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ZASSENHAUS ALGEBRAS

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ABSTRACT. Let K be a field and A a K-algebra. Let $\widehat{A} = \{\varphi \in \mathrm{End}_K(A) : \varphi(X) \subseteq X \text{ for all left ideals } X \text{ of } A\}$ be the ring of all K-linear transformations of the K-vectorspace A that leave all left ideals invariant. For many classes of finite dimensional algebras A we determine if $\widehat{A} = A$. In the last section, we consider the case of Leavitt path algebras.

1. Introduction. In 1967, Hans Zassenhaus published the following, remarkable

Theorem 1 [**10**]. *Let R be a ring with identity such that R^+, the additive group of R, is a free Abelian group of finite rank. Then there exists an Abelian group M such that $R = R \otimes_{\mathbf{Z}} \mathbf{Z} \subseteq M \subseteq R \otimes_{\mathbf{Z}} \mathbf{Q}$ and* $\mathrm{End}\,(M) = R$, *i.e., any homomorphism of M is the multiplication from the left by some element $r \in R$.*

This result was generalized to a larger class of rings in [**7**] and recently in [**8**]. The key to the proof is the construction of a (manageable) family $\mathcal{F}$ of left ideals of the ring R such that

$$\mathrm{End}\,(R, \mathcal{F}) = \{\varphi \in \mathrm{End}\,(R^+) : \varphi(X) \subseteq X \text{ for all } X \in \mathcal{F}\}$$

consists only of left multiplications by elements of R, i.e., $R = \mathrm{End}\,(R, \mathcal{F})$. Such a family $\mathcal{F}$ of left ideals was called a Zassenhaus family in [**8**]. Of course, the ring R has a Zassenhaus family of left ideals, if and only if the family of all left ideals is a Zassenhaus family, if and only if the family of all principal left ideals is a Zassenhaus family. It is easy to see that the ring $T = \left\{ \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} : a, b \in \mathbf{Q} \right\}$ has no Zassenhaus

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family of left ideals. One could say that rings with Zassenhaus families of left ideals have, in some sense, "enough" left ideals.

Given a ring R, we define

$$\widehat{R} = \left\{\varphi \in \mathrm{End}\,(R^+) : \varphi(X) \subseteq X \text{ for all left ideals } X \text{ of } R\right\}.$$

(Of course this notion can also be defined for right ideals [**8**]. Here we use left ideals, which allows us to write maps on the left). If $sR = \{0\}$ for $s \in R$ implies $s = 0$ ($1 \in R$ is sufficient for that), then the ring $\widehat{R}$ contains a copy of R, i.e., all left multiplications by elements of R. This allows one to iterate the $\widehat{R}$ construction by transfinite induction. We have an example in [**5**], where this transfinite, ascending chain of rings never terminates. The only other examples we have at this time have the property that $\widehat{\widehat{R}} = \widehat{R}$, i.e., $\widehat{\ }$ acts as some kind of "closure operator."

In this paper we want to restrict our attention to algebras $A = A_K$ over a field K. We will adjust our definitions to this new situation. Let $\mathrm{End}_K(A_K)$ denote the ring of all linear transformations of the K-vectorspace A_K. We define

$$\widehat{A_K} = \{\varphi \in \mathrm{End}_K(A) : \varphi(X) \subseteq X \text{ for all (algebra) left ideals } X \text{ of } A\}.$$

In this paper, disregarding the notations in [**8**], we call the algebra A_K a *Zassenhaus algebra*, if $\widehat{A_K} = A_K$. Note that $\widehat{A_K}$ has an identity even if A_K does not. Moreover, if $1 \in A_K$, then A_K is naturally embedded in

$$\widehat{A_K} = A_K \oplus \{\varphi \in \widehat{A_K} : \varphi(1) = 0\}.$$

Here we identify $a \in A$ with the map $(a\cdot) : A \to A$ where $(a\cdot)(x) = ax$ for all $x \in A$.

From now on, we will suppress the subscript K. All examples of K-algebras A that we know are either Zassenhaus algebras or $\widehat{A}$ is a Zassenhaus algebra, i.e., $\widehat{A} = \widehat{\widehat{A}}$. The purpose of this paper is to compute $\widehat{A}$, where A is some K-algebra. We are only able to obtain partial results.

Here is an easy example: Let D be a division ring of dimension n over K. Then D has only the trivial left ideals and $\widehat{D}$ is simply the ring of $n \times n$ matrices over K, and thus, if $n > 1$, is not equal to D. It was shown in [**5**] that in this case $\widehat{\widehat{D}} = \widehat{D}$.

Suppose that A is finite dimensional, and let $J(A)$ denote the Jacobson radical of A. The case when $J(A) = \{0\}$, i.e., A is a semi-simple algebra is easy to settle. This was done in [**5**] and we recall the result in Section 4. In Section 2, we deal with local algebras, i.e., $A/J(A)$ is a division ring. In that case if $A \neq K$ and K are infinite, then A is never a Zassenhaus algebra, but $\widehat{A}$ is a Zassenhaus algebra. The case when $A/J(A)$ is a ring direct product of division rings, i.e., A is a reduced algebra [**9**], appears to be quite difficult to deal with. We can only show that the reduced algebras that arise as path algebras of certain quivers are Zassenhaus algebras. If A is primary, i.e., $A/J(A)$ is a simple algebra, not a division ring, then A is Zassenhaus, as follows from earlier results. In Section 4 we use a representation of Artinian algebras as checkered matrix rings, cf. [**3**], to get some more general results. In our last section, we recall the definition of the Leavitt path algebra $L_K(\Gamma)$ over a directed graph Γ, cf. [**1**] and compute $\widehat{L_K(\Gamma)}$. These algebras, which attracted a lot of attention recently, see for example [**1, 2**], are somewhat similar to path algebras over quivers, but more involved. We will prove the following result that illustrates that the "$\widehat{\ }$ -construction" frequently behaves like some "closure operator."

Theorem 2. *Let K be an infinite field and $\Gamma = (V, E)$ a row finite, oriented graph. Then $\widehat{L_K(\Gamma)}$, with the finite topology, is a complete topological ring such that $\{\mathcal{O}_H : H$ a finite subset of $V\}$ is a basis of that topology and $L_K(\Gamma)$ is a dense subring of $\widehat{L_K(\Gamma)}$, which is a Zassenhaus algebra. Moreover, if Γ is finite, then $\widehat{L_K(\Gamma)} = L_K(\Gamma)$.*

2. Local algebras. Let K be a field and R a finite dimensional K-algebra. Recall that R is called a local K-algebra if $R/J(R) \cong D$, where D is a division ring and $J(R)$ is the Jacobson radical of R. Recall that $J(R)$ is a nilpotent ideal of R. We need some preliminary observations.

Proposition 1. *Let K be a field, $E = \mathrm{Mat}_{n\times n}(K)$ and $H = \mathrm{Mat}_{m\times n}(K)$ and $\beta \in \mathrm{Hom}_K(E, H)$ such that $\beta(x) \in Hx$ for all $x \in E$. Then there exists some $B \in H$ such that $\beta(x) = Bx$ for all $x \in E$.*

Proof. Let $\varepsilon^{(ij)}$ be the matrix with a 1 in the (i,j)-position and 0's everywhere else. Then there exists an $\ell_{\alpha i} \in K$ such that $\beta(\varepsilon^{(i1)}) = \sum_{\alpha=1}^{m} \ell_{\alpha i}\varepsilon^{(\alpha 1)}$. Let $1 < j \le n$. Then there exists an $\ell_{\alpha ij} \in K$ such that $\beta(\varepsilon^{(ij)}) = \sum_{\alpha=1}^{m} \ell_{\alpha ij}\varepsilon^{(\alpha j)}$.

Now $\beta(\varepsilon^{(i1)} + \varepsilon^{(ij)}) = A(\varepsilon^{(i1)} + \varepsilon^{(ij)})$ for some matrix $A = (a_{i\alpha}) \in H$.

Thus $\sum_{\alpha=1}^{m} \ell_{\alpha i}\varepsilon^{(\alpha 1)} + \sum_{\alpha=1}^{m} \ell_{\alpha ij}\varepsilon^{(\alpha j)} = A(\varepsilon^{(i1)} + \varepsilon^{(ij)}) = \sum_{\alpha=1}^{m} a_{\alpha i}\varepsilon^{(\alpha 1)} + \sum_{\alpha=1}^{m} a_{\alpha i}\varepsilon^{(\alpha j)}$.

