ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 4, 2011

WEYL GROUPS, DESCENT SYSTEMS AND BETTI NUMBERS

LEX E. RENNER

ABSTRACT. Let W be a Weyl group, and consider $W \subseteq Gl(V)$ acting in the usual way as a crystallographic reflection group on the rational vector space V. Associated with $\lambda \in V$ is a certain toric projective variety $X(J)$ with W-action. It turns out that $X(J)$ depends only upon the isotropy group W_J of λ. $X(J)$ is closely related to a certain family of *J-irreducible* reductive monoids. Using what is known about these monoids we assess the following issues. (a) The T-orbit structure of $X(J)$. (b) The exact combinatorial conditions on $\lambda \in V$ so that $X(J)$ is rationally smooth. (c) The Betti numbers of $X(J)$ in terms of a certain *augmented poset* $(W^J, \leq, \{\nu_s\})$.

0. Introduction. Let (W, S) be a finite Weyl group, and let $w \in W$. It is widely appreciated that the descent set

$$D(w) = \{s \in S \mid l(ws) < l(w)\}$$

determines an important chapter in the study of Coxeter groups. In [**13**] the author generalized some of these results to the situation where we replace W by $W^J = \{w \in W \mid l(ws) > l(w) \text{ for all } s \in J\}$. Here J is any proper subset of S. Associated with J is a certain torus embedding $X(J)$. We would like to calculate the Betti numbers of $X(J)$. The main point here is to find the proper generalization $S^J \subset W^J$ of the subset $S \subset W$. The resulting *descent system* (W^J, S^J) encodes all the relevant information about ascent and descent in W^J. One of the interesting outcomes of [**13**] is the determination of all subsets $J \subset S$ of S such that $X(J)$ is quasi-smooth in the sense of Danilov [**4**].

In this paper we use our results about (W^J, S^J) to study this torus embedding $X(J)$, which is possibly singular. To do this we employ a "monoid BB-theory." This provides a useful combinatorial replacement for the infinitesimal plus-minus method of "nonsingular" BB-theory. It allows us to properly analyze the cell structure of $X(J)$. Our discussion

Received by the editors on March 5, 2008, and in revised form on October 6, 2008.

DOI:10.1216/RMJ-2011-41-4-1329 Copyright ©2011 Rocky Mountain Mathematics Consortium

is guided by the W-structure and the T-structure of $X(J)$, which was determined by Putcha and the author in [**11**]. We use the results of [**11**] to obtain the structure and dimensions of the monoid BB-cells of $X(J)$ in terms of the descent system (W^J, S^J). These cells are well-behaved if $X(J)$ is rationally smooth. From there we can write down the Poincaré polynomial of $X(J)$ in terms of (W^J, S^J). We illustrate our method with several examples. The first one is (W, S) of type A_n, where $J = \{s_3, \dots, s_n\} \subset \{s_1, \dots, s_n\} = S$. The other one is (W, S) of type B_n, where $J = \{s_1, \dots, s_{l-1}\} \subset \{s_1, \dots, s_l\} = S$ and s_l corresponds to the short root.

1. The structure of $X(J)$ via J-irreducible monoids. Let W be a Weyl group, and let $r : W \to Gl(V)$ be the usual reflection representation of W. Along with this goes the *Weyl chamber* $C \subseteq V$ and the corresponding set of *simple reflections* $S \subseteq W$. W is generated by S, and C is a fundamental domain for the action of W on V. See [**7**] for details.

Let $\lambda \in C$. In this section we describe the face lattice F_λ of the rational polytope $P_\lambda = Conv(W \cdot \lambda)$, the convex hull of $W \cdot \lambda$ in V. It turns out that F_λ depends only on $W_\lambda = \{w \in W \mid w(\lambda) = \lambda\} = W_J = \langle s \mid s \in J \rangle$, where $J = \{s \in S \mid s(\lambda) = \lambda\}$. Thus we describe $F_\lambda = F_J$ completely in terms of $J \subseteq S$. Closely associated with these polytopes is a certain class of reductive algebraic monoids. We use what is known about these monoids to calculate F_J in terms of J and the underlying Dynkin diagram of W.

We also construct a certain projective variety $X(J)$ from the polytope P_λ. It turns out that $X(J)$ depends only on J, and not on λ.

We now recall some results first recorded in [**11**]. Throughout the paper we use the language and techniques of linear algebraic monoids. Luckily the main results and constructions have recently been assembled in [**12**]. Let M be an irreducible, normal algebraic monoid with reductive unit group G. We refer to such monoids as *reductive*. The reader can find any unproved statements about reductive monoids in [**12**].

If M is a reductive monoid with unit group G we let $B \subseteq G$ be a Borel subgroup of G and $T \subseteq B$ a maximal torus of G. $\overline{T}$ is the Zariski closure of T in M. $\overline{T}$ is a normal, affine torus embedding. The set of *idempotents* $E(\overline{T})$ of $\overline{T}$ is defined to be

$$E(\overline{T}) = \{e \in \overline{T} \mid e^2 = e\}.$$

There is exactly one idempotent in each T-orbit on $\overline{T}$. The set of these orbits is in one-to-one correspondence with the set of faces of a certain rational polytope. We let $E_1 = E_1(\overline{T}) = \{e \in E(\overline{T}) \mid \dim(Te) = 1\}$.

The $G \times G$-orbits of M are particularly interesting in this paper. Let

$$\Lambda = \{e \in \overline{T} \mid eB = eBe\}$$

be the *cross section lattice* of M relative to T and B. It is a basic fact that

$$M = \bigsqcup_{e \in \Lambda} GeG.$$

See [**10**].

As above we let $S \subseteq W$ be the set of simple reflections of W relative to T and B. We regard S as a graph with edges $\{(s,t) \mid st \neq ts\}$. Thus we may speak of the connected components of any subset of S.

A reductive monoid M with $0 \in M$ is called *J-irreducible* if $M \backslash \{0\}$ has exactly one minimal $G \times G$-orbit.

Theorem 1.1. *Let M be a reductive monoid. The following are equivalent.*

1. *M is J-irreducible.*
2. *There is an irreducible rational representation $\rho : M \to \mathrm{End}(V)$ which is finite as a morphism of algebraic varieties.*
3. *If $\overline{T} \subseteq M$ is the Zariski closure in M of a maximal torus $T \subseteq G$, then the Weyl group W of T acts transitively on the set of minimal nonzero idempotents of $\overline{T}$.*

See Corollary 6.8 of [**10**] and Lemma 7.8 of [**12**].

Notice in particular that one can construct, up to finite morphism, all J-irreducible monoids from irreducible representations of a semisimple group. Indeed, let G_0 be semisimple, and let $\rho : G_0 \to \mathrm{End}(V)$ be an irreducible representation. Define $M_1 \subseteq \mathrm{End}(V)$ to be the Zariski closure of $K^*\rho(G_0)$ where $K^* \subseteq \mathrm{End}(V)$ is the set of homotheties.

Finally, let $M(\rho)$ be the normalization of M_1. Then, according to Theorem 1.1, $M(\rho)$ is J-irreducible.

It turns out that, if M is J-irreducible, there is a unique minimal nonzero idempotent $e \in E(\overline{T})$ such that $eB = eBe$, where B is the given Borel subgroup containing T. If M is J-irreducible we say that M is *J-irreducible of type J* if, for this idempotent e,

$$J = \{s \in S \mid se = es\},$$

where S is the set of simple involutions relative to T and B. The set J can be determined in terms of any irreducible representation satisfying condition 2 of Theorem 1.1. Indeed, let $\lambda \in X(T)_+$ be any highest weight such that $\{s \in S \mid s^*(\lambda) = \lambda\} = J$. Then $M(\rho_\lambda)$ is J-irreducible of type J where ρ_λ is the irreducible representation of G_0 with highest weight λ. Furthermore, any two J-irreducible monoids with a finite morphism between them are of the same type. If e is the above-mentioned minimal idempotent then $L = e(V) \subseteq V$ is the one-dimensional $\rho_\lambda(B)$-stable subspace of V with weight λ. Finally, $P = \{g \in G_0 \mid \rho_\lambda(g)(L) = L\}$ is a parabolic subgroup of G_0 of type J.

We now describe the $G \times G$-orbit structure of a J-irreducible monoid of type $J \subset S$. The following result was first recorded in [**11**].

Theorem 1.2. *Let M be a J-irreducible monoid of type $J \subset S$.*

1. *There is a canonical one-to-one correspondence between the set of $G \times G$-orbits acting on $M\backslash\{0\}$ and the set of W-orbits acting on the set of idempotents of $\overline{T}$. This set is canonically identified with $\Lambda = \{e \in E(\overline{T}) \mid eB = eBe\}$.*

2. *$\Lambda\backslash\{0\} \cong \{I \subseteq S \mid$ no connected component of I is contained entirely in $J\}$ in such a way that e corresponds to $I \subseteq S$ if $I = \{s \in S \mid se = es \neq e\}$.*

3. *If $e \in \Lambda\backslash\{0\}$ corresponds to I, as in 2 above, then $C_W(e) = W_K$ where $K = I \cup \{s \in J \mid st = ts$ for all $t \in I\}$.*

See subsection 7.3 of [**12**] for a systematic discussion of J-irreducible monoids.

