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ON SUMS OF SQUARES OF PRIMES AND A kTH POWER OF PRIME

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ABSTRACT. In this article we consider the exceptional set of integers, not restricted by elementary congruence conditions, which cannot be represented as sums of two or three squares of primes and a kth power of prime for any integer $k \geq 2$. For example, we prove that with at most $O(N^{1-(1/3k2^{k-2})+\varepsilon})$ exceptions for $k \geq 4$, all positive integers $n \leq N$, satisfying the necessary congruence conditions, are the sum of two squares of primes and a kth of prime. This improves substantially the previous results in this direction.

1. Introduction. Let $\mathcal{A}_3$ be the set of all integers n which satisfy the conditions

- if k is odd, $n \not\equiv 0 \pmod 2$, $n \not\equiv 2 \pmod 3$;
- if k is even, $n \equiv 3 \pmod{24}$, $\begin{cases} n \not\equiv 0 \pmod 5 & \text{for } 4 \nmid k, \\ n \not\equiv 0, 2 \pmod 5 & \text{for } 4 \mid k, \end{cases}$

and $n \not\equiv 1 \pmod p$ for $p \equiv 3 \pmod 4$ and $(p-1) \mid k$. And also let $\mathcal{A}_4$ be the set of all integers n which satisfy

- if k is odd, $n \not\equiv 0 \pmod 3$;
- if k is even, $n \equiv 4 \pmod{24}$.

In 1938 Hua [**5**] proved that almost all $n \in \mathcal{A}_3$ are representable as sums of two squares of primes and a kth power of prime, from which it instantly follows that almost all $n \in \mathcal{A}_4$ are representable as sums of three squares of primes and a kth power of prime. More precisely, for any integer $k \geq 2$ and $j = 3, 4$, let

$$E_j(N) = |\mathcal{E}_j(N)|,$$

where

$$\mathcal{E}_j(N) = \{n \in \mathcal{A}_j : n \leq N, n \neq p_1^2 + \cdots + p_{j-1}^2 + p_j^k \text{ for any primes } p_u\}.$$

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Hua's result actually states that $E_j(N) \ll N(\log N)^{-A}$ for some positive constant A. Later, Schwarz's work [**19**] refined Hua's result by the same upper bound but for any $A > 0$. The upper bound of $E_3(N)$ was further improved by Leung and Liu [**8**] to $N^{1-\delta}$ for some computable but small positive δ depending on the constants in Deuring-Heilbronn phenomenon.

In particular, the case $k = 2$ of this problem has attracted many authors. Since 1998, when Liu and Zhan [**12**] found a new approach to increase the size of the "major arcs" in the application of the Hardy-Littlewood circle method, there has been a flurry of activity in this area; one may chart the developments in [**1, 2, 7, 9–11, 13, 14, 18, 22**].

In the case $k = 3$, Lü improved the upper bound of $E_3(N)$ to $N^{829/840+\varepsilon}$ in [**15**], and then to $N^{20/21+\varepsilon}$ in [**16**].

By using the circle method, we obtain the asymptotic formulae for the weighted number of solutions on the major arcs (see Propositions 2.1 and 5.1 for details) which will, as can be seen in the following sections, occupy the largest portion of this paper on applying the iterative method and the estimate for Dirichlet polynomials in [**9**]. While treating the minor arcs, we use the estimate in [**7**] to bound the exponential sums over primes (see Proposition 2.2). Collecting all these arguments, we make the following improvement.

Theorem 1. *Let $k \in \mathbf{N}$ and $E_3(N)$ be defined as above. Then, for all large N, we have*

$$E_3(N) \ll N^{1-(1/3k2^{k-2})+\varepsilon}, \quad \textit{for } k \geq 4.$$

Theorem 2. *Let $k \in \mathbf{N}$ and $E_4(N)$ be defined as above. Then, for all large N, we have*

$$E_4(N) \ll \begin{cases} N^{(19/42)+\varepsilon} & \textit{if } k = 3; \\ N^{(1/2)-(1/3k2^{k-2})+\varepsilon} & \textit{if } k \geq 4. \end{cases}$$

Remark. Applying the bound of exponential sums over primes in Proposition 2.2, we can obtain for $k = 2$, $E_3(N) \ll N^{(7/8)+\varepsilon}$ and $E_4(N) \ll N^{(3/8)+\varepsilon}$. The two estimates coincide with that of $E_3'(N)$

which was proved by Kumchev [**7**] and Ren [**18**] separately and that of $E_4'(N)$ established by Liu, Wooley and Yu [**11**], respectively, which are the optimal results by applying the circle method at present. Here, $E_j'(N)$ denotes the number of exceptions up to N for the problem with j squares. And, we will see in the next section, that the bound of exceptional sets is determined by the minor arc estimates.

Besides the above two aspects, we shall also investigate the representation of integers n as sums of a prime, a square of prime and a kth power of prime. Let $\mathcal{A}$ be the set of all integers n which satisfy

- if k is odd, $n \equiv 1 \pmod 2$;
- if k is even, $n \equiv 1$ or $3 \pmod 6$.

Then for any integer $k \geq 2$, we define

$$E(N) = |\{n \in \mathcal{A} : n \leq N, n \neq p_1 + p_2^2 + p_3^k \text{ for any primes } p_u\}|.$$

Recently, Wang [**21**] established that $E(N) \ll N^{(1/2)-(2/5k4^{k-1})+\varepsilon}$. Arguing similarly to the proofs of Theorems 1 and 2, we obtain the following.

Theorem 3. *Let $k \in \mathbf{N}$ and $E(N)$ be defined as above. Then, for all large N, we have*

$$E(N) \ll \begin{cases} N^{(3/8)+\varepsilon} & \text{if } k = 2; \\ N^{(19/42)+\varepsilon} & \text{if } k = 3; \\ N^{(1/2)-(1/3k2^{k-2})+\varepsilon} & \text{if } k \geq 4. \end{cases}$$

Very recently, Harman and Kumchev [**3**] introduced the sieve method into the present problem for the case $k = 2$ and proved that $E_3'(N) \ll N^{(17/20)+\varepsilon}$ and $E_4'(N) \ll N^{(7/20)+\varepsilon}$, as well as an upper bound $N^{(7/20)+\varepsilon}$ for the number of exceptions up to N for the problem with a prime and two squares of primes, at a cost of loss of the asymptotic formula for the number of solutions on the major arcs but replaced with a lower bound. By using a sieve method there is a greater flexibility in the application of exponential sum estimates than arises from a standard application of the circle method. We shall investigate the problem in the present paper combined with sieve method, and this will appear elsewhere.