This implies that $\ell_{\alpha i} = a_{\alpha i} = \ell_{\alpha ij}$ for all $1 < j \le n$. Now let $x = \sum_{i,j} x_{ij}\varepsilon^{(ij)} \in E$.

Then $\beta(x) = \sum_{i,j} x_{ij}\beta(\varepsilon^{(ij)}) = \sum_{i,j,\alpha} x_{ij}\ell_{\alpha i}\varepsilon^{(\alpha j)} = (\sum_i \ell_{\alpha i}x_{ij})_{\alpha,j} = Bx$ where $B = (\ell_{\alpha i}) \in H$. □

Proposition 2. *Let K be a field and W a finite dimensional K-vector space such that $\dim_K(W) < |K|$. Let T be a K-subalgebra of $\mathrm{End}_K(W)$ such that $1 \in T$. Then T is additively generated by non-singular elements x of T, i.e., $t \in T$ and $tx = 0$ implies $t = 0$.*

Proof. Let $t \in T$ be such that 0 is an eigenvalue of t. Then there exists some $0 \ne \mu \in K$ such that μ is not an eigenvalue of t. Let $s = t - \mu 1$. Then $s \in T$ and 0 is not an eigenvalue of the linear transformation s. This means that s is invertible in $\mathrm{End}_K(V)$ and it follows that s is non-singular in T. This means that $t = s + \mu 1$ and both summands are non-singular elements of T. □

Proposition 3. *Let K be a field and V, W finite dimensional K-vector spaces such that $\dim_K(W) < |K|$. Let $E = \mathrm{End}_K(V)$, $H = \mathrm{Hom}_K(V,W)$, and $1 \in T$ any subalgebra of $\mathrm{End}_K(W)$. Let $R = \begin{bmatrix} E & 0 \\ H & T \end{bmatrix}$. Then R is a Zassenhaus K-algebra.*

Proof. Note that $RR = \begin{bmatrix} E & 0 \\ H & T \end{bmatrix}\begin{bmatrix} E & 0 \\ H & T \end{bmatrix} = \begin{bmatrix} EE & 0 \\ HE+TH & TT \end{bmatrix} \subseteq R$, which shows that R is a K-algebra. Now we list some left ideals of R

(1) For $0 \ne \varepsilon \in E$ define $J_\varepsilon = R\begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E\varepsilon & 0 \\ H\varepsilon & 0 \end{bmatrix}$.

(2) For $\delta \in T$ define $L_\delta = R\begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & T\delta \end{bmatrix}$.

(3) For $h \in H$, define $L_h = R \begin{bmatrix} 0 & 0 \\ h & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ Th & 0 \end{bmatrix}$, and finally

(4) For $h \in H, \delta \in T$, define $J_{h,\delta} = R \begin{bmatrix} 0 & 0 \\ h & \delta \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 \\ xh & x\delta \end{bmatrix} : x \in T \right\}$.

Now let $\varphi : R \to R$ be a K-linear map leaving all left ideals of R invariant. Then φ is represented by maps $\alpha : E \to E, \beta : E \to H, \eta : H \to H$, and $\theta : T \to T$ such that $\varphi\left(\begin{bmatrix} x & 0 \\ y & z \end{bmatrix}\right) = \begin{bmatrix} \alpha(x) & 0 \\ \beta(x)+\eta(y) & \theta(z) \end{bmatrix}$ for all $x \in E, y \in H, z \in T$. The invariance of the left ideal in (1) and the fact that E is a Zassenhaus algebra by [**4**, Theorem 1] imply that there is some $e \in E$ such that $\alpha(x) = ex$ for all $x \in E$. The invariance of the ideals in (1) implies that $\beta(x) \in Hx$ for all $x \in E$. By Proposition 1 we get that there exists some $h_0 \in H$ such that $\beta(x) = h_0 x$ for all $x \in E$.

We now inspect the ideals in (4). We have: $\varphi\left(\begin{bmatrix} 0 & 0 \\ h & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ \eta(h) & \theta(1) \end{bmatrix} \in J_{h,1}$ for all $h \in H$. This implies $\eta(h) = \theta(1)h$ for all $h \in H$.

Let $\delta \in T$ be a non-singular element of T. Then

$$\varphi\left(\begin{bmatrix} 0 & 0 \\ h & \delta \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ \theta(1)h & \theta(\delta) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ t_{h,\delta}h & t_{h,\delta}\delta \end{bmatrix}$$

for some $t_{h,\delta} \in T$ and all $h \in H$. This implies that $(t_{a,\delta} - t_{b,\delta})\delta = 0$ for all $a, b \in H$. Since δ is non-singular, we infer $t_{a,\delta} - t_{b,\delta} = 0$ for all $a, b \in H$ and $t_{a,\delta} = t_\delta \in T$ depends only on $\delta \in T$. We now have: $\varphi\left(\begin{bmatrix} 0 & 0 \\ h & \delta \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ \theta(1)h & \theta(\delta) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ t_\delta h & t_\delta \delta \end{bmatrix}$ and thus $(\theta(1) - t_\delta)h = 0$ for all $h \in H = \mathrm{Hom}_K(V, W)$. Since $\theta(1) - t_\delta \in T \subseteq \mathrm{End}_K(W)$, we infer $\theta(1) = t_\delta$ for all $\delta \in T$. This shows that $\theta = \theta(1)\cdot$, since, by Proposition 2, T is additively generated by non-singular elements.

We conclude that φ is the multiplication from the left by $\begin{bmatrix} e & 0 \\ h & \theta(1) \end{bmatrix} \in R$. This shows that R is a Zassenhaus algebra. $\square$

Lemma 1. *Let K be a field and $1 \in R$ a K-algebra such that $R/J(R)$ is a division ring and $J(R)$ is nilpotent. Then $R = C \oplus J(R)$ as K-vectorspaces and each non-zero element of C is a unit of R.*

Proof. Let $0 \neq c \in C$. Then there exists a $d \in R$ such that $dc, cd \in 1 + J(R)$. Since $J(R)$ is nilpotent, dc and cd are units in R. This shows that c has a left inverse and a right inverse in R and thus is a unit in R. $\square$

We can now prove the main result of this section.

Theorem 3. *Let K be an infinite field, and let R be a finite dimensional, local K-algebra. If $R \neq K$, then R is not a Zassenhaus algebra, but $\widehat{R}$ is a Zassenhaus algebra.*

Proof. Let $R = C \oplus J(R)$ as K-vectorspaces and L a left ideal of R such that $L \subsetneqq J(R)$. Then there exists some $0 \neq c \in C$ and $j \in J(R)$ such that $c + j \in L$ and, by Lemma 1, c is a unit in R. Then $d = c^{-1}(c+j) \in (1 + J(R)) \cap L$ is a unit in R. This shows that $L = R$. Thus every proper ideal of R is contained in $J(R)$.

Let $T = \{\varphi \in \mathrm{End}_K(J(R)) : \varphi(X) \subseteq X$ for all left ideals $X \subseteq J(R)$ of $R\}$. It is now obvious that

$$\widehat{R} = \begin{bmatrix} \mathrm{End}_K(C) & 0 \\ \mathrm{Hom}_K(C, J(R)) & T \end{bmatrix}.$$

By Proposition 3, $\widehat{R}$ is a Zassenhaus algebra. If $\dim_K(C) > 1$, then $\mathrm{End}_K(C)$ has non-trivial idempotent elements, but R does not. This shows that $R \subsetneqq \widehat{R}$ in this case. Assume $\dim_K(C) = 1$, i.e., $C = K$. Then $\dim_K(\widehat{R}) = \dim_K(R) + \dim_K(T)$. Thus $R \subsetneqq \widehat{R}$ unless $T = \{0\}$, i.e., $J(R) = 0$ and thus $R \cong D$ is a division ring, which is not a Zassenhaus K-algebra unless $R = K$. This shows that K is the only local, finite dimensional K-algebra that is a Zassenhaus algebra. □

In some situations we can say a little more.

Let N be a nilpotent ring, $N^n \neq 0 = N^{n+1}$. Define $\mathfrak{a}_r(N) = \{x \in N : Nx = 0\} \trianglelefteq N$. Of course, $N^n \subseteq \mathfrak{a}_r(N)$.