Let M be a $\mathcal{J}$-irreducible monoid of type $J \subset S$, and assume that $\rho : M \to End(V)$ is an irreducible representation which is finite as a morphism. Let G_0 be the semisimple part of G with maximal torus $T_0 = G_0 \cap T$, and let $\rho_\lambda = \rho | G_0$, with highest weight $\lambda \in \mathcal{C}$, the rational Weyl chamber of G_0. Then, as above, $J = \{s \in S \mid s^*(\lambda) = \lambda\}$. Define

$$\mathcal{P}_\lambda = Conv\,(W \cdot \lambda)$$

the convex hull of $W \cdot \lambda$ in $X(T_0) \otimes \mathbf{Q}$.

We return to the situation of the beginning of this section, i.e., just reflection groups, no reductive monoids.

Corollary 1.3. *Let W be a Weyl group, and let $r : W \to Gl\,(V)$ be the usual reflection representation of W. Let $\mathcal{C} \subseteq V$ be the rational Weyl chamber, and let $\lambda \in \mathcal{C}$. Assume that $J = \{s \in S \mid s^*(\lambda) = \lambda\}$. Then the set of orbits of W on the face lattice $\mathcal{F}_\lambda$ of $\mathcal{P}_\lambda$ is in one-to-one correspondence with $\{I \subseteq S \mid$ no connected component of I is contained entirely in $J\}$.*

The subset $I \subseteq S$ corresponds to the unique face $F \in \mathcal{F}_\lambda$ with $I = \{s \in S \mid s(F) = F$ and $s|F \neq id\}$ whose relative interior F^0 has nonempty intersection with $\mathcal{C}$.

Let M be a $\mathcal{J}$-irreducible monoid of type $J \subset S$, and let $\overline{T}$ be the closure in M of a maximal torus T of G. By part b) of Theorem 5.4 of [**12**], $\overline{T}$ is a normal variety. We define

$$X(J) = (\overline{T} \backslash \{0\})/K^*.$$

The terminology is justified since $X(J)$ depends only on J and not on M. The set of $\mathcal{J}$-irreducible monoids associated with $X(J)$ can be identified with the set $\mathcal{C}^J = \{\lambda \in \mathcal{C} \mid C_S(\lambda) = J\}$. Torus embeddings are usually defined in terms of "fans," a collection of cones in the dual of $X(T)$. This is an ideal way to discuss cycles, divisors and sheaves on a torus embedding. However, we have nothing to gain by fostering that point of view. Our main purpose here is to uncover combinatorial information about $X(J)$.

We are very interested in the structure of $X(J)$, both geometrically and combinatorially, and the extent to which it is encoded by the descent system (W^J, S^J). Some of the key questions here are, "When is $X(J)$ rationally smooth?," "what are its Betti numbers?" and "how can they be described in terms of (W^J, S^J)?"

The situation $X(\phi)$ has been studied previously by several authors. Procesi [**8**] was the first to define and study this variety. In [**15**] Stanley proves that the h-vector of any simplicial, convex polytope is a symmetric, unimodal sequence. Stembridge [**16**] proves that the canonical representation of W on $H^*(X(\phi); \mathbf{Q})$ is a permutation representation and, with the help of Dolgachev-Lunts [**5**], he computes this representation. In [**3**] Brenti studies these descent polynomials (i.e., the Poincaré polynomials of $X(\phi)$) as analogues of the *Eulerian polynomials*. He also looks at the q-analogues of these polynomials.

2. Monoid BB-decompositions. BB-cells are often described in terms of the fixed point set along with the plus-minus decomposition of the tangent space at a fixed point. In many situations this is adequate. But in our situation, the infinitesimal method of [**1**] does not yield the desired information since we are dealing with singular varieties. The cells themselves are well-behaved topologically but the tangent space is not. Thus we are led to quantify the cells of $X(J)$ in terms of idempotents, $B \times B$-orbits and other natural monoid notions. We first assemble some useful information about $\mathcal{J}$-irreducible monoids.

If M is a reductive monoid, then there is a perfect analogue of the Bruhat decomposition. See Chapter 8 of [**12**]. If $R = \overline{N_G(T)}$ (Zariski closure) then, for any $x \in R$, $xT = Tx$. Thus we let

$$\mathcal{R} = R/T = R \setminus T.$$

$\mathcal{R}$ is a finite, inverse monoid with the agreeable property that

$$M = \bigsqcup_{r \in \mathcal{R}} BrB.$$

See Theorem 8.8 of [**12**].

Proposition 2.1. *Let M be $\mathcal{J}$-irreducible, and let $f \in E(\overline{T})$. Then there is a unique $e \in E_1(\overline{T})$ such that $eBf = eBe$. In particular, $ef = e$.*

Proof. By Proposition 6.27 of [**10**], fMf is a $\mathcal{J}$-irreducible monoid with identity element f and unit group H_f, while $fBf \subseteq H_f$ is a Borel subgroup of H_f with maximal torus $fT = fTf$. Since fMf is $\mathcal{J}$-irreducible, there is a unique $e \in E(\overline{fT})$ such that $efBf \subseteq fBfe$. But this implies that $eBf = eBe$ and conversely. See [**11**] for more information about $\mathcal{J}$-irreducible monoids. □

Let $\mathcal{R}^{\times} = \mathcal{R} \setminus \{0\}$.

Corollary 2.2. *For any $r \in \mathcal{R}^{\times}$ there is a unique $e \in E_1(\overline{R})$ such that $eBr = eBer \neq 0$.*

Proof. Write $r = f\sigma$ where $\sigma \in W$ and $f \in E(\overline{T})$. □

Corollary 2.3. *If $x \in M$ and $x \neq 0$, then there is a unique $e \in E_1$ such that $eBx = eBex \neq 0$.*

Proof. This follows directly from Proposition 2.1 and Corollary 2.2. □

For the remainder of this section we describe the BB-decomposition of $X(J)$ for an appropriate one-parameter subgroup of T. As we have pointed out, $X(J)$ is closely related to a certain family of $\mathcal{J}$-irreducible reductive monoids $\{M_\lambda \mid \lambda \in \mathcal{C}^J\}$, where $\mathcal{C}^J = \{\lambda \in \mathcal{C} \mid C_S(\lambda) = J\}$. We obtain a useful description of the decomposition of $X(J)$ by analyzing the corresponding decomposition of M. This leads to the important "augmented" poset structure of W^J that we shall discuss in the next section.

We first prove a technical lemma that allows us to describe, in some cases, a BB-decomposition in terms of the rank-one idempotents of a certain D-monoid $\overline{T}$.

Let X be a normal, projective variety and assume that $S = K^*$ acts on X. If $F_i \subset X^S$ is a connected component of the fixed point set X^S we define, following [**1**],

$$X_i = \{x \in X \mid \lim_{t \to 0}(t \cdot x) \in F_i\}.$$

This decomposes X as a disjoint union

$$X = \bigsqcup_i X_i$$

of locally closed subsets. Furthermore we have the BB-maps

$$\pi_i : X_i \longrightarrow F_i$$

defined by $\pi_i(x) = \lim_{t\to 0}(t \cdot x)$. See [**1**] for more details. In that paper the author assumes that X is nonsingular. Then he proves his much-celebrated results (see Theorem 4.3 of [**1**]).

However, many of his ideas can be extended to the nonsingular case. On the other hand, the purpose of our discussion is to describe this BB-*decomposition* in terms of the system of idempotents of an appropriate algebraic monoid.

Lemma 2.4. *Let $X \subseteq \mathbf{P}^N$ be a closed irreducible subvariety with affine cone $Y \subseteq K^{N+1}$. Let $T \subseteq Gl_{N+1}(K)$ be a torus containing the group of invertible scalar matrices $Z \subseteq T$. Assume that T acts on X. Let M be the closure of T in* $\mathrm{End}\,(K[Y])$. *For $e \in E_1 = E_1(M)$ let $eY \subseteq Y$ be the closed subset defined by*

$$eY = \{y \in Y \mid e(y) = y\}.$$

Let $X^T \subseteq X$ be the set of fixed points for the action of T on X. Assume that

$$X^T = \bigsqcup_{e\in E_1} eX$$

where $eX = (eY \backslash \{0\})/Z \subseteq X$.

Let $[x] \in X$, and let $V = \overline{T \cdot [x]} \subset X$ with cone $\widehat{V} = \overline{T \cdot x} \subseteq Y$. Define

$$M_V = \overline{\{t|\widehat{V} \mid t \in T\}} \subseteq \mathrm{End}\,(\widehat{V}).$$

Then $\theta : M \to M_V$, defined by $\theta(f) = f \mid \widehat{V}$, induces a surjection

$$E_1(M) \longrightarrow E_1(M_V)$$

such that $\theta(e) \neq 0$ if and only if $\widehat{V} \cap eY \neq 0$.

Proof. If $f \in E_1(M_V)$, we then let $fV = (f\widehat{V} \setminus \{0\})/Z \subseteq V$. By Proposition 3.22 of [**12**], $\dim(fM_V) = 1$ since M_V is a D-monoid. But M_V is identified bijectively with $\widehat{V}$ via the morphism

$$\varphi : M_V \longrightarrow \widehat{V},$$

defined by $\varphi(y) = yx$. Furthermore, $\varphi(zy) = z(\varphi(y))$ for any $y, z \in M_V$. Thus

$$1 = \dim(fM_V) = \dim(\varphi(fM_V)) = \dim(f\varphi(M_V)) = \dim(f\widehat{V}).$$

But $f\widehat{V}$ is also closed in $\widehat{V}$, irreducible and Z-stable. Hence $f\widehat{V} = Zv \cup \{0\}$ for some $v \in f\widehat{V}$. We conclude that

$$fV = \{[v]\},$$

a singleton. Furthermore, $fV \subseteq X^T$, so that (by assumption) $v \in eY \setminus \{0\}$, for some $e \in E_1M$. Hence we consider $g = \theta(e) = e \mid \widehat{V} \in M_V$. Then $g \neq 0$ since $g(v) = e(v) = v \neq 0$.