Notation. As usual, $\varphi(n)$, $\mu(n)$ and $\Lambda(n)$ stand for the functions of Euler, Möbius and von Mangoldt respectively. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote a Dirichlet character and the principal character modulo q, and we use the notation $\sum^*$ to denote sums over all primitive characters. For integers $a, b, \dots$ we denote by $[a, b, \dots]$ as their least common multiple. The letter N is a large integer, and $L = \log N$. And $r \sim R$ means $R < r \le 2R$. If there is no ambiguity, we express $(a/b) + \theta$ as $a/b + \theta$ or $\theta + a/b$. The same convention will be applied for quotients. The letters ε and A denote positive constants which are arbitrarily small and sufficiently large respectively, c denotes a positive constant which may vary at different places, and p, with or without subscripts, always denotes a prime number.

2. Outline of the method and proof of Theorem 1. We shall concentrate on proving Theorem 1. We will describe the straightforward modifications needed for Theorem 2 at the end of the paper and will suppress the proof of Theorem 3 altogether.

For any integer $k \ge 2$, and all $n \in \mathcal{A}_3$ with $n \sim N$, consider the quantity

$$r_3(n) = \sum_{\substack{n = p_1^2 + p_2^2 + p_3^k \\ N/2 < p_1^2,\, p_2^2,\, p_3^k \le N}} (\log p_1)(\log p_2)(\log p_3),$$

where p_1, p_2, p_3 are primes. Define further the exponential sum

$$S_k(\alpha) = \sum_{N/2 < p^k \le N} (\log p) e(p^k \alpha), \quad k \ge 2.$$

Then we have

$$r_3(n) = \int_0^1 S_2^2(\alpha) S_k(\alpha) e(-n\alpha)\, \mathrm{d}\alpha. \tag{2.1}$$

In order to apply the circle method, we set

$$P = N^{\theta}, \quad Q = \frac{N}{PL^B}, \tag{2.2}$$

where θ is a fixed number such that $0 < \theta < (1/3k)$ and B is a constant that will be determined in terms of a given constant A. By Dirichlet's lemma on rational approximation, each $\alpha \in [(1/Q), 1 + (1/Q)]$ may be written in the form

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \le \frac{1}{qQ} \tag{2.3}$$

for some integers a, q with $1 \le a \le q \le Q$ and $(a, q) = 1$. We denote by $\mathfrak{M}(a, q)$ the set of α satisfying (2.3) and write the major arcs $\mathfrak{M}$ for the union of all $\mathfrak{M}(a, q)$ with $1 \le a \le q \le P$ and $(a, q) = 1$. Then define the minor arcs by $\mathfrak{m} = [(1/Q), 1 + (1/Q)] \backslash \mathfrak{M}$. Now the formula (2.1) becomes

$$r_3(n) = \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right\} S_2^2(\alpha) S_k(\alpha) e(-n\alpha)\, \mathrm{d}\alpha.$$

To handle the integral on the major arcs, we will establish the following asymptotic formula in Sections 3 and 4. In view of this, we shall need some necessary notations. For $\chi \bmod q$ and $k \ge 2$, we define

$$C_k(\chi, a) = \sum_{h=1}^{q} \overline{\chi}(h) e\left(\frac{ah^k}{q}\right),$$
$$C_k(q, a) = C_k(\chi^0, a).$$

If $\chi_1, \ldots, \chi_j$, $j = 3$ or 4, are Dirichlet characters modulo q, then we write

$$B_j(n, q, \chi_1, \ldots, \chi_j) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) C_2(\chi_1, a) \cdots C_2(\chi_{j-1}, a) C_k(\chi_j, a),$$
$$B_3(n, q) = B_3(n, q, \chi^0, \chi^0, \chi^0),$$

and

$$\mathfrak{S}_3(n, P) = \sum_{q \le P} \frac{B_3(n, q)}{\varphi^3(q)}.$$

Proposition 2.1. *Let the major arcs $\mathfrak{M}$ be as above with P, Q determined by (2.2). Then for $n \sim N$, and $2B > A + 15k$ with $A > 0$*

sufficiently large, we have

$$\int_{\mathfrak{M}} S_2^2(\alpha)S_k(\alpha)e(-n\alpha)\,\mathrm{d}\alpha = \frac{1}{4k}\,P_3\,\mathfrak{S}_3(n,P) + O(N^{1/k}\log^{-A}N),$$

where

$$P_3 := \sum_{\substack{m_1+m_2+m_3=n\\ N/2<m_i\le N}} (m_1m_2)^{-1/2}m_3^{(1/k)-1} \asymp N^{1/k},$$

and $\mathfrak{S}_3(n,P) \gg \log^{-15k} N$ *for all* $n\in\mathcal{A}_3$ *with at most* $O(N^{(5/6)+\varepsilon})$ *or* $O(N^{1+\varepsilon}P^{-1/2})$ *exceptions according to whether* $k=2$ *or* $k\ge 3$.

Now we bound the contribution of $S_k(\alpha)$ from the minor arcs.