Claim 1. *Let $0 \neq L \trianglelefteq_\ell N$. Then $L \cap \mathfrak{a}_r(N) \neq 0$.*

Proof. Assume $L \cap \mathfrak{a}_r(N) = 0$. Let $0 \neq s_1 \in L$. Then $Ns_1 \neq 0$, and there is some $s_2 \in N$ such that $0 \neq s_2 s_1 \in L$. Assume that we already have elements $s_{k-1}, \dots, s_3, s_2 \in N$ such that $s_{k-1} \cdots s_3 s_2 s_1 = t \neq 0$. Then $0 \neq t \in L$ and thus not in $\mathfrak{a}_r(N)$. Thus there is some $s_k \in N$ such that $s_k t \neq 0$. For $k = n+1$ we get a contradiction, since N is nilpotent. □

Let D be some division ring such that the nilpotent ring N is a $D-D$-bimodule. Then the direct sum $S = D \oplus N$ is a ring with $J(S) = N$.

Let $m \in N, m \notin \mathfrak{a}_r(N)$, and $0 \neq \delta \in Nm \cap \mathfrak{a}_r(N)$. Then $S(m+\delta) = D(m+\delta) \oplus Nm$.

To prove this, let $0 \neq d \in D$, $n \in N$ such that $d(m+\delta) = nm$.

Then $(d-n)m = -d\delta$ and $m = (d-n)^{-1}d\delta \in \mathfrak{a}_r(N)$, a contradiction. We will also need

Proposition 4. *Let $1 \in R$ be a ring and M a free R-module of rank at least 2 and $\varphi \in \operatorname{End}_{\mathbf{Z}}(M)$ such that $\varphi(x) \in Rx$ for all $x \in M$. Then there is some $a \in R$ such that $\varphi(x) = ax$ for all $x \in M$.*

Proof. Let B be a basis of M, and let b_1, b_2 be two distinct elements of B. Then there are $r_1, r_2, r_3 \in R$ such that $\varphi(b_1) = r_1 b_1, \varphi(b_2) = r_2 b_2$, and $\varphi(b_1+b_2) = r_3(b_1+b_2)$. Since B is a basis, we infer $r_1 = r_3 = r_2$, and it follows that there exists some $a \in R$ such that $\varphi(b) = ab$ for all $b \in B$. Now let $s \in R$. Then $\varphi(b_1 + sb_2) = ab_1 + t(sb_2) = r(b_1 + sb_2)$ for some $r, t \in R$. We infer that $a = r$ and $ts = as$. This shows that $\varphi(sb) = a(sb)$ for all $s \in R, b \in B$, which implies that $\varphi(x) = ax$ for all $x \in M$. □

Proposition 5. *Let K be a field and D a division ring that is also a K-algebra. Let $S = D \oplus N$ be a ring with $N^2 = \{0\}$ and N a $D-D$-bimodule. Then $\widehat{S} \cong \begin{bmatrix} \operatorname{End}_K(D) & 0 \\ \operatorname{Hom}_K(D,N) & D \end{bmatrix}$ if* $\mathrm{do},_D(N) > 1$.

If $\dim_D(N) = 1$, then $\widehat{S} \cong \begin{bmatrix} \operatorname{End}_K(D) & 0 \\ \operatorname{End}_K(D) & \operatorname{End}_K(D) \end{bmatrix}$. In either case, $\widehat{\widehat{S}} = \widehat{S}$.

Proof. If $\dim_D(N) = 1$, then N is the only non-trivial left ideal of S. If $\dim_K(N) > 1$, then every non-trivial left ideal of S is contained in N and every left D-subspace of ${}_DN$ is a left ideal of S. The rest now follows easily from the above. □

Theorem 4. *Let the K-algebra D be a division ring and $S = D \oplus N$ a finite dimensional K-algebra such that $J(S) = N$ with $N^n = 0 \neq N^{n-1}$*

and $\dim_D(N) \geq 2$. *Moreover, assume that*

$(*)$ $A = \mathfrak{a}_r(N) \nsubseteq Nm$ *for all* $m \in N$.

Then $\widehat{S} = \begin{bmatrix} \mathrm{End}_K(D) & 0 \\ \mathrm{Hom}_K(D,N) & T \end{bmatrix}$, *where* T *is a* K*-algebra of the form* $T = D \oplus M$ *with* $M^{n-1} = \{0\}$.

Proof. Since each element in $S - N$ is a unit of S, each proper left ideal J of S is contained in N. This shows that $\widehat{S}$ has the proposed form, where $T = \{\varphi \in \mathrm{End}_K(N) : \varphi(x) \in Sx$ for all $x \in N\}$. Now let $A = \mathfrak{a}_r(N) \supseteq N^n$. Note that a subset J of $\mathfrak{a}_r(N)$ is a left ideal of S if and only if J is a subspace of the left D-vector space N. Now let $\varphi \in T$. By Proposition 4, there is some $d \in D$ such that $\varphi \restriction_A = d$ is the multiplication by $d \in D$. Now let $m \in N - A$ and $a \in A - Nm$. Then $S(m+a) = D(m+a) + N(m+a) = D(m+a) \oplus Nm$. Thus $\varphi(m+a) = d_1(m+a) + ym$ for some $d_1 \in D$ and $y \in N$. Moreover, $\varphi(m) = d_2m + zm$ for some $z \in N$, and thus $d_2m + zm + da = d_1(m+a) + ym$ and $(d_2 - d_1)m \equiv (d_1 - d)a \bmod Nm$. This implies $(d_2 - d_1)m \in Nm$ and thus $d_1 = d_2$. This shows that $(d_1 - d)a \in Nm$. If $d_1 - d \neq 0$, we get that $a \in Nm$, a contradiction to our choice of a. Thus, for $\psi = \varphi - d$ we have $\psi(x) \in Nx$ for all $x \in N$ and it follows that $\psi^{n-1} = 0$.

Note that $A \nsubseteq N^2$ implies the condition $(*)$ posed on A in this Theorem. □

Example 1. We would like to point out a natural example for algebras S that satisfy the hypotheses of Theorem 4. Pick a natural number n and a division K-algebra D. Let S be the ring of all lower triangular $n \times n$-matrices over D with the same element $d \in D$ down on the main diagonal. Then $J(S)$ is the set of all lower triangular matrices with only zeros on the main diagonal. Define

$$\psi = (a_{ij}) \in S \text{ by } a_{ij} = \begin{cases} 1 & \text{if } (i,j) = (n, n-1) \\ 0 & \text{otherwise.} \end{cases}$$

Then $S = D \oplus J(S)$ and $J(S)\psi = \{0\}$, i.e., $\psi \in \mathfrak{a}_r(J(S))$ but $\psi \notin (J(S))^2$. Now Theorem 4 tells the structure of $\widehat{S}$.

3. Reduced algebras. Recall that the finite dimensional K-algebra R is reduced [**9**, page 101], if $R/J(R) \cong \oplus_{i=1}^n D_i$ is a ring direct sum of division rings D_i. Unfortunately, this case seems to be quite difficult. We restrict our attention to path algebras over quivers. Recall that a quiver $Q = (V, E)$ is an oriented graph where V is the set of vertices and E is the set of edges (arrows). For $e \in E$, the element $t(e) \in V$ is the tail of e and $h(e) \in V$ is the head of e. A path $\pi = e_n e_{n-1} \cdots e_2 e_1$ in Q is a sequence of edges e_i such that $t(e_{i+1}) = h(e_i)$ for all $1 \leq i < n$ and $n \geq 1$. (We follow the custom in the topic that paths run from the right to the left.) The number n is the length of the path π. Define $h(\pi) = h(e_n)$ and $t(\pi) = t(e_1)$. Let Π be the set of all paths in Q. The path algebra of the quiver Q over the field K is the vector space $KQ = \oplus_{v\in V} Kv \oplus \oplus_{\pi\in\Pi} K\pi$ with a multiplication defined by the following relations:

For $\pi, \sigma \in \Pi$, we have

$$\pi \cdot \sigma = \begin{cases} \pi\sigma & \text{if } t(\pi) = h(\sigma) \\ 0 & \text{otherwise.} \end{cases}$$