Now $\theta : M \to M_V$ is dominant so $\theta(E_1(M)) \subseteq E_1(M_V) \cup \{0\}$. (Proof: if $\theta(e) \neq 0$ then $\theta(e) \in E(M_V) \setminus \{0\}$, so that $1 \leq \dim(\theta(e)\theta(T)) \leq \dim(eT) = 1$. Thus $\theta(e) \in E_1(M)$ by Proposition 3.22 of [**12**]). Thus we now have $v = f(v) = g(v)$, where $f, g \in E_1(M_V)$. But then $f(g(v)) = f(v) = v \neq 0$, so that $fg \neq 0$ in M_V. But also $f, g \in M_V$ are minimal nonzero idempotents where the partial ordering on idempotents is defined by

$$e_1 \leq e_2 \quad \text{if } e_1e_2 = e_1.$$

Thus $f = g$. But, by definition, $g = \theta(e)$. Hence $f = g = \theta(e)$.

Finally, notice that: if $e\widehat{V} \neq 0$, then $\theta(e) \neq 0$; and if $e\widehat{V} = 0$, then $\theta(e) = 0$. But also $e\widehat{V} = 0$ if and only if $\widehat{V} \cap eY = 0$ (since $e\widehat{V} = \widehat{V} \cap eY$). $\square$

Lemma 2.5. *Let $X, Y, M, V, \widehat{V}, Z$, etc., be as in Lemma* 2.4. *Let $\varphi : K^* \to T$ be such that $X^T = \{x \in X \mid \varphi(t)[x] = [x]$ for all $t \in K^*\}$, and let $X(e) = \{x \in X \mid \lim_{t\to 0}(\varphi_t([x])) \in eX\}$. Then the BB-map $\pi_e : X(e) \to eX$, $\pi_e([x]) = \lim_{t\to 0}(\varphi_t([x]))$, is determined by $\pi_e([x]) = [e(x)]$.*

Proof. Let $[x] \in X(e)$, and let $\lim_{t\to 0}(\varphi_t([x])) = [x]_0 \in V^T \cap eX$. Thus, by Lemma 2.4, $e \mid \widehat{V} \neq 0$. But then $ex \neq 0$ (since otherwise $etx = tex = 0$ for all $t \in T$, and thus $e \mid Tx = 0$ so that $e \mid \widehat{V} = 0$, since $Tx \subseteq \widehat{V}$ is dense, a contradiction).

But $e\widehat{V} = Ze(x) \cup \{0\}$, and so finally we obtain that

$$\lim_{t\to 0}(\varphi_t([x])) \in V^T \cap eX = \{[e(x)]\}.$$

This completes the proof. □

Lemma 2.6. *Let M be a $\mathcal{J}$-irreducible monoid with unit group G and maximal torus $T \subseteq G$. Let $Z \subseteq G$ be the connected component of the center of G. Then*

$$\{x \in M\backslash\{0\} \mid Zx = Tx\} = \bigcup_{e\in E_1(\overline{T})} eM.$$

Consequently, if $X = (M\backslash\{0\})/K^*$ and $eX = (eM\backslash\{0\})/K^*$, then

$$X^T = \bigsqcup_{e\in E_1} eX$$

for the action $T \times X \to X$ given by $(t,[x]) \rightsquigarrow [tx]$.

Proof. Let $x \in M \setminus \{0\}$ be such that $Zx = Tx$. Since $x \neq 0$ there is (by Corollary 2.3) a unique $e \in E_1$ such that $eBx = eBex \neq 0$. In particular $ex \neq 0$. By the Bruhat decomposition we can write $x = brb'$ where $b, b' \in B$ and $r \in \mathcal{R}$. Then we let $y = xb'^{-1} = br$. Write $r = fw$ where $f \in E(\overline{T})$ and $w \in W$. Then $fy = fbr = fbfr = fcr = fcw$ for some $c \in C_B(f)$. In particular $fy \in fG$. Thus, by Proposition 3.22 of [**12**], if $f \notin E_1$ then $\dim(Tfy) > 1$. Thus $Zfy \subsetneq Tfy$. Thus $Zy \subsetneq Ty$ since $\dim(Ty) \geq \dim(Tfy)$. This is impossible. We conclude that $f = e \in E_1$. Thus, if $t \in T$ and $tbe = be$, then $tebe = etbe = ebe$. In particular $te = e$. But $\dim\{t \in T \mid tbe = be\} = \dim\{t \in T \mid te = e\} = \dim T - 1$. In particular $T_e \subseteq \{t \in T \mid tbe = be\}$, and consequently $e \in \{t \in \overline{T} \mid tbe = be\}$. Thus $ebe = be$. Therefore $y \in eM$, and finally $x = yb' \in eM$. □

Theorem 2.7. *Let M be a $\mathcal{J}$-irreducible monoid with unit group G, connected center $Z \subseteq G$, Borel subgroup $B \subseteq G$ and maximal torus $T \subseteq B$. Let $B_u \subset B$ be the subgroup of unipotent elements of B. Choose a one-parameter subgroup $\lambda : K^* \to T$ such that*

1. $\lim_{t\to 0}(tut^{-1}) = 1$ *for all* $u \in B_u$.
2. $\{x \in M\backslash\{0\} \mid \lambda(t)x \in Zx \text{ for all } t \in K^*\} = \cup_{e\in E_1(\overline{T})} eM$.

Let $X = (M\backslash\{0\})/K^*$, and let

$$X = \bigsqcup_{e\in E_1} X(e)$$

be the BB-decomposition of X relative to λ so that $X(e) = \{x \in X \mid \lim_{t\to 0}[\lambda_t(x)] \in eX\}$.

Then, for all $e \in E_1(\overline{T})$,

$$X(e) = \{[y] \in X \mid eBy = eBey \subseteq eG\}.$$

Proof. Let $Y(e) = \{y \in M \setminus \{0\} \mid [y] \in X(e)\}$, and let $[y] \in X(e)$. Notice that $ey \neq 0$. Now, by definition, $[y]_0 = \lim_{t\to 0}(\lambda(t)[y]) \in eX$. But, by Lemma 2.5, $[y]_0 = [ey]$. By our assumptions on λ we obtain that, for all $u \in B_u$, $[y]_0 = [uy]_0$. Hence $[ey] = [euy]$. Therefore we obtain that $eBy = Zey = eBey$ and thus $X(e) \subseteq X_e = \{[y] \in X \mid eBy = eBey \subseteq eG\}$, since, by Corollary 2.2, we have the unique $e \in E_1$ with $eBy = eBey \neq 0$. Notice also that if $ey \neq 0$ then $ey \in eG$, since e is a minimal, nonzero idempotent.

Let $Y_e = \{y \in M \setminus \{0\} \mid eBy = eBey \subseteq eG\}$. Then $M\backslash\{0\}$ is the disjoint union of the Y_e as well as of the $Y(e)$ while, from above, $X(e) \subseteq X_e$. Thus, for each $e \in E_1$, $Y_e = Y(e)$. The conclusion follows. □

Remark 2.8. It is interesting to notice that the cells are independent of λ.

We now look at things on the level of the torus embedding $X(J)$.

As above, let M be $\mathcal{J}$-irreducible of type $J \subseteq S$ with connected center $Z \subseteq M$, and let $E_1 = E_1(\overline{T})$. Choose a one-parameter subgroup $\lambda : K^* \to T$ such that

1. $\lim_{t\to 0}(tut^{-1}) = 1$ for all $u \in B_u$.
2. $\{x \in \overline{T}\backslash\{0\} \mid \lambda(t)x \in Zx$ for all $t \in K^*\} = \cup_{e\in E_1(\overline{T})} eT$.

Recall that

$$X(J) = (\overline{T}\backslash\{0\})/K^*$$

and write

$$X(J) = \bigsqcup_{e\in E_1} X(J)(e)$$

as the BB-decomposition of $X(J)$ relative to λ.

Definition 2.9. Let $e, e'| \in E_1(\overline{T})$. We say that $e < e'$ if $eBe' \neq 0$ and $e \neq e'$.

By the results of [**13**] the poset $(E_1, \leq)$ is anti-isomorphic to the poset $(W^J, \leq)$. This allows us to obtain control of the BB-cell decomposition of $X(J)$ in terms of the idempotents of $\overline{T}$.

Definition 2.10. For $e \in E_1(\overline{T})$ we let

$$\mathcal{X}_e = \{f \in E(\overline{T}) \mid ef = e \text{ and } e'f = 0 \text{ for all } e' > e\}.$$

By Theorem 2.5 of [**13**],

$$E(\overline{T})\backslash\{0\} = \bigsqcup_{e\in E_1(\overline{T})} \mathcal{X}_e$$

and $X(J) = \sqcup_{e\in E_1(\overline{T})} X(J)(e)$ where $X(J)(e) = \sqcup_{f\in\mathcal{X}_e} T[f]$.

Theorem 2.11. *The following are equivalent.*

1. $[f] \in X(J)(e)$.
2. $fe' = 0$ *for all* $e' > e$ *and* $fe = e$.

Proof. This follows from Theorem 2.7 above and Theorem 2.5 of [**13**]. □

3. Rationally smooth Torus embeddings. In this section we describe, in terms of $J \subseteq S$, when $X(J)$ is a rationally smooth torus embedding.