Proposition 2.2. *Let the minor arcs* $\mathfrak{m}$ *be as above with* P,Q *determined by* (2.2). *For any integers* $k\ge 2$, *let*

$$\rho(k) = \begin{cases} 1/8 & \textit{if } k=2;\\ 1/14 & \textit{if } k=3;\\ 1/(3\cdot 2^{k-1}) & \textit{if } k\ge 4.\end{cases}$$

Then we have

$$\sup_{\alpha\in\mathfrak{m}} |S_k(\alpha)| \ll N^{(1/k)-(\rho(k)/k)+\varepsilon}.$$

Proof. Put

$$Q^* = \begin{cases} N^{3/2k} & \text{if } k=2;\\ N^{(k-2\rho(k))/(2k-1)} & \text{if } k\ge 3.\end{cases}$$

Then Theorem 3 of Kumchev [**7**] states that

$$\sup_{\alpha\in\mathfrak{m}} |S_k(\alpha)| \ll N^{(1/k)-(\rho(k)/k)+\varepsilon} + (q^\varepsilon N^{1/k}L^c)/\sqrt{q(1+N|\lambda|)} \tag{2.4}$$

for α satisfying

$$\alpha = \frac{a}{q}+\lambda, \quad 1\le q\le Q^*,\ (a,q)=1,\ |\lambda|\le \frac{1}{qQ^*}.$$

Now $Q^* < Q$, and hence the minor arcs can be written as $\mathfrak{m} = \mathfrak{k}_1 \cup \mathfrak{k}_2$, where

$$\mathfrak{k}_1 = \left\{\alpha : 1 \le q \le P,\ \frac{1}{qQ} \le |\lambda| \le \frac{1}{qQ^*}\right\} \tag{2.5}$$

and

$$\mathfrak{k}_2 \subset \left\{\alpha : P < q < Q^*, |\lambda| \le \frac{1}{qQ^*}\right\}.$$

In either case we have $q(1 + N|\lambda|) \gg P$, and hence (2.4) becomes

$$\begin{aligned}\sup_{\alpha\in\mathfrak{m}} |S_k(\alpha)| &\ll N^{(1/k)-(\rho(k)/k)+\varepsilon} + \frac{N^{(1/k)+\varepsilon}}{\sqrt{P}} \\ &\ll N^{(1/k)-(\rho(k)/k)+\varepsilon}.\end{aligned} \tag{2.6}$$

This completes the proof of Proposition 2.2. □

We will see that, as shown in Section 5, Proposition 2.2 holds for a still larger range of θ than that in (2.2).

Now we can establish Theorem 1 by applying the device introduced by Wooley [**22**].

Proof of Theorem 1. Introduce the generating function

$$Z(\alpha) := \sum_{n\in\mathcal{E}_3(N)} e(-\alpha n).$$

Clearly we have

$$\int_0^1 S_2^2(\alpha) S_k(\alpha) Z(\alpha)\,\mathrm{d}\alpha = 0.$$

Using Proposition 2.1, it follows that

$$\begin{aligned}\left|\int_{\mathfrak{m}} S_2^2(\alpha)S_k(\alpha)Z(\alpha)\,\mathrm{d}\alpha\right| &= \left|\int_{\mathfrak{M}} S_2^2(\alpha)S_k(\alpha)Z(\alpha)\,\mathrm{d}\alpha\right| \\ &= \sum_{n\in\mathcal{E}_3(N)} \int_{\mathfrak{M}} S_2^2(\alpha)S_k(\alpha)e(-\alpha n)\,\mathrm{d}\alpha \\ &\gg E\,N^{1/k}L^{-15k},\end{aligned} \tag{2.7}$$

where we abbreviate $E_3(N)$ to E. Then by Cauchy's inequality and Proposition 2.2, we deduce that

$$\begin{aligned} E &\ll N^{-(1/k)+\varepsilon}\Big\{\sup_{\alpha\in\mathfrak{m}}|S_k(\alpha)|\Big\}\int_0^1|S_2^2(\alpha)Z(\alpha)|\,\mathrm{d}\alpha \\ &\ll N^{-(1/k)+\varepsilon}\Big\{\sup_{\alpha\in\mathfrak{m}}|S_k(\alpha)|\Big\}H_1^{1/2}H_2^{1/2}, \end{aligned} \tag{2.8}$$

where

$$H_1=\int_0^1|S_2(\alpha)|^4\,\mathrm{d}\alpha,\qquad H_2=\int_0^1|Z(\alpha)|^2\,\mathrm{d}\alpha=E.$$

The integral H_1 can be estimated by Hua's lemma,

$$H_1\ll L^4\sum_{\substack{m_1^2+m_2^2=m_3^2+m_4^2\\ N/2<m_i^2\le N}}1\ll NL^c. \tag{2.9}$$

The assertion of Theorem 1 now follows from (2.8), (2.9) and Proposition 2.2. □

3. Preliminaries and estimation of J and K. Define $B_j(n,q,\chi_1,\dots,\chi_j)$ for $j=3$ or 4 as in Section 2, as well as $B_3(n,q)$ and $\mathfrak{S}_3(n,P)$. For the singular series $\mathfrak{S}_3(n,P)$, we can establish a lower bound for almost all n. Applying the argument in [**3**, Lemma 7] for $k=2$ and in [**8**, Lemmas 6.4 and 6.6] for $k\ge 3$, we can get the following.

Lemma 3.1. *For all $n\in\mathcal{A}_3$ with $n\sim N$, except for a subset of cardinality $O(N^{(5/6)+\varepsilon})$ or $O(N^{1+\varepsilon}P^{-1/2})$ exceptions, according to whether $k=2$ or $k\ge 3$, respectively, we have*

$$\mathfrak{S}_3(n,P)\gg L^{-15k}.$$

The following lemma, for which the proof is implied in [**8**, Lemma 6.7] and is now standard, plays an important role when we prove Proposition 2.1.

Lemma 3.2. *Let χ_j mod r_j with $j = 1, 2, 3$ be primitive characters, $r_0 = [r_1, r_2, r_3]$, and χ^0 the principal character modulo q. Then*

$$\sum_{\substack{q \le x \\ r_0 | q}} \frac{1}{\varphi^3(q)} |B_3(n, q, \chi_1\chi^0, \chi_2\chi^0, \chi_3\chi^0)| \ll r_0^{-(1/2)+\varepsilon} \log^{15k} x.$$

Furthermore, we define for $k \ge 2$

(3.1)
$$V_k(\lambda) = \sum_{N/2 < m^k \le N} e(m^k\lambda),$$
$$W_k(\chi, \lambda) = \sum_{N/2 < p^k \le N} (\log p)\chi(p)e(p^k\lambda) - \delta_\chi \sum_{N/2 < m^k \le N} e(m^k\lambda),$$

where $\delta_\chi = 1$ or 0 according as χ is principal or not; also, define

$$J_k(g) = \sum_{r \le P} [g, r]^{-(1/2)+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda| \le 1/(rQ)} |W_k(\chi, \lambda)|,$$
$$K_k(g) = \sum_{r \le P} [g, r]^{-(1/2)+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |W_k(\chi, \lambda)|^2 \, \mathrm{d}\lambda \right)^{1/2},$$

where $\sum^*$ means that the summation is over all primitive characters.