Here $\pi\sigma$ is the concatenation of the two paths. Moreover, for $v \in V$, $e \in E$ we have

$$ve = \begin{cases} e & \text{if } v = h(e) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad ev = \begin{cases} e & \text{if } v = t(e) \\ 0 & \text{otherwise.} \end{cases}$$

Finally,

$$uv = \begin{cases} u & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$$

The path π is closed if $t(\pi) = h(\pi)$. A loop is an edge e with $t(e) = h(e)$. We will always assume that Q is finite, i.e., the sets V and E are finite. In that case, it is well known that the K-algebra KQ has finite K-dimension if and only if Q is acyclic, i.e., there are no closed paths (loops) in Q. Moreover, $1 = \sum_{v\in V} v$ is the identity of KQ and V is a complete set of pairwise orthogonal idempotents in KQ. We are able to show

Theorem 5. *Let K be a field and Q a finite quiver. Then the following hold*:

(a) *If K is an infinite field, then KQ is a Zassenhaus algebra.*

(b) *If K is any field and Q has no loops, then KQ is a Zassenhaus algebra.*

(c) *If K is any field and Q is acyclic, then KQ is a reduced Zassenhaus algebra.*

Proof. To show (b), let $\varphi \in \widehat{KQ}$ be such that $\varphi(1) = 0$. We will show that $\varphi = 0$. Let $v \in V$. Then $KQ = (KQ)v \oplus (KQ)(1-v)$ and $\varphi(-v) = \varphi(1-v) \in (KQ)v \cap (KQ)(1-v) = \{0\}$. This shows that $\varphi(v) = 0$ for all $v \in V$. Now let $e \in E$ and $u = h(e)$. Then $\varphi(e) = \varphi(u+e) \in (KQ)(u+e) \cap (KQ)e$. Let π be some path. Then $\pi e \neq 0 \neq \pi u$ if and only if $t(\pi) = u$ and the two products are both 0 otherwise. Assume $t(\pi) = h(e) = u$ and thus $\pi(u+e) = \pi + \pi e \neq 0$. If some linear combination $\sum_{i=1}^n k_i(\pi_i + \pi_i e) \in (KQ)e$, then $\sum_{i=1}^n k_i \pi_i \in (KQ)e$, and it follows that $\pi_i = \sigma_i e$ for some $\sigma_i \in \Pi$. But then $\pi_i e = \sigma_i ee \neq 0$, and thus $h(e) = t(e)$ and e is a loop. This shows that $(KQ)(u+e) \cap (KQ)e = \{0\}$, and it follows that $\varphi(e) = 0$ for all $e \in E$. Now let $\mu = \lambda e_1$ be a path of length $n \geq 2$ and λ a path of length $n-1$. By induction hypothesis, we may assume that $\varphi(\lambda) = 0$, and thus $\varphi(\lambda + \mu) = \varphi(\mu) \in (KQ)(\lambda+\mu) \cap (KQ)\mu$. As before, assume that $\sum_{i=1}^n k_i \pi_i(\lambda + \mu) \in (KQ)\mu$. Note that

$$\pi_i(\lambda+\mu) = \pi_i(\lambda + \lambda e_1) = \begin{cases} \pi_i\lambda + \pi_i\mu & \text{if } t(\pi_i) = h(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $\pi_i \lambda = \sigma_i \mu$ for some $\sigma_i \in \Pi$, which implies that $\lambda = \lambda_i e_1$ for some $\lambda_i \in \Pi$. It follows that $\mu = \lambda_i e_1 e_1 \neq 0$, which, as before, shows that the edge e_1 is a loop. Thus, $(KQ)(\lambda+\mu) \cap (KQ)\mu = \{0\}$, and it follows that $\varphi(\mu) = 0$ for all $\mu \in \Pi$.

To prove (a), we need one more step. Assume e is a loop and $t(e) = v = h(v)$. Then $1 \in K[e]$, the subalgebra of KQ generated by e over K, is just a polynomial ring. Let k_i, $i = 1, 2, 3, \dots$ be distinct elements in the infinite field K such that $k_1 = 0$. Let $g_n = (k_1 + e)(k_2 + e) \cdots (k_n + e) \in K[e]$. By induction over n, we will prove that $\cap_{1 \leq i \leq n}(k_i + e)KQ = g_n KQ$. If $n = 1$, there is nothing to show. Note that $\cap_{1 \leq i \leq n+1}(k_i + e)KQ = g_n KQ \cap (k_{n+1} + e)KQ$. Clearly, g_n and $k_{n+1} + e$ are relatively prime in $K[e]$, and by the Euclidean algorithm there are $\alpha, \beta \in K[e]$ such that $1 = g_n\alpha + (k_i + e)\beta$. Let $y \in \cap_{1 \leq i \leq n+1}(k_i + e)KQ$. Then there are $a, b \in KQ$ such that

$y = g_n a = (k_{n+1} + e)b$. It follows that $y = g_n \alpha y + (k_i + e)\beta y = g_n \alpha (k_{n+1} + e)a + (k_{n+1} + e)\beta g_n b = g_{n+1}(\alpha a + \beta b) \in g_{n+1}KQ$. It is now easy to see that $\cap_{i\in \mathbf{N}}(k_i + e)KQ = \{0\}$. Assume that w is an element in that intersection. Let m be the maximal number of occurrences of the edge e in a representation of the element w. Now $w \in g_n KQ$ for $n > m$ in which e^n occurs. (Note that if π is some path then $g_n \pi = 0 \Leftrightarrow e\pi = 0 \Leftrightarrow e^j \pi = 0$ for all j.) Now if $\varphi \in \widehat{KQ}$ such that $\varphi(1) = 0$, then $\varphi(e) = \varphi(k_i + e) \in \cap_{i\in \mathbf{N}}(k_i + e)KQ = \{0\}$. Thus $\varphi(e) = 0$ for all loops e as well, and the rest follows as in the proof of (b).

To show (c), let $J = \sum_{e\in E}(KQ)e$. If $\pi \in \Pi$ is a path such that some edge $e \in E$ shows up twice, then this path contains a closed path, but Q is assumed to be acyclic. Thus every edge shows up at most once in every path. This shows that $|E|$ is an upper bound for the length of all paths in Q, and therefore J is nilpotent and $(KQ)/J \cong \oplus_{v\in V} Kv$ is a ring direct product of copies of the field K. This shows that KQ is a reduced algebra and Zassenhaus by (b). $\square$

Let Δ be the quiver with a single vertex v and a single loop e; then $K\Delta$ is isomorphic to the polynomial ring $K[x]$ over K in the single indeterminate x. We have already shown in [**5**] that $K[x]$ is Zassenhaus if the field K is infinite. On the other hand, if K is the finite field with q elements, then $\varphi : K[x] \to K[x]$ with $\varphi(f) = f^q$ for all $f \in K[x]$ is a ring endomorphism of $K[x]$, usually referred to as a Frobenius homomorphism. Clearly, $\varphi \in \widehat{K[x]}$ but is not a multiplication by some element in $K[x]$. Therefore, $K[x]$ need not be Zassenhaus algebra, if the field K is finite.

While all finite dimensional path algebras over (acyclic) quivers are reduced Zassenhaus algebras, it is easy to see that there are many reduced algebras that are not Zassenhaus algebras: Let K be an infinite field. Take two copies of a local K-algebra $S \neq K$, and consider the ring direct product $R = S \times S$. Then R is reduced, not local and not a Zassenhaus algebra by Theorem 4. On the other hand, $\widehat{R}$ is a Zassenhaus algebra. We would like to pose the following

Conjecture 1. *Let R be a reduced, finite dimensional K-algebra over the infinite field K. Then $\widehat{R}$ is a Zassenhaus algebra.*

4. Checkered matrix algebras. First we recall the case of a semisimple algebra. This case was covered in [**6**]. We include it here for the sake of completeness. Let R be a finite dimensional, semisimple K-algebra. It is well known that $R = \oplus_{i=1}^{k} e_i R_i$ is a ring direct sum of simple algebras R_i and the e_i are central, orthogonal idempotents. Moreover, there are division algebras D_i over K and $n_i \geq 1$ such that $R_i \cong \mathrm{Mat}_{n_i \times n_i}(D_i)$. Note that for any left ideal I of R we have $I = \oplus_{i=1}^{k}(I \cap R_i)$. This shows that $\widehat{R} = \oplus_{i=1}^{k} \widehat{R_i}$ is a ring direct sum and $R = \widehat{R}$ if and only if $R_i = \widehat{R_i}$ for all $1 \leq i \leq k$. By [**6**], R_i is a Zassenhaus algebra if $n_i > 1$. If $n_i = 1$, then $R_i = D_i$ is a Zassenhaus algebra if and only if $D_i = K$. On the other hand, $\widehat{D_i}$ is a Zassenhaus algebra by [**5**]. This proves:

Theorem 6. *Same notations as above. Then*

(1) *$\widehat{R}$ is a Zassenhaus algebra.*

(2) *R is a Zassenhaus algebra if and only if $D_i = K$ whenever $n_i = 1$.*

(3) *If K is algebraically closed, then R is always a Zassenhaus algebra.*

Other easy cases are primary algebras. Recall that a finite dimensional K-algebra R is called primary if $R/J(R)$ is a simple algebra. By a well-known result, cf. [**9**, page 98], there exists a local algebra B and $n \geq 1$ such that $R \cong \mathrm{Mat}_{n \times n}(B)$. Note that R is local if and only if $n = 1$. By [**5, 6**], we have:

Theorem 7. *Let R be a finite dimensional, primary, not local K-algebra. Then R is a Zassenhaus algebra.*

Let K be a field and A a left (or right) Artinian K-algebra. Then there exists a reduced algebra B, the basic algebra of A, and a sequence $\overrightarrow{e} = (e_1, e_2, \ldots, e_r)$ of primitive, orthogonal idempotents of B such that $1 = \sum_{i=1}^{r} e_i$. Moreover, there exists a sequence $\overrightarrow{n} = (n_1, n_2, \ldots, n_r)$ of positive integers such that A is isomorphic to a ring of $r \times r$ block matrices $M_{\overrightarrow{n}}(B, \overrightarrow{e})$ where the (i,j)-block is $\mathrm{Mat}_{n_i \times n_j}(e_i B e_j)$, i.e.,

$$M_{\overrightarrow{n}}(B, \overrightarrow{e}) = \begin{bmatrix} \mathrm{Mat}_{n_1 \times n_1}(e_1 B e_1) & \mathrm{Mat}_{n_1 \times n_2}(e_1 B e_2) & \cdots & \mathrm{Mat}_{n_1 \times n_r}(e_1 B e_r) \\ \mathrm{Mat}_{n_2 \times n_1}(e_2 B e_1) & \mathrm{Mat}_{n_2 \times n_2}(e_2 B e_2) & \cdots & \mathrm{Mat}_{n_2 \times n_r}(e_2 B e_r) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Mat}_{n_r \times n_1}(e_r B e_1) & \mathrm{Mat}_{n_r \times n_2}(e_r B e_2) & \cdots & \mathrm{Mat}_{n_r \times n_r}(e_r B e_r) \end{bmatrix}.$$

This algebra is called a checkered matrix algebra, cf. [**9**, page 98]. Note that the algebras mentioned in Theorems 6 and 7 are special cases of checkered matrix algebras. By [**3**], the following holds:

Theorem 8 [**3**]. *Let K be a field and A a left (or right) Artinian K-algebra. Then there exists a reduced, left (or right) Artinian algebra B and $\overrightarrow{e}$, $\overrightarrow{n}$ as above such that A is isomorphic to $M_{\overrightarrow{n}}(B, \overrightarrow{e})$. Moreover, B, $\overrightarrow{e}$ and $\overrightarrow{n}$ are uniquely determined by A.*

Note that B is local if and only if $r = 1$. If $r = 1 = n_1$, then $A \cong B$ is local. If $r = 1 < n_1$, then A is primary, not local, cf. [**9**, page 98]. Moreover, $e_i B e_i$ is a local algebra for all $1 \leq i \leq r$.

Our goal is to prove

Theorem 9. *With the notations from above, $M_{\overrightarrow{n}}(B, \overrightarrow{e})$ is a Zassenhaus algebra if $r \geq 2$ and $n_i \geq 2$ for all $1 \leq i \leq r$.*

First we need:

Lemma 2. *Let T, S be rings, $1 \in S$, such that T is also a right S-module. Let $M = \mathrm{Mat}_{m \times m}(T)$, $m \geq 2$, and $R = \mathrm{Mat}_{n \times m}(S)$. Let $\beta \in \mathrm{Hom}_{\mathbf{Z}}(R, M)$ be such that $\beta(x) \in Mx$ for all $x \in R$. Then there exists some $B \in M$ such that $\beta(x) = Bx$ for all $x \in R$.*

Proof. First, note that M is a right R-module. Let $\varepsilon^{(ik)}$ be the matrix in R which has 1 in the (i, k)-position and 0s everywhere else. For $t \in T$, we use the same notation for the matrix $t\varepsilon^{(ik)} \in M$ that has t in the (i, k)-position and zeros elsewhere. There exist elements $\ell_{\alpha i}^{(**)} \in T$ such that, for $s \in S$, we have $\beta(\varepsilon^{(ik)}) = \sum_\alpha \ell_{\alpha i}^{(ki)} \varepsilon^{(\alpha k)}$ and $\beta(s\varepsilon^{(ij)}) = \sum_\alpha \ell_{\alpha i}^{(js)} s\varepsilon^{(\alpha j)}$. Moreover, there exists some matrix $A^{(s)} = (a_{\alpha i}^{(s)})_{\alpha i} \in M$ such that $\beta(\varepsilon^{(ik)} + s\varepsilon^{(ij)}) = A^{(s)}(\varepsilon^{(ik)} + s\varepsilon^{(ij)}) = \sum_\alpha a_{\alpha i}^{(s)} \varepsilon^{(\alpha k)} + \sum_\alpha a_{\alpha i}^{(s)} s\varepsilon^{(\alpha j)} = \sum_\alpha \ell_{\alpha i}^{(ki)} \varepsilon^{(\alpha k)} + \sum_\alpha \ell_{\alpha i}^{(js)} s\varepsilon^{(\alpha j)}$. Now assume that $1 \leq j \neq k \leq m$. We compare coefficients and get: $a_{\alpha i}^{(s)} = \ell_{\alpha i}^{(k1)}$ and $a_{\alpha i}^{(s)} s = \ell_{\alpha i}^{(js)}$. We infer $\ell_{\alpha i}^{(k1)} s = \ell_{\alpha i}^{(js)}$ for all $s \in T$ and $k \neq j$. This shows that there exists a matrix $B = (\ell_{\alpha i}) \in M$ such that $\beta(s\varepsilon^{(ij)}) = Bs\varepsilon^{(ij)}$ for all $1 \leq i, j \leq m$, and the conclusion follows. □

Now we are ready to prove Theorem 9:

Proof. Let $M = M_{\overrightarrow{n}}(B, \overrightarrow{e})$, and assume that $r \geq 2$ and $n_i \geq 2$ for all $1 \leq i \leq r$. Moreover, let $M_{ij} = \mathrm{Mat}_{n_i \times n_j}(e_i B e_j)$ for $1 \leq i, j \leq r$.

Assume $\varphi \in \mathrm{End}_{\mathbf{Z}}(M)$ such that $\varphi(X) \subseteq X$ for all $X \trianglelefteq_\ell M$. Since $M = \bigoplus_{1 \leq i,j \leq n} M_{ij}\varepsilon^{(ij)}$, φ is represented by a matrix $[\varphi_{ij,\alpha\beta}]$ where $\varphi_{ij,\alpha\beta} : M_{\alpha\beta} \to M_{ij}$. Let $m_{\alpha\alpha} \in M_{\alpha\alpha}$. Note that $M m_{\alpha\alpha}\varepsilon^{(\alpha\alpha)} = \oplus_{i=1}^r M_{i\alpha} m_{\alpha\alpha}\varepsilon^{(i\alpha)} \trianglelefteq_\ell M$ for all $1 \leq \alpha \leq r$. This shows that $\varphi_{kj,i\alpha} = 0$ whenever $j \neq \alpha$. Also $\varphi_{k\alpha,i\alpha}(M_{i\alpha}m_{\alpha\alpha}) \subseteq M_{k\alpha}m_{\alpha\alpha}$ for all i, k, α. Especially $\varphi_{i\alpha,\alpha\alpha} : M_{\alpha\alpha} \to M_{i\alpha}$ such that $\varphi_{i\alpha,\alpha\alpha}(x) \in M_{i\alpha}x$ for all $x \in M_{\alpha\alpha}$. Lemma 2 shows that there is some $t_{i\alpha} \in M_{i\alpha}$ such that $\varphi_{i\alpha,\alpha\alpha}(x) = t_{i\alpha}x$ for all $x \in M_{\alpha\alpha}$.