Definition 3.1. We refer to $J \subset S$ as *combinatorially smooth* if $X(J)$ is a rationally smooth torus embedding.

The following theorem indicates exactly how to detect this very interesting condition.

Theorem 3.2. *Let $\lambda \in \mathcal{C}$, and let $J = \{s \in S \mid s^*(\lambda) = \lambda\}$. The following are equivalent.*

1. *$J \subset S$ is combinatorially smooth.*
2. *$\mathcal{P}_\lambda$ is a simple polytope.*
3. *There are exactly $|S|$ edges of $\mathcal{P}_\lambda$ meeting at λ.*
4. *$J = \{s \in S \mid s(\lambda) = \lambda\}$ has the properties*

(a) *If $s \in S\backslash J$, and $J \not\subseteq C_W(s)$, then there is a unique $t \in J$ such that $st \neq ts$. If $C \in \pi_0(J)$ is the unique connected component of J with $t \in C$, then $C\backslash\{t\} \subseteq C$ is a setup of type $A_{l-1} \subseteq A_l$.*

(b) *For each $C \in \pi_0(J)$ there is a unique $s \in S\backslash J$ such that $st \neq ts$ for some $t \in C$.*

The reader is directed to Theorem 3.2 of [**13**] for the proof. Using Theorem 3.2 one can obtain a complete list of the different possibilities for each Weyl group (W, S). See Corollary 3.5 of [**13**]. For example, if (W, S) is a Weyl group of type E_8, there are 22 different, combinatorially smooth subsets $J \subseteq S$.

Assume now that Y is a rationally smooth, projective torus embedding for the action $T \times Y \to Y$. Let $F \subseteq Y$ be the set of T-fixed points. Choose a generic 1-psg $\lambda : K^* \to T$ so that Y has BB-decomposition

$$Y = \bigsqcup_{\alpha \in F} Y_\alpha.$$

Proposition 3.3. *Let $U_\alpha = \{x \in Y \mid \alpha \in \overline{Tx}\}$. Then Y_α is the closure in U_α of a T-orbit. In particular, Y_α is irreducible.*

Proof. Now T acts on U_α and $\alpha \in U_\alpha$ is the unique fixed point for this action. Since Y is combinatorially smooth, so too is U_α. Thus there exists a finite, dominant, flat T-equivariant morphism

$$p_\alpha : U_\alpha \longrightarrow K^n$$

where $n = \dim(Y)$, and K^n has the usual structure of an affine torus embedding for the n-torus. Thus we may write $p_\alpha(x) = (x_1, \dots, x_n)$ and $\lambda_t(x_1, \dots, x_n) = (t^{a_1}x_1, \dots, t^{a_n}x_n)$. Thus, by definition of the BB-cell,

$$\begin{aligned} Y_\alpha &= \{x \in U_\alpha \mid x_i = 0 \text{ if } a_i < 0\} \\ &= p_\alpha^{-1}(\{(x_1, \dots, x_n) \in K^n \mid x_i = 0 \text{ if } a_i < 0\}). \end{aligned}$$

This is the closure of a T-orbit in U_α since p_α induces a bijection on T-orbits. □

Let X be a normal, affine torus embedding with 0. Thus X is a D-monoid in the sense of Chapter 3 of [**12**]. We phrase our results in terms of D-monoids, idempotents and so on, since this is the most convenient method to obtain the desired information about the cells of $X(J)$. We need a combinatorial substitute for the infinitesimal method that often accompanies the BB-method applied to K^*-actions on nonsingular varieties. We then provide a "dictionary" in Remark 3.7 to help the reader translate between the results about (W^J, S^J) and the results about $X(J)$. If $e \in E_1(X)$ let

$$\mathcal{U}_e = \{x \in X \mid ex \neq 0\}.$$

$\mathcal{U}_e$ is a D-monoid with unit group T and minimal idempotent e. Let $E_2(\mathcal{U}_e) = E_2(X) \cap E(\mathcal{U}_e)$. If $X \setminus \{0\}$ is rationally smooth then, for any $e \in E_1(X)$,

$$|E_2(\mathcal{U}_e)| = \dim(X) - 1.$$

Indeed, this is one characterization of the property "$X \setminus \{0\}$ is rationally smooth." See Theorem 3.2 above for a formal result for the cases of interest in this paper.

Lemma 3.4. *Assume that $X \setminus \{0\}$ is rationally smooth and let $\mathcal{U} \subseteq X$ be an open subset such that*

a) $\mathcal{U}$ *is of the form* $\cup_{e'\in B}\mathcal{U}_{e'}$, *so that* $B = E_1(\mathcal{U}) \subseteq E_1(X)$.

b) $e \in E_1(X)\backslash B$.

Then the following are equivalent.

1. $\mathcal{U}_e\backslash\mathcal{U} = f\mathcal{U}_e$ *for some (unique)* $f \in E(\mathcal{U}_e)$.

2. $\mathcal{U}_e \cap \mathcal{U} = \cup_{g\in A}\mathcal{U}_g$, *where* $A = \{g \in E_2(X) \mid ge = e$ *and* $ge' = e'$ *for some* $e' \in B\}$.

Furthermore, $\dim(f\mathcal{U}_e) = \mid E_2(\mathcal{U}_e)\backslash A| + 1 = \dim(X) - |A|$.

Proof. Assume 1, so that $\mathcal{U}_e\backslash\mathcal{U} = f\mathcal{U}_e$, where $fe = e$. Then $\mathcal{U}_e \cap \mathcal{U} = \mathcal{U}_e\backslash f\mathcal{U}_e$. But

$$\mathcal{U}_e\backslash f\mathcal{U}_e = \bigcup_{g\in N}\mathcal{U}_g$$

where $N = \{g \in E(X) \mid ge = e$ and $gf < g\}$ (since $gf < g$ if and only if $g \not\leq f$). But

$$\bigcup_{g\in N}\mathcal{U}_g = \bigcup_{g\in C}\mathcal{U}_g,$$

where $C = \{g \in E_2(X) \mid ge = e$ and $gf < g\}$ (since, for $g \in E(\mathcal{U})$, $gf < f$ if and only if there exists $g' \in E_2(\mathcal{U}_e)$ such that $g' \leq g$ and $g'f < g$, and notice also that $\mathcal{U}_g \subseteq \mathcal{U}_{g'}$). But

$$\bigcup_{g\in C}\mathcal{U}_g = \bigcup_{g\in A}\mathcal{U}_g$$

where $A = \{g \in E_2(X) \mid ge = e$ and $ge' = e'$ for some $e' \in E_1(\mathcal{U})\}$. This last displayed equality holds since for $g \in E_2(\mathcal{U}_e)$, $ge' = e'$ for some unique $e' \in E_1(X)\backslash\{e\}$. But if also $g \in \mathcal{U} = \cup_{h\in B}\mathcal{U}_h$, then $e' \in \mathcal{U}$.

Conversely, assume 2, so that $\mathcal{U}_e \cap \mathcal{U} = \cup_{g\in A}\mathcal{U}_g$ where $A = \{g \in E_2(X) \mid ge = e$ and $ge' = e'$ for some $e' \in E_1(\mathcal{U})\}$. Then

$$\mathcal{U}_e\backslash(\mathcal{U}\cap\mathcal{U}_e) = \bigcup_{h\in C} h\mathcal{U}_e$$

where $C = \{h \in E(\mathcal{U}_e) \mid gh < g$ for all $g \in A\}$. But one observes that $C = \{h \in E(\mathcal{U}_e) \mid gh = e$ for all $g \in A\}$, since for any $g \in A$ and

$h \in E(\mathcal{U}_e)$ either $hg = g$ or else $hg = e$. But then C has a unique maximal element

$$f = \bigvee_{g \in A^c} g$$

the *join* of all $g \in A^c = E_2(\mathcal{U}_e)\backslash A$. The reason for this is the fact that $E(\mathcal{U}_e)$ is a Boolean algebra with atoms $E_2(\mathcal{U}_e)$. This follows from Theorem 3.2. Indeed, this is another way to characterize the property "$X \setminus \{0\}$ is rationally smooth."

The dimension of $f\mathcal{U}_e$ is one plus the length of a maximal chain in the interval $[e, f] \subseteq E(X)$. This is as stated because $\dim(X) = |E_2(\mathcal{U}_e)| + 1$ and $E(\mathcal{U}_e)$ is a Boolean lattice with $E_2(\mathcal{U}_e)$ as atoms while, for $g \in E_2(\mathcal{U}_e)$, $fg = g$ if and only if $g \notin A$. $\square$

We return to the situation where $X(J)$ comes from a Weyl group (W, S). We assume also that $X(J)$ is rationally smooth. Since the cone X on $X(J)$ is the closure of a maximal torus in some $\mathcal{J}$-irreducible monoid M_λ ($\lambda \in \mathcal{C}^J$) we can apply the results of Section 2 to X and $X(J)$. If $x \in X$ is nonzero we write $[x] \in X(J)$. Let

$$X(J) = \bigsqcup_{e \in E_1} X(J)(e)$$

be a BB-decomposition of $X(J)$ for some 1-psg as in Theorem 2.11.

Recall, from Definition 2.9, the ordering $\leq$ on $E_1(\overline{T})$. This ordering is discussed in detail in Section 2 of [**13**]. It is a major key to the success of the descent structure (W^J, S^J).

Let
$$\widehat{X}(J)(e) = \{y \in \overline{T} \mid [y] \in X(J)(e)\}.$$
Notice that $\widehat{X}(J)(e) \subseteq \mathcal{U}_e$.