Our Proposition 2.1 depends on the following three lemmas.

Lemma 3.3. *Let P, Q be as in (2.2). We have*

$$J_k(g) \ll g^{-(1/2)+\varepsilon} N^{1/k} L^c.$$

Lemma 3.4. *Let P, Q be as in (2.2). For $g = 1$, Lemma 3.3 can be improved to*

$$J_k(1) \ll N^{1/k} L^{-A}.$$

where $A > 0$ is arbitrary.

Lemma 3.5. *Let P, Q be as in (2.2). We have*

$$K_k(g) \ll g^{-(1/2)+\varepsilon} N^{(1/k)-(1/2)} L^c.$$

We will not present in detail the proofs of these three lemmas, since they are similar to those of [**14**, Lemmas 2.2–2.4]. We will only give a proof of Lemma 3.3. To this end we shall need the following hybrid estimate for Dirichlet polynomials which was established by Liu [**9**, Lemma 2.1].

Let $X^{2/5} < Y \le X$ and $M_1, \dots, M_{10}$ be positive integers such that

$$2^{-10}Y \le M_1 \cdots M_{10} < X, \quad \text{and} \quad 2M_6, \dots, 2M_{10} \le X^{1/5}. \tag{3.2}$$

For $j = 1, \dots, 10$, define

$$a_j(m) = \begin{cases} \log m & \text{if } j = 1, \\ 1 & \text{if } j = 2, \dots, 5, \\ \mu(m) & \text{if } j = 6, \dots, 10, \end{cases} \tag{3.3}$$

where $\mu(n)$ is the Möbius function. Then we define the functions

$$f_j(s, \chi) = \sum_{m \sim M_j} \frac{a_j(m)\chi(m)}{m^s}$$

and

$$F(s, \chi) = f_1(s, \chi) \cdots f_{10}(s, \chi), \tag{3.4}$$

where χ is a Dirichlet character and s a complex variable.

Lemma 3.6. *Let $F(s, \chi)$ be defined as above. Then for any $1 \le R \le X^2$ and $T > 0$,*

$$\sum_{\substack{r \sim R \\ d \mid r}} \sum_{\chi \bmod r}^{*} \int_T^{2T} \left| F\left(\frac{1}{2} + \mathrm{i}t\right) \right| \mathrm{d}t \ll \left\{ \frac{R^2}{d} T + \frac{R}{d^{1/2}} T^{1/2} X^{3/10} + X^{1/2} \right\} \log^c X.$$

Proof of Lemma 3.3. Let

$$\widehat{W}_k(\chi, \lambda) = \sum_{N/2 < m^k \le N} (\Lambda(m)\chi(m) - \delta_\chi) e(m^k \lambda).$$

Then

$$W_k(\chi,\lambda) - \widehat{W}_k(\chi,\lambda) \ll N^{1/(2k)}.$$

This contributes to $J_k(g)$ an amount

$$\begin{aligned}
&\ll g^{-(1/2)+\varepsilon} N^{1/2k} \sum_{\substack{d|g\\ d\le P}} \sum_{\substack{r\le P\\ d|r}} \left(\frac{r}{d}\right)^{-(1/2)+\varepsilon} r \\
&\ll g^{-(1/2)+\varepsilon} N^{1/2k} \sum_{\substack{d|g\\ d\le P}} \sum_{m\le P/d} m^{-(1/2)+\varepsilon} md \\
&\ll g^{-(1/2)+\varepsilon} N^{1/2k} P^{3/2+\varepsilon} \ll g^{-(1/2)+\varepsilon} N^{1/k} L^{-A},
\end{aligned}$$

where we have used $[g,r](g,r) = gr$ and (2.2). Thus Lemma 3.3 is a consequence of the estimate

(3.5)
$$\sum_{r\sim R} [g,r]^{-(1/2)+\varepsilon} \sum_{\chi \bmod r}^{*} \max_{|\lambda|\le 1/(rQ)} |\widehat{W}_k(\chi,\lambda)| \ll g^{-(1/2)+\varepsilon} N^{1/k} L^c,$$

where $R \le P$ and $c > 0$ is some constant.

It is easy to establish (3.5) for $R < 1$. In fact, in this case we must have $r = 1$, and hence the left-hand side of (3.5) is

$$\ll g^{-(1/2)+\varepsilon} \sum_{N/2 < m^k \le N} \log m \ll g^{-(1/2)+\varepsilon} N^{1/k} L,$$

which is obviously acceptable. It therefore remains to show (3.5) in the case $R \ge 1$.

To the sum

$$\sum_{(N/2)^{1/k} < m \le u} \Lambda(m)\chi(m), \tag{3.6}$$

we apply Heath-Brown's identity (see [**4**, Lemma 1]) for $k = 5$, which states that, for $m \le X$,

$$\Lambda(m) = \sum_{j=1}^{5} \binom{5}{j} (-1)^{j-1} \sum_{\substack{m_1\cdots m_{2j}=m\\ m_{j+1},\dots,m_{2j}\le X^{1/5}}} (\log m_1)\mu(m_{j+1})\cdots\mu(m_{2j}).$$

We set $X = N^{1/k}$ in the above formula. Also, in (3.2) we take $X = N^{1/k}$ and $Y = (N/2)^{1/k}$; define $a_j(m), f_j(s,\chi)$ and $F(s,\chi)$ as in (3.3) and (3.4). Therefore (3.6) is a linear combination of $O(L^{10})$ terms, each of which is of the form

$$\Sigma(u;\mathbf{M}) = \sum_{\substack{m_1\sim M_1 \\ 2^{-10}(N/2)^{1/k}<m_1\cdots m_{10}\le u}} \cdots \sum_{m_{10}\sim M_{10}} a_1(m_1)\chi(m_1) \cdots a_{10}(m_{10})\chi(m_{10}),$$

where $\mathbf{M}$ denotes the vector $(M_1, M_2, \ldots, M_{10})$ with M_j as in (3.2).