Now define left ideals $J_{\alpha,\beta,s} = M(\varepsilon^{(\alpha\alpha)} + s\varepsilon^{(\alpha\beta)})$ for all $s \in M_{\alpha\beta}$ and $1 \leq \alpha \neq \beta \leq r$. Note that $J_{\alpha,\beta,s} = \oplus_{i=1}^r M_{i\alpha}(\varepsilon^{(i\alpha)} + s\varepsilon^{(i\beta)})$, and we get $\varphi(\varepsilon^{(\alpha\alpha)} + s\varepsilon^{(\alpha\beta)}) = \sum_{i=1}^r t_{i\alpha}\varepsilon^{(i\alpha)} + \sum_{i=1}^r \varphi_{i\beta,\alpha\beta}(s\varepsilon^{(\alpha\beta)}) = \sum_{i=1}^r x_{i\alpha}(\varepsilon^{(i\alpha)} + s\varepsilon^{(i\beta)})$. We infer that $t_{i\alpha} = x_{i\alpha}$ and $\varphi_{i\beta,\alpha\beta}(s\varepsilon^{(\alpha\beta)}) = t_{i\alpha}s\varepsilon^{(i\beta)}$, which shows that $\varphi_{i\beta,\alpha\beta} = t_{i\alpha} \in M_{i\alpha}$ is independent of β. Now $\varphi(1) = \varphi(\sum_{\alpha=1}^r \varepsilon^{(\alpha\alpha)}) = \sum_\alpha \varphi(\varepsilon^{\alpha\alpha}) = \sum_\alpha \sum_i \varphi_{i\alpha,\alpha\alpha}(\varepsilon^{\alpha\alpha}) = \sum_\alpha \sum_i t_{i\alpha}\varepsilon^{(i\alpha)} = 0$ implies that $t_{i\alpha} = 0$ for all $1 \leq i, \alpha \leq r$, which implies that $\varphi = 0$, and we have that M is a Zassenhaus algebra. □

5. Leavitt path algebras. We want to consider the case of Leavitt path algebras. We adopt the notations commonly used in this topic as in [**1**]. A directed graph $\Gamma = (V, E, r, s)$ where V is the set of vertices and E is the set of edges of Γ. Moreover, $r : E \to V$ and $s : E \to V$ are functions such that the edge (or arrow) e goes from $s(e)$, the source (or tail) of e to $r(e)$, the range (or head) of e. We will always assume that the set $s^{-1}(v)$ is finite, i.e., Γ is row finite. A sequence $\mu = e_1e_2\cdots e_n$ of edges is a path, if $s(e_{i+1}) = r(e_i)$ for all $1 \leq i \leq n-1$. (We follow the tradition in this area that paths run from the left to the right.) Let K be a field. We define the Leavitt path algebra $L_K(\Gamma)$ to be the K-algebra generated by the elements of the set $V \cup E \cup E^*$ where $E^* = \{e^* : e \in E\}$, $s(e^*) = r(e)$ and $r(e^*) = s(e)$. The elements of E are called real edges, and the elements of E^* are called ghost edges. We list the familiar relations for the generators of $L_K(\Gamma)$:

(0) For all $u, v \in V$ we have $uv = \delta_{u,v}u$, i.e., the vertices are orthogonal idempotents.

(1) $s(e)e = e = er(e)$ for all $e \in E$.

(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E$.

(3) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E$.

(4) If $v \in V$ and $s^{-1}(v) \neq \varnothing$, then $v = \sum_{e \in s^{-1}(v)} ee^*$.

Recall that $e \in E$ is a loop if $s(e) = r(e)$. By [**2**, Corollary 3.7] the algebra $L_K(\Gamma)$ is finite dimensional if and only if Γ is a direct product of full matrix rings over K if and only if Γ is finite and acyclic. Thus all finite dimensional Leavitt path algebras are Zassenhaus algebras. Our goal is to investigate the infinite dimensional case.

By [**1**, 1.5. Lemma] the K-vector space $L_K(\Gamma)$ is spanned by the vertices of Γ and paths of the form $\pi = e_{i_1} \cdots e_{i_\sigma} e^*_{j_1} \cdots e^*_{j_\tau}$ such that $\sigma, \tau \geq 0, \sigma + \tau > 0$ and the e's are real edges of Γ.

We define $\Delta(e_{i_1} \cdots e_{i_\sigma} e^*_{j_1} \cdots e^*_{j_\tau}) = \sigma - \tau$.

Since a Leavitt path algebra has an involution induced by *, we may pick sides at our convenience, and we define $\widehat{L_K(\Gamma)} = \{\varphi \in \mathrm{End}_K(L_K(\Gamma)) : \varphi(X) \subseteq X$ for all right ideals X of $L_K(\Gamma)\}$. We find that right ideals are a little more intuitive in this setting.

Observe that for any edge e we have $e^*e^* = e^*r(e^*)s(e^*)e^* = e^*0e^* = 0$ if e is not a loop.

Note that $L_K(\Gamma)$ is a $\mathbf{Z}$-graded algebra, cf., [**1**, 1.7. Lemma]: Define $A_n = \sum\{ke_{i_1} \cdots e_{i_\sigma} e^*_{j_1} \cdots e^*_{j_\tau} : \sigma + \tau > 0, e_{i_s} \in E, e^*_{j_t} \in E^*, k \in K, \sigma - \tau = n\}$. Then $L_K(\Gamma) = \bigoplus_{n \in \mathbf{Z}} L_n$, where $L_0 = KV + A_0$ and $L_n = A_n$ for $n \neq 0$.

Claim 2. *Let e be an edge (real or ghost) that is not a loop and $u = r(e)$ a vertex. Then $(u + e)L_K(\Gamma) \cap eL_K(\Gamma) = \{0\}$.*

Proof. Let $x \in L_K(\Gamma)$ be such that $(u + e)x \in eL_K(\Gamma)$ and thus $ux = ey$ for some $y \in L_K(\Gamma)$. Then $0 = e^*ux = e^*ey = u \cdot y$ because $u \neq r(e^*) = s(e)$. Now $0 = e(u \cdot y) = (eu)y = ey$, and it follows that $ux = 0$, and thus $0 = e(ux) = ex$. This shows that $(u + e)x = 0$, and the claim follows for real edges e.

We will show that the same proof works for ghost edges as well, which is not surprising, since * induces an involution.

Let $x \in L_K(\Gamma)$ be such that $(u + e^*)x \in e^*L_K(\Gamma)$ and $r(e^*) = s(e) = u$. Then $ux = e^*y$ for some $y \in L_K(\Gamma)$ and $e^*e^* = 0$. Thus, $e^*x = e^*ux = e^*e^*y = 0$. This implies $ux \in e^*L_K(\Gamma)$, and thus $ux = e^*t$ for some $t \in L_K(\Gamma)$. This implies $ux = u \cdot ux = ue^*t = 0$ since $s(e^*) = r(e) \neq s(e) = u$. This shows $(u+e^*)x = 0$, and the claim follows. □

Claim 3. *Let $e \in E$ be an edge (real or ghost) such that e is a loop, i.e., $s(e) = v = r(v)$ for some vertex v. Let $K[e]$ denote the K-subalgebra generated by v and e. Then $K[e]$ is isomorphic to the polynomial ring $K[x]$ over K with indeterminate x.*

Proof. There exists the natural K-algebra epimorphism $\gamma : K[x] \to K[e]$ with $\gamma(1) = v$ and $\gamma(x) = e$. Then $\gamma(1) \in L_0$ and $0 \neq \gamma(x^k) \in L_k$ for all $k > 0$. If follows from the $\mathbf{Z}$-grading of $L_K(\Gamma)$ that γ is injective as well. □

Claim 4. *Let $k_1, k_2, \ldots$ be distinct elements of K and $e \in E$ a loop with $s(e) = v = r(e)$. Then*

$$\bigcap_{i=1}^{n}(k_i v + e)L_K(\Gamma) = \left(\prod_{i=1}^{n}(k_i v + e)\right)L_K(\Gamma).$$