Theorem 3.5. *Let $e \in E_1$, and let $\mathcal{U} = \cup_{e' > e} \mathcal{U}_{e'}$. Then*

$$\widehat{X}(J)(e) = \mathcal{U}_e \backslash \mathcal{U} = f\mathcal{U}_e,$$

where $f \in E(X)$ is the unique smallest idempotent with $fh = h$ for all $h \in E_2(\mathcal{U}_e)\backslash A$. In particular, $\dim(\widehat{X}(J)(e)) = |E_2(\mathcal{U}_e)\backslash A| + 1 =$

$|S| - |A| + 1$. *In this case,* $A = \{g \in E_2(X) \mid ge = e$ *and* $ge' = e'$ *for some* $e' > e\}$.

Proof. We first show that $\mathcal{U}_e \subseteq \cup_{e' \geq e} \widehat{X}(J)(e')$. So let $f \in E(\mathcal{U}_e)$. If $fe' = 0$ for all $e' > e$ then $f \in \widehat{X}(J)(e)$. This follows from Theorem 2.11. If $fe' = e'$ for some $e' > e$ choose this e' maximally, so that $fe'' = 0$ for all $e'' > e'$. Hence, $f \in \widehat{X}(J)(e')$. Thus we obtain that

$$\mathcal{U}_e = \bigcup_{f \geq e} Tf \subseteq \bigcup_{e' \geq e} \widehat{X}(J)(e').$$

In particular, $\mathcal{U}_e \backslash \mathcal{U} \subseteq \cup_{e' \geq e} \widehat{X}(J)(e')$. On the other hand, $(\mathcal{U}_e \backslash \mathcal{U}) \cap \widehat{X}(J)(e') = \phi$ for $e' > e$, since $(\mathcal{U}_e \backslash \mathcal{U}) \cap \mathcal{U}_{e'} = \phi$ for $e' > e$. Thus $\mathcal{U}_e \backslash \mathcal{U} \subseteq \widehat{X}(J)(e)$. Conversely, $\widehat{X}(J)(e) \subseteq \mathcal{U}_e \backslash \mathcal{U}$ since $\widehat{X}(J)(e) \cap \mathcal{U} = \phi$ (By Theorem 2.11, if $f \in X(J)(e)$ then $fe' = 0$ for all $e' > e$). We conclude that

$$\widehat{X}(J)(e) = \mathcal{U}_e \backslash \mathcal{U}.$$

But we know from Proposition 3.3 that $\widehat{X}(J)(e)$ is irreducible and (therefore) of the form

$$\widehat{X}(J)(e) = f\mathcal{U}_e$$

where $f \in E(\mathcal{U}_e)$. Hence Lemma 3.4 applies and we obtain

$$f = \bigvee_{g \in A^c} g$$

where $A = \{g \in E_2(X) \mid ge = e$ and $ge' = e'$ for some $e' > e\}$.

The dimension formula follows directly from the corresponding formula in Lemma 3.4 using the fact (from Theorem 3.2) that $|S| = |E_2(\mathcal{U}_e)|$. □

We remind the reader of the ordering $<$ on E_1. See Section 2 of [**13**] for more details.

From Theorem 2.11 the following are equivalent. 1. $[f] \in X(J)(e)$.

2. $fe' = 0$ for all $e' > e$ and $fe = e$.

Recall that, for $e \in E_1(X)$,

$$\Gamma(e) = \{g \in E_2(X) \mid ge = e, \text{ and } ge' = e' \text{ for some } e' < e\}.$$

Notice that

$$\Gamma(e) = E_2(\mathfrak{U}_e) \setminus A$$

where A is as in Theorem 3.5. See Corollary 2.13 and Theorem 2.17 of [**13**].

Theorem 3.6. *Assume that $J \subseteq S$ is combinatorially smooth. For $e \in E_1$ recall that $X(J)(e) = \{[x] \in X(J) |$ such that $ex \neq 0$ and $e'x = 0$ for all $e' > e\}$ and, as above, let $\widehat{X}(J)(e) = \{y \in X \mid [y] \in X(J)(e)\}$. Then*

$$\widehat{X}(J)(e) = \mathfrak{U}_e \backslash \mathfrak{U} = f\mathfrak{U}_e$$

as in Theorem 3.5, *and* $\dim(X(J)(e)) = |\Gamma(e)|$.

Proof. This follows from Theorem 3.5 and Corollary 2.13 of [**13**] since $\dim(X(J)(e)) = \dim(\widehat{X}(J)(e)) - 1 = |E_2(\mathfrak{U}_2) \setminus A| = |\Gamma(e)|$. □

Recall from Definition 2.21 of [**13**] the *augmented poset* $(W^J, \leq, \{\nu_s\})$. By definition, $(W^J, \leq)$ is the usual Bruhat poset (which is canonically isomorphic to the poset $(E_1, \leq)$ defined in Definition 2.9), and

$$\nu_s(w) = |A_s^J(w)|$$

where $A_s^J(w)$ is the *ascent set* associated with $s \in S \backslash J$ and $e = we_1w^{-1}$. See Section 2 of [**13**] for more details. Theorem 3.6 says that, if $J \subset S$ is combinatorially smooth, then we can recover the dimensions of the BB-cells $\{X(J)(e) \subset X(J) \mid e \in E_1\}$ from $(W^J, \leq, \{\nu_s\})$. If $\nu(w) = \sum_s \nu(s)$ then from part 3 of Theorem 2.23 of [**13**] we obtain that

$$\nu(w) = |\Gamma(e)|.$$

Remark 3.7. Table 3.8 provides the reader with a summary-translation between the $X(J)$ jargon and the Bruhat poset jargon. On the one hand it is convenient and useful to grind away with monoids, idempotents and orbits to accumulate useful quantitative information the about cells of $X(J)$, but on the other hand it is useful to take inventory of our progress and state it in terms of $X(J)$ and $(W^J, \leq)$. Let

$$\Lambda^\times = \{I \subset S \mid \text{ no component of } I \text{ is contained in } J\},$$

and for $I \in \Lambda^\times$ let $I^* = I \cup \{t \in J \mid ts = st \text{ for all } s \in I\}$. Let $C_w \subset X(J)$ be the BB-cell of $w \in W^J$.The "picture" here is this. W^J is canonically identified with the set of fixed points $X(J)^T$ of T acting on $X(J)$. The set of one-dimensional T-orbits $\mathcal{O}_1(X(J))$ of $X(J)$ is identified with $\{(u,v) \in W^J \times W^J \mid u < v \text{ and } u^{-1}v \in S^J W_J\}$. If $(u,v) \in W^J \times W^J$ and $u^{-1}v \in S^J W_J$, then either $v < u$ or else $u < v$. The question of whether $v < w$ or $w < v$ is coded in the "descent system" (W^J, S^J). For each $w \in W^J$ the BB-cell C_w is constituted as follows.

$$C_w = \bigsqcup_{A \in \mathcal{O}(w)} A$$

where $\mathcal{O}(w) = \{A \subseteq X(J) \mid A = Tx \text{ for some } x \in X(J),\ w(x_0) \in \overline{A}$ and $v(x_0) \notin \overline{A}$ if $v < w\}$. A T-orbit $A \subseteq X(J)$ is in C_w if and only if any one-dimensional T-orbit of $\overline{A}$ has $w(x_0)$ in its closure. See Theorem 2.11.

$X(J)$ *jargon*	W^J *jargon*
$x_0 \in X(J)^T$	$1 \in W^J$
$x = w(x_0) \in X(J)^T$	$w \in W^J$
The T-orbit $A \subset X$ with $\overline{A} \cap X^T = W_I x_0$	$I \in \Lambda^\times$
The set of T-orbits (on $X(J)$) of dim $= 1$	$\{(u,v) \in W^J \times W^J$ such that $u < v$ and $u^{-1}v \in S^J W_J\}$
The set of T-orbits of dim $= 1$ with $x_0 \in \overline{A}$	$S^J = (W_J(S \setminus J)W_J) \cap W^J$
The set of T-orbits of dim $= 1$ in C_w	$A^J(w) = \{r \in S^J \mid w < wr\}$
The set of T-orbits on $X(T)$	$\{(w,I) \mid I \in \Lambda^\times,\ w < ws \text{ if } s \in I^*\}$

4. The Poincaré polynomial of $X(J)$.

Definition 4.1. Let X be a complex algebraic variety. The *Poincaré polynomial* of X is the polynomial $P(X,t)$ with the signed Betti numbers of X as coefficients.

$$P(X,t) = \sum_{i \geq 0} (-1)^i \dim_{\mathbf{Q}}[H^i(X;\mathbf{Q})]t^i.$$

Assume that $J \subseteq S$ is combinatorially smooth. In this section we describe the Poincaré polynomial of $X(J)$ in terms of the augmented poset $(W^J, \leq, \{\nu_s\})$.

By assumption J is combinatorially smooth. Thus, by the results of [**4**], the Betti numbers of $X(J)$ can be calculated by calculating the h-polynomial. Let

$$f_i = \text{the number of codimension } (i+1) - \text{orbits of } X(J)$$

where $i = -1, 0, \ldots, n-1$. The *h-polynomial* is defined by insisting that

$$\sum_{i=0}^{n} h_i t^{n-i} = \sum_{i=-1}^{n-1} f_i (t-1)^{n-i-1}.$$

Notice, in particular, that $f_{-1} = 1$. A simple calculation yields that

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{k-i} f_{i-1}.$$

By Theorem 10.8, Remark 10.9 and Proposition 12.11 of [**4**], the Poincaré polynomial of $X(J)$ is given by

$$P(X(J), t) = \sum_k h_k t^{2k}.$$

On the other hand we can describe the h-polynomial of $X(J)$ in terms of the augmented poset $(W^J, \leq, \{\nu_s\})$. This is the main point of the entire discussion.