By Perron's summation formula (see for example, [**17**, Theorem 2, page 98] or [**20**, Lemma 3.12]) and then shifting the contour to the left, the above $\Sigma(u;\mathbf{M})$ is

$$\begin{aligned} &= \frac{1}{2\pi \mathrm{i}} \int_{1+1/L-\mathrm{i}T}^{1+1/L+\mathrm{i}T} F(s,\chi)\frac{u^s-(N/2)^{s/k}}{s}\,\mathrm{d}s + O\bigg(\frac{N^{1/k}L^2}{T}\bigg) \\ &= \frac{1}{2\pi \mathrm{i}}\bigg\{\int_{1+1/L-\mathrm{i}T}^{1/2-\mathrm{i}T} + \int_{1/2-\mathrm{i}T}^{1/2+\mathrm{i}T} + \int_{1/2+\mathrm{i}T}^{1+1/L+\mathrm{i}T}\bigg\} + O\bigg(\frac{N^{1/k}L^2}{T}\bigg), \end{aligned}$$

where T is a parameter satisfying $2 \le T \le N^{1/k}$. The integral on the two horizontal segments above can be easily estimated as

$$\begin{aligned} &\ll \max_{1/2\le\sigma\le 1+1/L} |F(\sigma\pm\mathrm{i}T,\chi)|\frac{u^\sigma}{T} \\ &\ll \max_{1/2\le\sigma\le 1+1/L} N^{(1-\sigma)/k}L\frac{u^\sigma}{T} \\ &\ll \frac{N^{1/k}L}{T} \end{aligned}$$

on using the trivial estimate

$$\begin{aligned} F(\sigma\pm\mathrm{i}T,\chi) &\ll |f_1(\sigma\pm\mathrm{i}T,\chi)|\cdots|f_{10}(\sigma\pm\mathrm{i}T,\chi)| \\ &\ll (M_1^{1-\sigma}L)M_2^{1-\sigma}\cdots M_{10}^{1-\sigma} \\ &\ll N^{(1-\sigma)/k}L. \end{aligned}$$

Thus,

$$\begin{aligned} \Sigma(u;\mathbf{M}) = \frac{1}{2\pi}\int_{-T}^{T} F\bigg(\frac{1}{2}+\mathrm{i}t,\chi\bigg)\frac{u^{1/2+\mathrm{i}t}-(N/2)^{(1/2+\mathrm{i}t)/k}}{1/2+\mathrm{i}t}\,\mathrm{d}t \\ + O\bigg(\frac{N^{1/k}L^2}{T}\bigg). \end{aligned}$$

Now recalling $r > 1$, we have $\delta_\chi = 0$ for all χ mod r in the definition of $\widehat{W}_k(\chi, \lambda)$, and consequently,

$$\begin{aligned}\widehat{W}_k(\chi, \lambda) &= \sum_{N/2 < m^k \le N} \Lambda(m)\chi(m)e(m^k\lambda) \\ &= \int_{(N/2)^{1/k}}^{N^{1/k}} e(u^k\lambda)\,\mathrm{d}\Bigg\{\sum_{(N/2)^{1/k} < m \le u} \Lambda(m)\chi(m)\Bigg\}.\end{aligned}$$

Hence $\widehat{W}_k(\chi, \lambda)$ is a linear combination of $O(L^{10})$ terms, each of which is of the form

$$\begin{aligned}&\int_{(N/2)^{1/k}}^{N^{1/k}} e(u^k\lambda)\,\mathrm{d}\Sigma(u; \mathbf{M}) \\ &\qquad = \frac{1}{2\pi}\int_{-T}^{T} F\left(\frac{1}{2} + \mathrm{i}t, \chi\right) \int_{(N/2)^{1/k}}^{N^{1/k}} u^{-1/2+\mathrm{i}t} e(u^k\lambda)\,\mathrm{d}u\,\mathrm{d}t \\ &\qquad\quad + O\left(\frac{N^{1/k}L^2}{T}(1 + |\lambda|N)\right).\end{aligned}$$

By taking $T = N^{1/k}$ and changing variables in the inner integral, we deduce from the above formulae that

$$\begin{aligned}|\widehat{W}_k(\chi, \lambda)| \ll L^{10} \max_{\mathbf{M}} \Bigg| &\int_{-T}^{T} F\left(\frac{1}{2} + \mathrm{i}t, \chi\right) \\ &\times \int_{N/2}^{N} v^{(1/2k)-1} e\left(\frac{t}{2k\pi}\log v + \lambda v\right) \mathrm{d}v\,\mathrm{d}t\Bigg| + N^{1/(3k)},\end{aligned} \tag{3.7}$$

where the maximum is taken over all $\mathbf{M} = (M_1, M_2, \ldots, M_{10})$. Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{t}{2k\pi}\log v + \lambda v\right) &= \frac{t}{2k\pi v} + \lambda, \\ \frac{\mathrm{d}^2}{\mathrm{d}v^2}\left(\frac{t}{2k\pi}\log v + \lambda v\right) &= -\frac{t}{2k\pi v^2},\end{aligned}$$

by Lemmas 4.3 and 4.4 in [**20**], the inner integral in (3.7) can be estimated as

$$\begin{aligned}&\ll \left(\frac{N}{2}\right)^{(1/2k)-1} \min\left\{\frac{N}{\sqrt{|t|+1}}, \frac{N}{\min\limits_{N/2<v\le N}|t + 2k\pi\lambda v|}\right\} \\ &\ll \begin{cases} N^{1/2k}/\sqrt{|t|+1} & \text{if } |t| \le T_*, \\ N^{1/2k}/|t| & \text{if } T_* < |t| \le T, \end{cases}\end{aligned} \tag{3.8}$$

where

$$T_* = 4k\pi N/(RQ).$$

Here the choice of T_* is to ensure that $|t + 2k\pi\lambda v| > |t|/2$ whenever $|t| > T_*$; in fact,

$$|t + 2k\pi\lambda v| \geq |t| - 2k\pi|v|/(rQ) > |t|/2 + T_*/2 - 2k\pi N/(RQ) \geq |t|/2.$$