Proof. By induction on n. If $n = 1$, there is nothing to show. Note that $K[e]$ is commutative. Let $n > 1$, and assume that the claim holds for $n - 1$. Let $\pi = \prod_{i=1}^{n-1}(k_i v + e)$. By Claim 3, $K[e]$ is a polynomial ring and π and $k_n v + e$ are relatively prime in $K[e]$. Thus we have $\alpha, \beta \in K[e]$ such that $\pi\alpha + (k_n v + e)\beta = v$, and note that $\pi(k_n v + e)L_K(\Gamma) \subseteq \cap_{i=1}^{n}(k_i v + e)L_K(\Gamma) = \pi L_K(\Gamma) \cap (k_n v + e)L_K(\Gamma)$. Let $w \in \cap_{i=1}^{n}(k_i v + e)L_K(\Gamma)$. Then there exist $c, d \in L_K(\Gamma)$ with $w = \pi c = (k_n v + e)d$. This implies $\pi\alpha d + (k_n v + e)\beta d = vd = \pi\alpha d + \beta(k_n v + e)d = \pi\alpha d + \beta w$ and thus $vd = \pi\alpha d + \beta w = \pi\alpha d + \beta\pi c = \pi\alpha d + \pi\beta c = \pi(\alpha d + \beta c)$. We infer that $w = (k_n v + e)d = (k_n v + e)vd = (k_n v + e)\pi(\alpha d + \beta c) \in \pi(k_n v + e)L_K(\Gamma)$. □

We now can prove the crucial

Claim 5. (a) *Let $k_1, k_2, \ldots$ be distinct non-zero elements of K and $e \in E$ a real loop with $s(e) = v = r(e)$. Then $\cap_{i=1}^{\infty}(k_i v + e)L_K(\Gamma) = \{0\}$.*

(b) *Let $e^* \in E^*$ be a ghost loop. Let $h_n \in K[e]$ be polynomials in the polynomial ring $K[e]$ such that the polynomials $f_n = eh_n + v$, $n \in \mathbf{N}$ are pairwise relatively prime. Then $\cap_{i=1}^{\infty}(h_n + e^*)L_K(\Gamma) = \{0\}$.*

Proof. (a) We will utilize the $\mathbf{Z}$-grading of $L_K(\Gamma)$ mentioned above. Assume $0 \neq w \in \cap_{i=1}^{\infty}(k_i v + e)L_K(\Gamma)$. There is some $m \geq 0$ such that $w \in \oplus_{k=-m}^{m} L_k$. Since w is a linear combination of paths ρ with $\Delta(\rho) = \sigma - \tau$ we may also assume that $\tau < m$ for all the paths used in the representation of w. Choose a natural number $n > 2m$. Then $0 \neq w \in \cap_{i=1}^{n}(k_i v + e)L_K(\Gamma)$ and

$$\bigcap_{i=1}^{n}(k_i v + e)L_K(\Gamma) = \left(\prod_{i=1}^{n}(k_i v + e)\right)L_K(\Gamma)$$

by Claim 4. Let $g_n = \prod_{i=1}^{n}(k_i v + e)$ and note that $g_n = a_0 v + a_1 e + \cdots + a_{n-1}e^{n-1} + e^n$ for some $a_i \in K$ and $a_0 \neq 0$. Now let $\rho = e_{i_1} \cdots e_{i_\sigma} e_{j_1}^* \cdots e_{j_\tau}^*$ be a path such that $s(\rho) = v$ and $(a_0 v + a_1 e + \cdots + a_{n-1}e^{n-1} + e^n)\rho \in \oplus_{k=-m}^{m} L_k$. This means that $-m \leq n + \Delta(\rho) \leq m$ and $\Delta(\rho) \leq m - n < m - 2m = -m$ follows. On the other hand, $a_0 v\rho = a_0\rho \in L_{\Delta(\rho)}$ and $\Delta(\rho) < -m$. We infer that

$$w \notin \cap_{i=1}^{n}(k_i v + e)L_K(\Gamma) = \left(\prod_{i=1}^{n}(k_i v + e)\right)L_K(\Gamma)$$

and the claim follows in the case that the loop e is a real edge.

Now we consider the case where e^* is a loop which is a ghost edge with $s(e^*) = r(e) = v = s(e) = r(e^*)$. Here the approach in (a) does not work anymore, since $(k_i v + e^*)L_K(\Gamma) \supseteq (k_i v + e^*)fL_K(\Gamma) = (k_i f + e^* f)L_K(\Gamma) = (k_i f)L_K(\Gamma) = fL_K(\Gamma)$ for any edge $f \neq e$ but with $s(f) = v$. This shows that the intersection we looked at above is not $\{0\}$ at all, but rather large.

We now prove (b), maintaining the notations from above.

Note that $\cap_{i=1}^{\infty}(h_i + e^*)L_K(\Gamma) = \cap_{i=1}^{\infty}(e^* e h_i + e^*)L_K(\Gamma) = \cap_{i=1}^{\infty}e^*(eh_i + v)L_K(\Gamma) = \cap_{i=1}^{\infty}e^* f_i L_K(\Gamma)$. An argument similar to the one used in the proof of Theorem 4 (a), as well as just above, shows that $\cap_{i=1}^{\infty} f_i L_K(\Gamma) = \{0\}$. Moreover, for $g_n = f_1 f_2 \cdots f_n$ it follows that $\cap_{i=1}^{n} f_i L_K(\Gamma) = g_n L_K(\Gamma)$ and if $y \in L_K(\Gamma)$ such that $e^* g_n y =$

0, then $g_n y = 0$ and it easily follows that $\cap_{i=1}^{\infty}(h_i + e^*)L_K(\Gamma) = \cap_{i=1}^{\infty} e^* f_i L_K(\Gamma) = e^*(\cap_{i=1}^{\infty} f_i L_K(\Gamma)) = e^*\{0\} = \{0\}$. $\square$

We are now ready for the pivotal

Claim 6. *Assume that the field K is infinite. Let $\psi \in \widehat{L_K(\Gamma)}$, v a vertex such that $\psi(v) = 0$ and π a path with $r(\pi) = v$. Then $\psi(\pi) = 0$.*

Proof. Let $\pi = e_n e_{n-1} \cdots e_2 e_1$ where the e_i are edges or ghost edges. We induct over n. Let $n = 1$ and thus $\pi = e_1 = e$ some real edge. If e is not a loop, then $\psi(e) = \psi(v + e) \in (v + e)L_K(\Gamma) \cap eL_K(\Gamma) = \{0\}$ by Claim 2. Fix S, an infinite set of non-zero elements of K.

Assume that e is a real loop and $r(e) = v = s(e)$. Then $\psi(kv + e) = \psi(e) \in \cap_{k\in S}(kv + e)L_K(\Gamma) = \{0\}$ by Claim 5. This settles the case $n = 1$. By the induction hypothesis, we may assume that for $\pi' = e_{\ell-1}e_{\ell-2} \cdots e_2 e_1$ we have $\psi(\pi') = 0$, where ℓ is some natural number such that $2 \le \ell \le n$. For easier notations, set $e = e_\ell$. Firstly, assume that e is not a loop. Then $\psi(\pi' + e\pi') = \psi(e\pi') \in (\pi' + e\pi')L_K(\Gamma) \cap e\pi' L_K(\Gamma) = (v + e)(\pi' L_K(\Gamma)) \cap e(\pi' L_K(\Gamma)) \subseteq (v + e)L_K(\Gamma) \cap eL_K(\Gamma) = \{0\}$ by Claim 2. Thus $\psi(e\pi) = 0$ as well. On the other hand, if e is a real loop, then $\psi(k\pi' + e\pi') = \psi(e\pi') \in \cap_{k\in S}(k\pi' + e\pi')L_K(\Gamma) = \cap_{k\in S}(kv + e)\pi' L_K(\Gamma) \subseteq \cap_{k\in S}(kv + e)L_K(\Gamma) = \{0\}$ by Claim 5 (a). This shows that $\psi(e\pi') = 0$ and eventually $\psi(\pi) = 0$.