Theorem 4.2. *Assume that $X(J)$ is rationally smooth. Then the Poincaré polynomial of $X(J)$ is*

$$P(X(J), t) = \sum_{w \in W^J} t^{2\nu(w)}.$$

Proof. By Theorem 3.6 there is one monoid BB-cell $X(J)(e)$ for each $e \in E_1$. Furthermore, $\dim X(J)(e) = |\Gamma(e)| = |A^J(w)| = \nu(w)$, where $w \in W^J$ is such that $wew^{-1} = e_1$. But, since $X(J)$ is rationally smooth, each $X(J)(e)$ is a union of T-orbits in such a way that the h-polynomial $h(e)$ of $X(J)(e)$ is given by

$$h(e) = \sum_{A \subset A^J(w)} t^{|A|}$$

A simple calculation yields that $h(e) = t^{\nu(w)}$. But $X(J) = \sqcup_{e \in E_1} X(J)(e)$, and so the h-polynomial of $X(J)$ is given by

$$h(t) = \sum_{e \in E_1} h(e) = \sum_{w \in W^J} t^{\nu(w)}. \qquad \square$$

Example 4.3. Assume that $J = \phi$, and let $X = X(\phi)$. We want to compute

$$P(X,t) = \sum_{e \in E_1} t^{2\nu(e)}$$

in this case. Let $\Lambda = \{e_1\}$. In this case $W \cong E_1$ via $w \rightsquigarrow e_w$ if $e_w = we_1w^{-1}$. By the results of [**13**],

$$E_2 \cong \{(w, ws) \in W \times W \mid s \in S,\ l(w) < l(ws)\}$$

via $g \rightsquigarrow (w, ws)$ if $g = wg_sw^{-1} = wsg_ssw^{-1}$ for some unique $g_s \in \Lambda_2 = \{g_s \mid s \in S\}$ and $w \in W$ with $l(w) < l(ws)$. Thus, with this identification, $\{g \in E_2 \mid ge_w = e_w\} \cong \{(v, vs) \in W \times W \mid s \in S,\ l(v) < l(vs)$ and $w \in \{v, vs\}\}$, and hence

$$\Gamma(e_w) \cong \{s \in S \mid l(w) < l(ws)\}.$$

Thus $\nu(e_w) = |\{s \in S \mid l(w) < l(ws)\}| = |S| - |D(w)|$, where $D(w) = \{s \in S \mid l(w) > l(ws)\}$. We let $d(w) = |D(w)|$. By Poincaré duality $\sum_{e \in E_1} t^{2\nu(e)} = \sum_{w \in W} t^{2d(w)}$. Thus, for convenience, we compute $\sum_{w \in W} t^{2d(w)}$. By Theorem 7.2.1 of [**2**] we have (taking into account the doubling of degrees) that

$$P(X,t) = \sum_{I \subseteq S} t^{2|S \setminus I|}(t^2 - 1)^{|I|}|W^I|$$

where $W^I = \{w \in W \mid D(w) \subseteq S \setminus I\}$. This sum is called the *Eulerian polynomial* of W.

If (W, S) is the Coxeter group of type A_{n-1} then $W \cong S_n$, the symmetric group on n letters. Define $A(n,k) = |\{w \in S_n \mid |D(w)| = k+1\}|$, the Eulerian numbers. Thus, for the associated variety X,

$$P(X,t) = \sum_{k=-1}^{n-2} A(n,k)t^{2(k+1)}.$$

These *Eulerian numbers* $A(n,k)$ are known to satisfy the following recurrence relations.

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k).$$

The first few Eulerian polynomials:

1	1
2	$1+t^2$
3	$1+4t^2+t^4$
4	$1+11t^2+11t^4+t^6$
5	$1+26t^2+66t^4+26t^6+t^8$
6	$1+57t^2+302t^4+302t^6+57t^8+t^{10}$
7	$1+120t^2+1191t^4+2416t^6+1191t^8+120t^{10}+t^{12}$
8	$1+247t^2+4293t^4+15619t^6+15619t^8+4293t^{10}+247t^{12}+t^{14}$
9	$1+502t^2+14608t^4+88234t^6+156190t^8+88234t^{10}+14608t^{12}+502t^{14}+t^{16}$

Similar formulas can be derived for the Coxeter groups of type B and D. See [**3**] for more details.

Example 4.4. In this example we list the Poincaré polynomials associated with combinatorially smooth polyhedra of type A_3. Here $S = \{s_1, s_2, s_3\}$ with $s_1s_2 \neq s_2s_1$ and $s_2s_3 \neq s_3s_2$.

J	Associated Polyhedron	Poincaré Polynomial of $X(J)$
$\{s_1, s_2\}$	tetrahedron	$1+t^2+t^4+t^6$
$\{s_1\}$	truncated tetrahedron	$1+5t^2+5t^4+t^6$
$\{s_2, s_3\}$	tetrahedron	$1+t^2+t^4+t^6$
$\{s_3\}$	truncated tetrahedron	$1+5t^2+5t^4+t^6$
ϕ	permutahedron	$1+11t^2+11t^4+t^6$

Example 4.5. In this example we list the Poincaré polynomials associated with combinatorially smooth polyhedra of type C_3. Here $S = \{s_1, s_2, s_3\}$ with $s_1s_2 \neq s_2s_1$ and $s_2s_3 \neq s_3s_2$. $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ and α_3 is the long simple root.

J	Associated Polyhedron	Poincaré Polynomial of $X(J)$
$\{s_1, s_2\}$	cube	$1 + 3t^2 + 3t^4 + t^6$
$\{s_1\}$	truncated cube	$1 + 11t^2 + 11t^4 + t^6$
$\{s_3\}$	truncated octahedron	$1 + 11t^2 + 11t^4 + t^6$
ϕ	rhombitruncated cuboctahedron	$1 + 23t^2 + 23t^4 + t^6$

Example 4.6. In this example we discuss the Poincaré polynomial of $X(J)$ where $(W, S) = (S_{n+1}, \{s_1, s_2, \ldots, s_n\})$ is the Weyl group of type A_n $(n \geq 2)$ and $J = J_n = \{s_3, s_4, \ldots, s_n\}$. Using Theorem 1.2 we obtain that

$$\begin{aligned}\Lambda \backslash \{0\} \cong & \{e(k, n-k-1) \mid k = 1, \ldots, n\} \\ & \sqcup \{e(k, n-k-2) \mid k = 0, \ldots, n-1\},\end{aligned}$$

where

$$e(i, j) \geq e(k, l) \text{ if } i \geq j \text{ and } i + j \geq k + l.$$

Furthermore, the number of T-orbits on $X(J)$ is as follows.

\# of k-dimensional orbits $= (n-k+1)\binom{n+1}{k+1}$, if $0 < k < n$.

\# of n-dimensional orbits $= 1$.

\# of 0-dimensional orbits $= n(n+1)$.

Thus we might calculate the Poincaré polynomial from the f-polynomial as described at the beginning of this section. The calculation would be somewhat interesting and we would obtain that

$$\begin{aligned}P(X(J), t) = & t^{2n} + (n+2)t^{2(n-1)} + (n+2)t^{2(n-2)} + \cdots \\ & + (n+2)t^4 + (n+2)t^2 + 1.\end{aligned}$$

However, this does not illustrate our new method. We want to illustrate the calculation of $P(X(J), t)$ using the structure of (W^J, S^J). To this end we proceed as follows.

As in the above example, we let $(W_n, S_n) = \langle s_1, s_2, \ldots, s_n \rangle$ $(n \geq 2)$, and let $J_n = \{s_3, s_4, \ldots, s_n\} \subset S_n$. A simple calculation shows that

$$W_n^J = \{(s_p \cdots s_1)(s_q \cdots s_2)\} \cup \{s_p \cdots s_1\} \cup \{s_q \cdots s_2\}$$

where $1 \leq p \leq n$ and $2 \leq p \leq n$. Furthermore, by Theorem 4.2 of [**13**], $S_2^J = \{s_1, s_2\}$ (noting that $J_2 = \phi$), and for $n > 2$,

$$\begin{aligned} S_n^J &= \{s_1, s_2, s_3s_2, s_4s_3s_2, \dots, s_ns_{n-1}\cdots s_3s_2\} \\ &= S_{n-1}^J \cup \{s_ns_{n-1}\cdots s_3s_2\}. \end{aligned}$$

For this example we shall change notation slightly from that of Theorem 4.2 of [**13**]. We simply write S_n^J for $S_{s_2}^J \sqcup S_{s_1}^J$. In all cases we use J instead of J_n, since no confusion results if we think of J as "everything but s_1 and s_2." Notice also that

$$\begin{aligned} W_n^J = W_{n-1}^J &\cup \{(s_n \cdots s_1)(s_p \cdots s_2)\} \cup \{s_n \cdots s_1\} \\ &\cup \{(s_q \cdots s_1)(s_n \cdots s_2)\} \cup \{s_n \cdots s_2\} \end{aligned}$$

where $2 \leq p \leq n$ and $1 \leq q \leq n$. Recall, from Definition 2.19 of [**3**], the descent set $A_n^J(w) = \{r \in S^J \mid w < wr\}$ of $w \in W_n^J$. The point of this example is to calculate $A_n^J(w)$ for each $w \in W_n^J$. Our approach here is inductive. If $w \in W_{n-1}^J$, we construct $A_n^J(w)$ from $A_{n-1}^J(w)$. If $w \in W_n^J \setminus W_{n-1}^J$ we calculate $A_n^J(w)$ directly. The following proposition records the necessary steps.