Therefore it follows from (3.7) and (3.8) that (3.5) is a consequence of the following two estimates: For $R \leq P$ and $0 < T_1 \leq T_*$,

$$\sum_{r\sim R} [g,r]^{-(1/2)+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2} + \mathrm{i}t, \chi\right)\right| \mathrm{d}t \ll g^{-(1/2)+\varepsilon} N^{1/2k} (T_1+1)^{1/2} L^c; \tag{3.9}$$

while for $R \leq P$ and $T_* < T_2 \leq T$,

$$\sum_{r\sim R} [g,r]^{-(1/2)+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} \left| F\left(\frac{1}{2} + \mathrm{i}t, \chi\right)\right| \mathrm{d}t \ll g^{-(1/2)+\varepsilon} N^{1/2k} T_2 L^c. \tag{3.10}$$

To show (3.9), we note that $[g,r](g,r) = gr$. Then the left hand side of (3.9) is

$$\ll g^{-(1/2)+\varepsilon} \sum_{\substack{d|g\\ d\leq R}} \left(\frac{R}{d}\right)^{-(1/2)+\varepsilon} \sum_{\substack{r\sim R\\ d|r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2} + \mathrm{i}t, \chi\right)\right| \mathrm{d}t.$$

By Lemma 3.6, the above quantity can be estimated as

$$\begin{aligned} &\ll g^{-(1/2)+\varepsilon} \sum_{\substack{d|g\\ d\leq R}} \left(\frac{R}{d}\right)^{-(1/2)+\varepsilon} \\ &\qquad\qquad \times \left(\frac{R^2}{d} T_1 + \frac{R}{d^{1/2}} T_1^{1/2} N^{3/10k} + N^{1/2k}\right) L^c \\ &\ll g^{-(1/2)+\varepsilon} \tau(g) \big\{ R^{(3/2)+\varepsilon} T_1 + R^{(1/2)+\varepsilon} T_1^{(1/2)+\varepsilon} N^{3/10k} \\ &\qquad\qquad + N^{1/2k} \big\} L^c \\ &\ll g^{-(1/2)+\varepsilon} N^{1/2k} (T_1+1)^{1/2} L^c, \end{aligned}$$

provided that $R \le P = N^\theta$ with $\theta < 1/3k$. This establishes (3.9).

Similarly, we can prove (3.10) for $R \le P = N^\theta$ with $\theta < 1/3k$, by taking $T = T_2$ in Lemma 3.6. Lemma 3.3 thus follows from (3.9) and (3.10). $\square$

4. Proof of Proposition 2.1. With Lemmas 3.3–3.5 known, we can use the iterative idea to prove Proposition 2.1.

Proof of Proposition 2.1. For $q \le P$ and $N/2 < p^k \le N$, $k \ge 2$, we have $(q,p) = 1$. Therefore we can rewrite the exponential sum $S_k(\alpha)$ as

$$\begin{aligned} S_k\left(\frac{a}{q}+\lambda\right) &= \frac{C_k(q,a)}{\varphi(q)}V_k(\lambda) + \frac{1}{\varphi(q)}\sum_{\chi \bmod q} C_k(\chi,a)W_k(\chi,\lambda) \\ &=: a_k + b_k. \end{aligned}$$

Thus

$$\begin{aligned} (4.1)\quad & \int_{\mathfrak{M}} S_2^2(\alpha)S_k(\alpha)e(-n\alpha)\,\mathrm{d}\alpha \\ &= \int_{\mathfrak{M}} (a_2+b_2)^2(a_k+b_k)e(-n\alpha)\,\mathrm{d}\alpha \\ &= \int_{\mathfrak{M}} \big(a_2^2a_k + 2a_2b_2a_k + b_2^2a_k + a_2^2b_k + 2a_2b_2b_k + b_2^2b_k\big)e(-n\alpha)\,\mathrm{d}\alpha \\ &=: I_{10} + 2I_{11} + I_{12} + I_{20} + 2I_{21} + I_{22}, \end{aligned}$$

where

$$\begin{aligned} I_{1j} = \sum_{q \le P}\frac{1}{\varphi^3(q)} & \sum_{\substack{a=1\\(a,q)=1}}^{q} C_2^{2-j}(q,a)C_k(q,a)e\left(-\frac{an}{q}\right) \\ & \times \int_{-1/qQ}^{1/qQ} V_2^{2-j}(\lambda)V_k(\lambda)\left\{\sum_{\chi \bmod q} C_2(\chi,a)W_2(\chi,\lambda)\right\}^j e(-n\lambda)\,\mathrm{d}\lambda \end{aligned}$$

and

$$I_{2j} = \sum_{q \le P} \frac{1}{\varphi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C_2^{2-j}(q,a) e\Big(-\frac{an}{q}\Big)$$
$$\times \int_{-1/qQ}^{1/qQ} V_2^{2-j}(\lambda) \Big\{ \sum_{\chi \bmod q} C_2(\chi, a) W_2(\chi, \lambda) \Big\}^j$$
$$\times \Big\{ \sum_{\chi \bmod q} C_k(\chi, a) W_k(\chi, \lambda) \Big\} e(-n\lambda)\, \mathrm{d}\lambda.$$

We shall prove that I_{10} gives the main term and others the error term. We first compute the main term I_{10}. Applying [**6**, Lemma 8.8] to $V_k(\lambda)$, we get for $k \ge 2$,

$$\begin{aligned} V_k(\lambda) &= \int_{N/2^{1/k}}^{N^{1/k}} e(u^k \lambda)\, \mathrm{d}u + O(1) \\ &= \frac{1}{k} \int_{N/2}^{N} v^{1/k-1} e(v\lambda)\, \mathrm{d}v + O(1) \\ &= \frac{1}{k} \sum_{N/2 < m \le N} e(m\lambda) m^{(1/k)-1} + O(1). \end{aligned} \tag{4.2}$$