Now assume that $e_1 = e^*$ is a ghost edge. We use the notations in Claim 5 (b). Note that we know already that $\psi(h_n) = 0$ since $h_n \in K[e]$ and e is a real loop. This implies that $\psi(e^*) = \psi(h_n + e^*) \in \cap_{i=1}^{\infty}(h_i + e^*)L_K(\Gamma) = \{0\}$ by Claim 5 (b). Thus $\psi(e^*) = 0$ and we can induct just as in the case of a real loop to obtain that $\psi(\pi) = 0$. $\square$

Remark 1. Let R be a subalgebra of the algebra of all linear transformations of some K-vector space V. Let $\mathcal{F}$ be the family of all finite subsets of V. For any $F \in \mathcal{F}$, define $\mathcal{O}_F = \{\varphi \in R : \varphi(x) = 0$ for all $x \in F\}$. We write the composition of the maps in $\mathrm{End}_K(V)$ such that the sets $\mathcal{O}_F$ are all right ideals of R. The set $\{\mathcal{O}_F : F \in \mathcal{F}\}$ is a basis of the so-called finite topology on R, which turns R into a Hausdorff topological ring. The completion $\overline{R}$ of R is again a ring of linear transformations of V. Now let A be a K-algebra and $R = \widehat{A} = \{\varphi \in \mathrm{End}_K(A) : \varphi(X) \subseteq X$ for all right ideals X of $A\}$ equipped with the finite topology. Let $\psi \in \overline{R} = \widehat{\widehat{A}}$, be the completion of $\widehat{A}$. Let X be a right ideal of A and $x \in X$. Then there exists some $\varphi \in \widehat{A}$ such that $\psi(x) = \varphi(x) \in X$. This shows $\psi \in \widehat{A}$ and we have that $\widehat{A}$ is closed in the finite topology. We define $\cdot A = \{\cdot a : a \in A\}$ where $\cdot a : A \to A$ is the map defined by $\cdot a(x) = xa$ for all $x \in X$. Note that $\cdot A \subseteq \widehat{A}$.

The ring $\widehat{L_K(\Gamma)}$ equipped with the finite topology becomes a topological ring. We define $\mathcal{O}_H = \{\varphi \in \widehat{L_K(\Gamma)} : \varphi(v) = 0$ for all $v \in H\}$ for any finite set H of vertices of Γ. If $\varphi \in \mathcal{O}_H$, then Claim 6 implies that $\varphi(x) = 0$ for any $x \in L_K(\Gamma)$ whose paths end in a vertex $v \in H$. This shows that $\{\mathcal{O}_H : H$ a finite set of vertices $\}$ is a basis of the finite topology of $\widehat{L_K(\Gamma)}$. For any finite set H of vertices of Γ we define the "local unit" $v_H = \sum_{v\in H} v$. Let $\psi \in \widehat{L_K(\Gamma)}$. Then $\psi(v_H) = v_H t_H$ for some $t_H \in L_K(\Gamma)$. Moreover, for $v \in H$, we have $\psi(v) = vs_v$ for some $s_v \in L_K(\Gamma)$. This shows that $vt_H = vs_v$ for all $v \in H$. Define $\varphi_H = \psi - \cdot t_H \in \widehat{L_K(\Gamma)}$ and it follows that $\varphi_H \in \mathcal{O}_H$. This shows that $\cdot L_K(\Gamma)$ is dense in $\widehat{L_K(\Gamma)}$. Moreover, ψ is the limit of the Cauchy sequence $\{t_H\}_H$. Note that $id_{L_K(\Gamma)} = \lim\{v_H\}_H$. We have proved most of the following

Theorem 10. *Let K be an infinite field and $\Gamma = (V, E)$ a row finite, oriented graph. Then $\widehat{L_K(\Gamma)}$, with the finite topology, is a complete topological ring such that $\{\mathcal{O}_H : H$ a finite subset of $V\}$ is a basis of that topology and $\cdot L_K(\Gamma)$ is a dense subring of $\widehat{L_K(\Gamma)}$, which is a Zassenhaus algebra. Moreover, if Γ is finite, then $\widehat{L_K(\Gamma)} = L_K(\Gamma)$.*

Proof. Now we insist on our convention that maps are written on the right. It is easy to see that $\widehat{L_K(\Gamma)}$ is a Zassenhaus algebra. Let $r \in L_K(\Gamma)$ and H be the finite set of vertices in which all the paths (monomials) representing r end. Then we have that $(\cdot r)\widehat{L_K(\Gamma)} = (\cdot r)(L_K(\Gamma) + \mathcal{O}_H) = (\cdot r)L_K(\Gamma)$ by Claim 6, and $\cdot r$ is contained in

$(\cdot r)L_K(\Gamma)$. Let $\eta \in \widehat{\widehat{L_K(\Gamma)}}$. Then $\eta((\cdot r)\widehat{L_K(\Gamma)}) = \eta((\cdot r)L_K(\Gamma)) \subseteq (\cdot r)\widehat{L_K(\Gamma)} = (\cdot r)L_K(\Gamma)$, and it follows that $\eta(\cdot L_K(\Gamma)) \subseteq \cdot L_K(\Gamma)$ and moreover $\eta \restriction_{L_K(\Gamma)} \in \widehat{L_K(\Gamma)}$ and is continuous in the finite topology because the $\mathcal{O}_H$ are right ideals of $\widehat{L_K(\Gamma)}$. By the density of $\cdot L_K(\Gamma)$ in $\widehat{L_K(\Gamma)}$ we get that $\eta \in \widehat{L_K(\Gamma)}$. We have that every $\psi \in \widehat{L_K(\Gamma)}$ is determined by its action on V, the set of vertices of Γ. Moreover, for any $a \in L_K(\Gamma)$, the map $\cdot a$ has the property that $(v)(\cdot a) = va = 0$ for all but finitely many $v \in V$. $\square$

Remark 2. It is not hard to see that $L_K(\Gamma) = \oplus_{v\in V} vL_K(\Gamma)$ as right $L_K(\Gamma)$-module and $\widehat{L_K(\Gamma)} = \prod_{v\in V} vL_K(\Gamma)$ is the unrestricted Cartesian product of the K-vectorspaces $vL_K(\Gamma)$. We write $(va_v)_{v\in V} = \sum_{v\in V}\sum va_v$ for any $a_v \in L_K(\Gamma)$. Let $x \in L_K(\Gamma)$. Then $x(va_v)_{v\in V} = x\sum_{v\in V} va_v = \sum_{v\in V} xva_v \in xL_K(\Gamma)$ is a finite sum since the paths used in the presentation of $x \in L_K(\Gamma)$ end only in a finite set of vertices. Moreover, for $\sum_{v\in V} va_v, \sum_{\mu\in V} \mu b_v \in \widehat{L_K(\Gamma)}$ we have $(\sum_{v\in V} va_v)(\sum_{\mu\in V} \mu a_\mu) = \sum_{v\in V} v(\sum_{\mu\in V} a_v\mu b_\mu)$. The latter sum is finite. This sheds some light on the structure of $\widehat{L_K(\Gamma)}$. Note that $\sum_{v\in V} v \in \widehat{L_K(\Gamma)}$ is the identity of $\widehat{L_K(\Gamma)}$.

REFERENCES

1. G. Abrams and G.A. Pino, *The Leavitt path algebra of a graph*, J. Algebra **293** (2005), 319–334.

2. G. Abrams, G.A. Pino and M.S. Molina, *Finite dimensional Leavitt path algebras*, J. Pure Appl. Algebra **209** (2007), 753–762.

3. R. Brauer and E. Weiss, *Non-commutative rings*, Part I, Harvard Press, Cambridge, 1950.

4. J. Buckner and M. Dugas, *Quasi-localizations of Z*, Israel J. Math. **160** (2007), 349–370.

5. ———, *Left rigid rings*, J. Algebra **309** (2007), 192–206.

6. ———, *Group algebras with Zassenhaus families of right ideals*, Houston J. Math. **35** (2009), 691–703.

7. M.C.R. Butler, *On locally free torsion-free rings of finite rank*, J. London Math. Soc. **43** (1968), 202–216.

8. M. Dugas and R. Göbel, *An extension of Zassenhaus' theorem on endomorphism rings*, Fund. Math. **194** (2007), 239–251.

9. R.S. Pierce, *Associative algebras*, Grad. Texts Math., Springer-Verlag, New York, 1982.

10. H. Zassenhaus, *Orders as endomorphism rings of modules of the same rank*, J. London Math. Soc. **42** (1967), 180–182.

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