Proposition 4.7. *Let (W, S) and $J \subset S$ be as above.*

1. *If $w \in W_{n-1}^J$ then $s_n \cdots s_1 \in A_n^J(w)$. Thus $A_n^J(w) = A_{n-1}^J(w) \cup \{s_n \cdots s_2\}$.*
2. $A_n^J(s_n \cdots s_1) = \{s_2, s_3s_2, \dots, s_ns_{n-1}\cdots s_2\}$.
3. $A_n^J(s_n \cdots s_1s_2) = \{s_3s_2, s_4s_3s_2, \dots, s_ns_{n-1}\cdots s_2\}$.
   $A_n^J(s_n \cdots s_1s_3s_2) = \{s_4s_3s_2, s_5s_4s_3s_2, \dots, s_ns_{n-1}\cdots s_2\}$
   $\vdots$
   $A_n^J((s_n \cdots s_1)(s_{n-1}s_{n-2}\cdots s_2)) = \{s_ns_{n-1}\cdots s_2\}$
   $A_n^J((s_n \cdots s_1)(s_ns_{n-1}\cdots s_2)) = \phi$.
4. $A_n^J((s_p \cdots s_1)(s_n \cdots s_2)) = \{s_1\}$ *if* $1 \leq p < n$.
   $A_n^J(s_n \cdots s_2) = \{s_1\}$.

Proof. This is a calculation using part 3 of Theorem 2.23 of [**13**] as our guide. This allows us to calculate $A^J(w)$ by considering the products wr where $w \in W^J$ and $r \in S^J$, and then writing $wr = vc$

where $v \in W^J$ and $c \in W_J = \langle s_3, \ldots, s_n \rangle$. From there it is easy to decide, by inspection, whether $w > v$ or $v > w$. For visual convenience, we write the decomposition $wr = vc$ as $wr = v[c]$.

We first consider 1. Let $w = (s_p s_{p-1} \cdots s_1)(s_q s_{q-1} \cdots s_2)$ where $0 < p < n$ and $1 < q < n$. Then

$$w(s_n s_{n-1} \cdots s_2) = (s_p s_{p-1} \cdots s_1)(s_n s_{n-1} \cdots s_2)[s_{q+1} \cdots s_3].$$

If $w = s_p s_{p-1} \cdots s_1$, then

$$w(s_n s_{n-1} \cdots s_2) = (s_p s_{p-1} \cdots s_1)(s_q s_{q-1} \cdots s_2).$$

If $w = s_q s_{q-1} \cdots s_2$, then

$$w(s_n s_{n-1} \cdots s_2) = (s_n s_{n-1} \cdots s_2)[s_{q+1} s_q \cdots s_3].$$

Thus, in all cases, $s_n \cdots s_2 \in A_n^J(w)$. Therefore 1 holds.

We next consider 2. For $n \geq q \geq p+1 \geq 3$ let $w = (s_n s_{n-1} \cdots s_1)(s_p s_{p-1} \cdots s_2)$. Then

$$w(s_q s_{q-1} \cdots s_2) = (s_n s_{n-1} \cdots s_2)(s_q s_{q-1} \cdots s_2)[s_{p+1} s_p \cdots s_3].$$

Thus $s_q s_{q-1} \cdots s_2 \in A_n^J(w)$. If $n \geq p \geq q$ again let $w = (s_n s_{n-1} \cdots s_1)(s_p s_{p-1} \cdots s_2)$. Then

$$w(s_q s_{q-1} \cdots s_2) = (s_n s_{n-1} \cdots s_1)(s_{q-1} s_{q-2} \cdots s_2)[s_{p+1} s_p \cdots s_3].$$

Thus $s_q s_{q-1} \cdots s_2 \notin A_n^J(w)$ in this case. Finally, one checks that $(s_n \cdots s_1 s_p \cdots s_2)(s_1) = s_{p-1} \cdots s_1 s_n \cdots s_2$. Hence $s_1 \notin A_n^J(w)$ since the length goes down by one. Thus

$$A_n^J(s_n \cdots s_1 s_p \cdots s_2) = \{s_{p+1} \cdots s_2, s_{p+2} \cdots s_2, ..., s_n s_{n-1} \cdots s_2\},$$

and consequently 2 holds.

Lastly we consider 3. If $w = (s_n s_{n-1} \cdots s_1)(s_n s_{n-1} \cdots s_2)$, then

$$ws_1 = (s_{n-1} s_{n-2} \cdots s_1)(s_n s_{n-1} \cdots s_2).$$

Thus $s_1 \notin A_n^J(w)$ in this case. If $p < n$ and $w = (s_p s_{p-1} \cdots s_1)(s_n s_{n-1} \cdots s_2)$, then

$$ws_1 = (s_{p+1} s_p \cdots s_1)(s_n s_{n-1} \cdots s_2).$$

Thus $s_1 \in A_n^J(w)$ in this case. If $q < n$ and $w = (s_p s_{p-1} \cdots s_1)$ $(s_n s_{n-1} \cdots s_2)$, then

$$w(s_q s_{q-1} \cdots s_2) = (s_p s_{p-1} \cdots s_1)(s_{q-1} s_{q-2} \cdots s_2)[s_n s_{n-1} \cdots s_3].$$

Thus $s_q s_{q-1} \cdots s_2 \notin A_n^J(w)$ in this case. We conclude that $A_n^J(w)$ is empty if $w = (s_n s_{n-1} \cdots s_1)(s_n s_{n-1} \cdots s_2)$, and $A_n^J(w) = \{s_1\}$ if $w = (s_p s_{p-1} \cdots s_1)(s_n s_{n-1} \cdots s_2)$ and $p < n$. This completes the proof. □

Corollary 4.8. *Let* $(W_n, S_n) = \langle s_1, s_2, \dots, s_n \rangle$ $(n \geq 2)$, *where* $S_n = \{s_1, s_2, \dots, s_n\}$. *As above, we also let* $J = \{s_3, s_4, \dots, s_n\} \subset S_n$ *and* $X_n(J)$ *be the associated torus embedding. Then*

$$\begin{aligned} P(X_n(J), t) = t^{2n} + (n+2)t^{2(n-1)} + (n+2)t^{2(n-2)} + \cdots \\ + (n+2)t^4 + (n+2)t^2 + 1. \end{aligned}$$

Proof. The case $n = 2$ is left to the reader, and the case $n = 3$ is already included in Example 4.4. By induction, we have that

$$\begin{aligned} P(X_{n-1}(J), t) = t^{2(n-1)} + (n+1)t^{2(n-2)} + (n+1)t^{2(n-3)} + \cdots \\ + (n+1)t^4 + (n+1)t^2 + 1. \end{aligned}$$

To obtain $P(X_n(J), t)$ from $P(X_{n-1}(J), t)$ we multiply $P(X_{n-1}(J), t)$ by t^2 and then add t^{2k} for each $w \in W_n^J \setminus W_{n-1}^J$ with $|A_n^J(w)| = k$, according to the requirements of Proposition 4.7. Consequently, we obtain that

$$P(X_n(J), t) = t^2 P(X_{n-1}(J), t) + (t^{2(n-1)} + \cdots + t^4 + t^2 + 1) + nt^2.$$

The three summands in this expression for $P(X_n(J), t)$ result, in order, from parts 1, 2 and 3 of Proposition 4.7. A simple calculation then yields the desired result. □

Figure 1 provides a visual illustration of how the ascent numbers of W_n^J reproduce recursively. The numbers along the bottom two edges of the diagram correspond to the elements of $W_4^J \setminus W_3^J$ (according to parts 2 and 3 of Proposition 4.7). The other numbers correspond to the elements of $W_3^J \subset W_4^J$ (according to part 1 of Proposition 4.7).

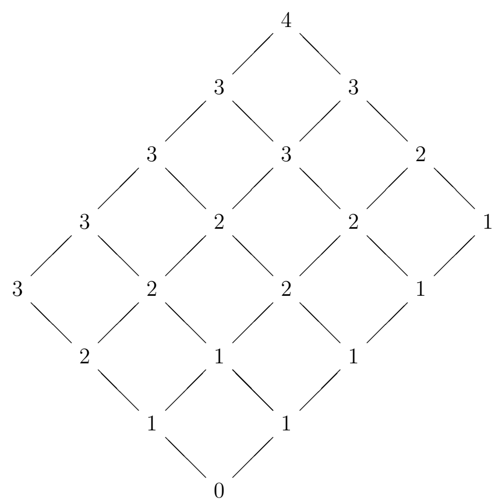

FIGURE 1. The ascent numbers for W_4^J.

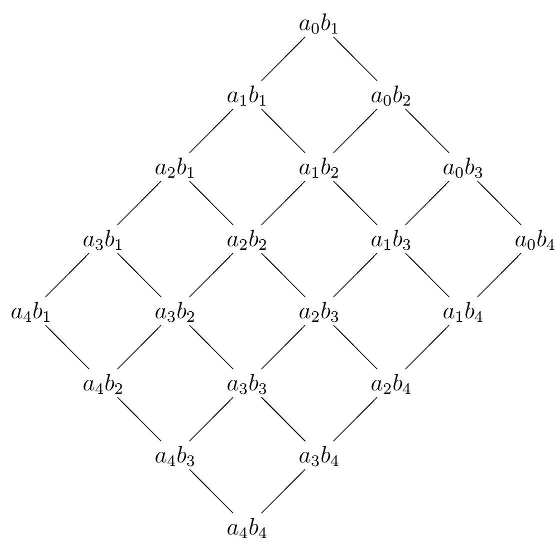

FIGURE 2. W_4^J.