Substituting this into I_{10} we see that

(4.3)

$$I_{10} = \frac{1}{4k} \sum_{q \le P} \frac{B_3(n,q)}{\varphi^3(q)} \int_{-1/qQ}^{1/qQ} \Big(\sum_{N/2 < m \le N} e(m\lambda) m^{-1/2} \Big)^2$$
$$\times \Big(\sum_{N/2 < m \le N} e(m\lambda) m^{1/k-1} \Big) e(-n\lambda)\, \mathrm{d}\lambda$$
$$+ O\Big(\sum_{q \le P} \frac{|B_3(n,q)|}{\varphi^3(q)} \int_{-1/qQ}^{1/qQ} \Big| \sum_{N/2 < m \le N} e(m\lambda) m^{-1/2} \Big|$$
$$\times \Big| \sum_{N/2 < m \le N} e(m\lambda) m^{(1/k)-1} \Big| \mathrm{d}\lambda \Big).$$

Using the following elementary estimates

$$\sum_{N/2<m\le N} e(m\lambda)m^{(1/k)-1} \ll \left(\frac{N}{2}\right)^{(1/k)-1} \min\left(N, \frac{1}{|\lambda|}\right) \tag{4.4}$$

and Lemma 3.2 with $r_0 = 1$, the O-term in (4.3) can be estimated as

$$\begin{aligned}
&\ll \sum_{q\le P} \frac{|B_3(n,q)|}{\varphi^3(q)} \left\{ \int_0^{N^{-1}} N^2 \left(\frac{N}{2}\right)^{(1/k)-(3/2)} \mathrm{d}\lambda \right. \\
&\qquad\qquad \left. + \int_{N^{-1}}^{\infty} \left(\frac{N}{2}\right)^{(1/k)-(3/2)} \frac{\mathrm{d}\lambda}{\lambda^2} \right\} \\
&\ll N^{(1/k)-(1/2)} L^{15k}.
\end{aligned}$$

Now we extend the integral in the main term of (4.3) to $[-1/2, 1/2]$; by a similar argument we see that the resulting error is

$$\begin{aligned}
\ll L^{15k} \int_{1/PQ}^{1/2} \left(\frac{N}{2}\right)^{(1/k)-2} \frac{\mathrm{d}\lambda}{\lambda^3} &\ll \left(\frac{N}{2}\right)^{(1/k)-2} P^2 Q^2 L^{15k} \\
&\ll N^{1/k} L^{-A},
\end{aligned}$$

provided that $2B > A + 15k$. Thus (4.3) becomes

$$I_{10} = \frac{1}{4k} P_3\, \mathfrak{S}_3(n, P) + O\big(N^{1/k} L^{-A}\big), \tag{4.5}$$

where P_3 is defined in Proposition 2.1 and satisfies $P_3 \asymp n^{1/k}$ with $n \sim N$, $A > 0$ is sufficiently large by noting Lemma 3.1.

For the other terms in (4.1), we begin with I_{22} which is the most complicated one. Reducing the characters in I_{22} into primitive characters, we have

$$
\begin{aligned}
|I_{22}| &= \Bigg| \sum_{q\le P} \frac{1}{\varphi^3(q)} \sum_{\chi_1 \bmod q} \cdots \sum_{\chi_3 \bmod q} B_3(n,q,\chi_1,\chi_2,\chi_3) \\
&\quad \times \int_{-1/qQ}^{1/qQ} W_2(\chi_1,\lambda) W_2(\chi_2,\lambda) W_k(\chi_3,\lambda) e(-n\lambda)\,\mathrm{d}\lambda \Bigg| \\
&\le \sum_{r_1\le P} \cdots \sum_{r_3\le P} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \\
&\quad \cdots \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \sum_{\substack{q\le P\\ r_0|q}} \frac{|B_3(n,q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)|}{\varphi^3(q)} \\
&\quad \times \int_{-1/qQ}^{1/qQ} \big|W_2(\chi_1\chi^0,\lambda)\big| \, \big|W_2(\chi_2\chi^0,\lambda)\big| \, \big|W_k(\chi_3\chi^0,\lambda)\big| \,\mathrm{d}\lambda,
\end{aligned}
$$

where χ^0 is the principal character modulo q, $r_0 = [r_1, r_2, r_3]$ depending on r_1, r_2, r_3, and the sum $\sum^*$ is over all primitive characters. For $q \le P$ and $N/2 < p^k \le N$, $k \ge 2$, we have $(q,p) = 1$. Using this and (3.1), we have $W_k(\chi_j\chi^0, \lambda) = W_k(\chi_j, \lambda)$ for the primitive characters χ_j above. Thus by Lemma 3.2, we obtain

$$
\begin{aligned}
|I_{22}| &\le \sum_{r_1\le P} \cdots \sum_{r_3\le P} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \\
&\quad \cdots \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \int_{-1/(r_0Q)}^{1/(r_0Q)} \big|W_2(\chi_1,\lambda)\big| \, \big|W_2(\chi_2,\lambda)\big| \, \big|W_k(\chi_3,\lambda)\big| \,\mathrm{d}\lambda \\
&\quad \times \sum_{\substack{q\le P\\ r_0|q}} \frac{|B_3(n,q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)|}{\varphi^3(q)} \\
&\ll L^c \sum_{r_1\le P} \cdots \sum_{r_3\le P} r_0^{-(1/2)+\varepsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \cdots \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \\
&\quad \times \int_{-1/(r_0Q)}^{1/(r_0Q)} \big|W_2(\chi_1,\lambda)\big| \, \big|W_2(\chi_2,\lambda)\big| \, \big|W_k(\chi_3,\lambda)\big| \,\mathrm{d}\lambda.
\end{aligned}
$$

In the last integral, we take out $|W_2(\chi_1, \lambda)|$ and then use Cauchy's inequality to get

(4.6)
$$
\begin{aligned}
|I_{22}| \ll L^c \sum_{r_1\le P} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda|\le 1/(r_1Q)} |W_2(\chi_1,\lambda)| \\
\times \sum_{r_2\le P} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \left(\int_{-1/(r_2Q)}^{1/(r_2Q)} |W_2(\chi_2,\lambda)|^2 \,\mathrm{d}\lambda\right)^{1/2} \\
\times \sum_{r_3\le P} r_0^{-(1/2)+\varepsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \left(\int_{-1/(r_3Q)}^{1/(r_3Q)} |W_k(\chi_3,\lambda)|^2 \,\mathrm{d}\lambda\right)^{1/2}.
\end{aligned}
$$

Now we introduce an iterative procedure to bound the above sums over r_1, r_2, r_3, consecutively.

