A_\varphi \mu = \mu$; thus, $A_\varphi a\mu = a\mu$, $a > 0$ and so, $S_{a\mu}$ is an eigenfunction of C_φ associated to some unimodular eigenvalue λ_a, for all $a > 0$. If we prove $\lambda_a \neq 1$, for some $a > 0$, the proof is over. Note that

$$\lambda_a = \frac{S_{a\mu} \circ \varphi(0)}{S_{a\mu}(0)} = \exp\left(-ia \sum_{n=-\infty}^{+\infty} \frac{2|\varphi(0)| \sin(\operatorname{Arg}(\varphi^{[n]}(u)) - \operatorname{Arg}(\varphi(0)))}{|\varphi^{[n]}(u) - \varphi(0)|^2} w_n\right).$$

Assume the fixed points of φ are ± 1. Then φ leaves invariant the upper and lower semicircles, respectively. It also leaves invariant the diameter of endpoints ± 1, for which reason $\varphi(0)$ is a nonzero real number. Thus, all numbers $\varphi^{[n]}(u)$ are on the same semicircle as u, and $u \neq \pm 1$. Therefore, the numbers, $\sin(\operatorname{Arg}(\varphi^{[n]}(u)) - \operatorname{Arg}(\varphi(0))) = \sin(\operatorname{Arg}(\varphi^{[n]}(u)))$ are either all positive or all negative, depending on the sign of $\varphi(0)$. At any rate, this makes the quantity

$$\sum_{n=-\infty}^{+\infty} \frac{2|\varphi(0)| \sin(\operatorname{Arg}(\varphi^{[n]}(u)) - \operatorname{Arg}(\varphi(0)))}{|\varphi^{[n]}(u) - \varphi(0)|^2} w_n$$

nonzero, for which reason the eigenvalues λ_a cannot equal 1 for all $a > 0$.

If the fixed points of φ are not ± 1, then there is a disk automorphism α so that $\alpha^{-1} \circ \varphi \circ \alpha = \psi$ is a hyperbolic disk automorphism with fixed points ± 1. Given the induced operator similarity: $C_\alpha C_\varphi C_\alpha^{-1} = C_\psi$, it follows that C_φ has unimodular eigenvalues, not equal to 1, associated to singular inner eigenfunctions, because C_ψ has that property. □

The lemma above is a refinement of [**14**, Example 3.4]. We included it with its proof, though, for two reasons. On one hand, for the sake of completeness; on the other, because the result in [**14**] does not establish the fact that the eigenvalue under consideration, is different from 1. That fact is essential to us in this paper, where the lemma above, combined with our previous considerations, allows us to extend the result in the lemma to arbitrary composition operators whose symbol is an inner function of hyperbolic type. Indeed:

Theorem 4. *Let φ be an inner function of hyperbolic type. Then $W(C_\varphi)$ is a disk,* (*open or closed*) *centered at the origin, having radius larger than* 1.

Proof. Without loss of generality, assume 1 is the Denjoy-Wolff point of φ. By Theorem 2, there is σ an analytic map of $\mathbf{U}$ into the right half-plane satisfying (8) and, according to Lemma 3, the map $u := (\sigma - 1)/(\sigma + 1)$ may be assumed inner. It is straightforward to check that

$$u \circ \varphi = \phi \circ u \quad \text{where} \quad \phi(z) = \frac{z + r}{1 + rz} \quad \text{and } r = \frac{K - 1}{K + 1}. \tag{14}$$

As one can readily see, ϕ is a hyperbolic disk automorphism so, by Lemma 4, there is $1 \neq \lambda \in \partial \mathbf{U}$, so that the operator C_ϕ has a singular inner eigenfunction S_ν associated to the eigenvalue λ. By (14), $S_\nu \circ u$ is a zero-free inner function; thus, a singular inner function that is an eigenfunction of C_φ associated to the eigenvalue λ. By Proposition 2, $W(C_\varphi)$ must be a disk about 0. The fact that the radius of that disk is greater than 1 follows exactly as in the proof of Theorem 3. □

As a final comment, we wish to mention the fact that some authors trace the introduction of Aleksandrov measures as early as Clark's paper [**8**] and therefore call the aforementioned measures, Aleksandrov-Clark measures, [**19**].

REFERENCES

1. L.V. Ahlfors, *Conformal invariants*, in *Topics in geometric function theory*, McGraw-Hill, New York, 1973.

2. A.B. Aleksandrov, *Multiplicity of boundary values of inner functions*, Izv. Akad. Nauk Armyan. SSR **22** (1987), 490–503 (in Russian).

3. I.N. Baker and Ch. Pommerenke, *On the iteration of analytic functions in a half-plane* II, J. London Math. Soc. **20** (1979), 255–258.

4. P.S. Bourdon, V. Matache and J.H. Shapiro, *On convergence to the Denjoy-Wolff point*, Illinois J. Math. **49** (2005), 405–430.

5. P.S. Bourdon and J.H. Shapiro, *Cyclic phenomena for composition operators*, Mem. Amer. Math. Soc. **125** (1997), 1–105, American Math. Society, Providence, R.I.

6. ———, *The numerical ranges of automorphic composition operators*, J. Math. Anal. Appl. **251** (2000), 839–854.

7. J.A. Cima and A. Matheson, *Essential norms of composition operators and Aleksandrov measures*, Pacific J. Math. **179** (1997), 59–64.

8. D.N. Clark, *One dimensional perturbations of restricted shifts*, J. Anal. Math. **25** (1972), 169–191.

9. C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, FL, 1995.

10. P.L. Duren, *Theory of* H^p *spaces*, Academic Press, New York, 1970. Reprinted Dover, 2000.

11. J.B. Garnet, *Bounded analytic functions*, Second edition, Springer-Verlag, New-York, 2007.

12. P.R. Halmos, *A Hilbert space problem book*, Second edition, Springer-Verlag, New York, 1982.

13. K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, 1962.

14. V. Matache, *The eigenfunctions of a certain composition operator*, Contemp. Math. **213** (1998), 121–136, American Mathematical Society, Providence, RI.

15. ———, *A short proof of a characterization of inner functions in terms of the composition operators they induce*, Rocky Mountain J. Math. **35** (2005), 1723–1726.

16. ———, *Numerical ranges of composition operators*, Linear Algebra Appl. **331** (2001), 61–74.

17. ———, *Distances between composition operators*, Extracta Math. **22** (2007), 19–33.

18. E.A. Nordgren, *Composition operators*, Canad. J. Math. **20** (1968), 442–449.

19. A. Poltoratski and D. Sarason, *Aleksandrov-Clark measures*, in *Recent advances in operator theory*, Contemp. Math. **393**, American Mathematical Society, Providence, RI, 2006, 1–14.

20. Ch. Pommerenke, *On the iteration of analytic functions in a half-plane* I, J. London Math Soc. **19** (1979), 439–447.

21. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.

22. J.H. Shapiro, *Composition operators and classical function theory*, Springer-Verlag, Berlin, 1993.

23. J.H. Shapiro and C. Sundberg, *Compact composition operators on* L^1, Proc. Amer. Math. Soc. **108** (1990), 443–449.

24. G. Valiron, *Sur l'itération des fonctions holomorphes dans un demiplan*, Bull. Sci. Math. **55** (1931), 105–128.

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