Notice that the diagram depicts only part of the Bruhat order relation on W_4^J. For example, the relation $a_0 b_1 < a_2 b_0$ is not depicted.

Figure 2 indicates the correspondence between the nodes of the diagram and the elements of W_4^J. Recall that $W_4^J = \{a_i b_j \mid i = 0, 1, 2, 3, 4 \text{ and } j = 0, 1, 2, 3\}$, where $a_0 = 1$, $a_1 = s_1$, $a_2 = s_2 s_1$, $a_3 = s_3 s_2 s_1$, $a_4 = s_4 s_3 s_2 s_1$, $b_1 = 1$, $b_2 = s_2$, $b_3 = s_3 s_2$ and $b_4 = s_4 s_3 s_2$, and $S^J = \{s_1, s_2, s_3 s_2, s_4 s_3 s_2\}$.

It is interesting to calculate the augmented poset of W^J in this example. Write

$$A_n^J(w) = A_{s_1}^J(w) \bigsqcup A_{s_2}^J(w)$$

as in Definition 2.19 of [**13**]. If $w \in W^J$ then $w = a_p$, $w = b_q$, or else $w = a_p b_q$. Here $a_p = s_p \cdots s_1$ $(1 \leq p \leq n)$ and $b_q = s_q \cdots s_2$ $(2 \leq q \leq n)$. If we adopt the useful convention $a_0 = 1$ and $b_1 = 1$, then we can write

$$W^J = \{a_p b_q \mid 0 \leq p \leq n \text{ and } 1 \leq q \leq n\}$$

with uniqueness of decomposition. Let $w = a_p b_q \in W^J$. Then

a) $A_{s_1}^J(a_p b_q) = \{s_1\}$ if $p < q$.
$A_{s_1}^J(a_p b_q) = \phi$ if $q \leq p$.
$\nu_{s_1}(a_p b_q) = 1$ if $p < q$ and $\nu_{s_1}(a_p b_q) = 0$ if $q \leq p$.

b) $A_{s_2}^J(a_p b_q) = \{s_{q+1} \cdots s_2, \ldots, s_n \cdots s_2\}$ if $q < n$.
$A_{s_2}^J(a_p b_q) = \phi$ if $q = n$.
$\nu_{s_2}(a_p b_q) = n - q$.

It is interesting to compute the "Euler polynomial"

$$H(t_1, t_2) = \sum_{w \in W^J} t_1^{\nu_1(w)} t_2^{\nu_2(w)}$$

of the augmented poset $(W^J, \leq, \{\nu_1, \nu_2\})$ (we write ν_i for ν_{s_i}). A simple calculation yields

$$H(t_1, t_2) = t_1 t_2^{n-1} + \left(\sum_{k=2}^{n} (k t_2^{k-1} + (n + 2 - k) t_1 t_2^{k-2}) \right) + 1.$$

Example 4.9. In this example we consider the root system of type B_l. Let E be a real vector space with orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_l\}$. Then

$\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\} \cup \{\varepsilon_i + \varepsilon_j \mid i \neq j\} \cup \{\varepsilon_i\}$, and
$\Delta = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\} = \{\alpha_1, \dots, \alpha_l\}$.

Let $S = \{s_1, s_2, \dots, s_{l-1}, s_l\}$ be the corresponding set of simple reflections. Here we consider the case

$$J = \{s_1, \dots, s_{l-1}\}.$$

We first calculate $W^J = \{w \in W \mid w(\alpha_i) \in \Phi^+ \text{ for all } 1 \leq i \leq l-1\}$. This leads to a simple calculation and we obtain

$$W^J \cong \{1 \leq i_1 < i_2 < \cdots < i_k \leq l\},$$

via

$$w(\varepsilon_v) = \varepsilon_{i_v} \text{ for } 1 \leq i \leq k,$$

and

$$w(\varepsilon_{k+v}) = -\varepsilon_{j_v} \text{ for } 1 \leq v \leq l-k,$$

where $l \geq j_1 > j_2 > \cdots > j_{l-k} \geq 1$ (so that $\{1, \dots, l\} = \{i_1, i_2, \dots, i_k\} \sqcup \{j_1, j_2, \dots, j_{l-k}\}$).

Let

$$\lambda = 2\lambda_l = \varepsilon_1 + \cdots + \varepsilon_l = \alpha_1 + 2\alpha_2 \cdots + l\alpha_l.$$

By Proposition 4.1 of [**9**], for $v, w \in W^J$, $w \leq v$ if and only if $w(\lambda) - v(\lambda)$ is a sum of positive roots. A simple calculation yields that

$$w \leq v \text{ if and only if } m_w(i) \leq m_v(i) \text{ for all } i = 1, \dots, l,$$

where

$$m_w(i) = |\{j \leq i \mid w(\varepsilon_v) = -\varepsilon_j \text{ for some } v = 1, \dots, l\}|.$$

Let $M(w) = \{j \mid w(\varepsilon_v) = -\varepsilon_j \text{ for some } v = 1, \dots, l\}$. If $M(w) \subset M(v)$ then also $M(v)^c \subset M(w)^c$ (complement of sets) and we obtain that

$$w(\lambda) - v(\lambda) = A + B$$

where

$$A = \sum_{j \in M(w)^c} (\alpha_j + \alpha_{j+1} + \cdots + \alpha_l) - \sum_{j \in M(v)^c} (\alpha_j + \alpha_{j+1} + \cdots + \alpha_l),$$

and

$$B = \sum_{j \in M(v)} (\alpha_j + \alpha_{j+1} + \cdots + \alpha_l) - \sum_{j \in M(w)} (\alpha_j + \alpha_{j+1} + \cdots + \alpha_l).$$

Thus $M(w) \subset M(v)$ implies that $w \leq v$, at least for elements of W^J.

We now wish to calculate $A^J(w)$ for each $w \in W^J$. Recall that

$$A^J(w) = \{r \in S^J \mid w < wr\}$$

and

$$S^J = \{s_l, s_{l-1}s_l, \dots, s_i s_{i-1} \cdots s_{l-1}s_l, \dots, s_1 \cdots s_l\}.$$

Let $w \in W^J$ correspond, as above, to $i_1 < \cdots < i_k$ and $j_1 > \cdots > j_{l-k}$. Let $r_i = s_i \cdots s_l \in S^J$. One checks that

$$M(wr) = M(w) \cup \{j\} \text{ if } i \leq k,$$

and

$$M(wr) = M(w) \setminus \{j\} \text{ if } i > k.$$

Hence by our previous calculations $w < wr_i$ if and only if $i \leq k$. Thus we obtain

$$A^J(w) = \{s_k \cdots s_l, \dots, s_1 \cdots s_l\} = \{r \in S^J \mid w < wr\}.$$

Thus if $w \in W^J$ we obtain

$$\nu(w) = |\{j \mid w(\varepsilon_v) = \varepsilon_j \text{ for some } v\}|.$$

We can use this information to calculate the Poincaré polynomial of $X(J)$. An easy calculation yields

$$P(X(J), t) = \sum_{w \in W^J} t^{2\nu(w)} = \sum_{A \subset \{1, \dots, l\}} t^{2|A|} = (1 + t^2)^l.$$

REFERENCES

1. A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Annals Math. **98** (1973), 480–497.

2. A. Bjorner and F. Brenti, *Combinatorics of coxeter groups*, Grad. Texts Math. **231**, Springer Verlag, New York, 2005.

3. F. Brenti, *q-Eulerian poynomials arising from Coxeter groups*, European J. Combinatorics **15** (1994), 417–441.

4. V.I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97–154.

5. I. Dolgachev and V. Lunts, *A character formula for the representation of a Weyl group on the cohomology of the associated toric variety*, J. Algebra **168** (1994), 741–772.

6. F.D. Grosshans, *The variety of points which are not semi-stable*, Illinois J. Math. **26** (1982), 138–148.

7. J.E. Humphreys, *Introduction to Lie algebras and representation theory*, third edition, GTM **9**, Springer, New York, 1980.

8. C. Procesi, *The toric variety associated to Weyl chambers*, in *Mots*, M. Lothaire, ed., Hermés, Paris, 1990.

9. R.A. Proctor, *Bruhat lattices, plane partition generating functions and miniscule representations*, European J. Combinatorics **5** (1984), 331–350.

10. M.S. Putcha, *Linear algebraic monoids*, LNM, Cambridge University Press, Cambridge, 1988.

11. M.S. Putcha and L.E. Renner, *The system of idempotents and lattice of* $\mathcal{J}$*-classes of reductive algebraic monoids*, J. Algebra **116** (1988), 385–399.

12. L.E. Renner, *Linear algebraic monoids*, Encycl. Math. Sci. **134**, Springer, New York, 2005.

13. ———, *Descent systems for Bruhat posets*, J. Algebraic Combin. **29** (2009), 413–435.

14. L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), 255–264.

15. R.P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics and geometry*, Annals New York Acad. Sci. **576** (1989), 500–534.

16. J.R. Stembridge, *Some permutation representations of Weyl groups associated with the cohomology of toric varieties*, Adv. Math. **106** (1994), 244–301.

Department of Mathematics, University of Western Ontario, London, N6A 5B7, Canada
Email address: lex@uwo.ca