We first estimate the above sum over r_3 in (4.6) via Lemma 3.5. Since $r_0 = [r_1, r_2, r_3] = [[r_1, r_2], r_3]$, the sum over r_3 in (4.6) is

$$
\begin{aligned}
&= \sum_{r_3\le P} [[r_1,r_2],r_3]^{-(1/2)+\varepsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \left(\int_{-1/(r_3Q)}^{1/(r_3Q)} |W_k(\chi_3,\lambda)|^2 \,\mathrm{d}\lambda\right)^{1/2} \\
&= K_k([r_1,r_2]) \ll [r_1,r_2]^{-(1/2)+\varepsilon} N^{(1/k)-(1/2)} L^c.
\end{aligned}
$$

This contributes to the sum over r_2 of (4.6) in amount

$$
\begin{aligned}
&\ll N^{(1/k)-(1/2)} L^c \sum_{r_2\le P} [r_1,r_2]^{-(1/2)+\varepsilon} \\
&\quad \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \left(\int_{-1/(r_2Q)}^{1/(r_2Q)} |W_2(\chi_2,\lambda)|^2 \,\mathrm{d}\lambda\right)^{1/2} \\
&= N^{(1/k)-(1/2)} L^c K_2(r_1) \ll r_1^{-(1/2)+\varepsilon} N^{(1/k)-(1/2)} L^c,
\end{aligned}
$$

where we have used Lemma 3.5 again. Inserting this last bound into (4.6), we can bound the sum over r_1 and find that

(4.7)
$$
\begin{aligned}
|I_{22}| &\ll N^{(1/k)-(1/2)} L^c \sum_{r_1\le P} r_1^{-(1/2)+\varepsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda|\le 1/(r_1Q)} |W_2(\chi_1,\lambda)| \\
&= N^{(1/k)-(1/2)} L^c J_2(1) \ll N^{1/k} L^{-A},
\end{aligned}
$$

where we have used Lemma 3.4 in the last step.

For the estimation of the terms $I_{11}, I_{12}, I_{20}, I_{21}$, noting (4.2) and (4.4) we get

$$\left(\int_{-1/Q}^{1/Q} |V_k(\lambda)|^2 \,\mathrm{d}\lambda\right)^{1/2} \ll \left(\left(\frac{N}{2}\right)^{(2/k)-2} \frac{N}{2}\right)^{1/2} \ll N^{(1/k)-(1/2)}.$$

Using this estimate and Lemmas 3.3–3.5, we argue similarly to the treatment of I_{22} and obtain

$$|I_{11}| + |I_{12}| + |I_{20}| + |I_{21}| \ll N^{1/k} L^{-A}. \tag{4.8}$$

Proposition 2.1 now follows from (4.1), (4.5), (4.7) and (4.8). □

5. Proof of Theorem 2. We now outline the modifications necessary to our previous argument. We keep the same major and minor arcs decomposition but merely replacing θ in (2.2) by

$$0 < \theta < \frac{2}{5k}. \tag{5.1}$$

Then for all $n \in \mathcal{A}_4$ with $n \sim N$, consider

$$\begin{aligned} r_4(n) &:= \sum_{\substack{n = p_1^2 + p_2^2 + p_3^2 + p_4^k \\ N/2 < p_1^2, p_2^2, p_3^2, p_4^k \le N}} (\log p_1)(\log p_2)(\log p_3)(\log p_4) \\ &= \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right\} S_2^3(\alpha) S_k(\alpha) e(-n\alpha)\,\mathrm{d}\alpha. \end{aligned}$$

Moreover, define $J_k(g)$ and $K_k(g)$ as in Section 3 with the exponent of $[g, r]$ there, i.e. $-(1/2) + \varepsilon$, replaced by $-1 + \varepsilon$. By the same treatment, we can estimate J and K to get the desired upper bounds as shown in Lemmas 3.3–3.5 with the exponent of the variable g replaced also by $-1 + \varepsilon$, and the range of θ, arising in (5.1), will be determined simultaneously. And the saving $-1 + \varepsilon$ plays a key role in the argument to obtain this larger range of θ. Then following the proof of Proposition 2.1, we can get the asymptotic formula of $r_4(n)$ on the major arcs.

Proposition 5.1. *Let the major arcs $\mathfrak{M}$ be defined as in Section 2 with P, Q determined by (2.2) but θ replaced by (5.1). Then for $n \sim N$, we have*

(5.2)
$$\int_{\mathfrak{M}} S_2^3(\alpha) S_k(\alpha) e(-n\alpha)\,\mathrm{d}\alpha = \frac{1}{8k} P_4\, \mathfrak{S}_4(n) + O(N^{(1/2)+(1/k)} \log^{-A} N),$$

where

$$P_4 := \sum_{\substack{m_1+m_2+m_3+m_4=n \\ N/2<m_i\le N}} (m_1 m_2 m_3)^{-1/2} m_4^{(1/k)-1} \asymp N^{(1/2)+(1/k)},$$

and

$$\mathfrak{S}_4(n) := \sum_{q=1}^{\infty} \frac{B_4(n,q,\chi^0,\chi^0,\chi^0,\chi^0)}{\varphi^4(q)}$$

which satisfies $\mathfrak{S}_4(n) \gg 1$ for $n \in \mathcal{A}_4$.

We remark that $\mathfrak{S}_4(n) \gg 1$ for $n \in \mathcal{A}_4$ (the argument is similar to Lemma 3.1). Combining (5.2) with the similar treatment in Section 2, we find that Proposition 2.2 also holds for the larger range of θ in (5.1) and hence Theorem 2 follows as required.

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