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A HAMILTONIAN MODEL AND SOLITON PHENOMENON FOR A TWO-MODE KdV EQUATION

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ABSTRACT. We present a Hamiltonian model for a two-mode KdV equation. By using the theory of functional bivector and prolongation of the vector field, the Hamiltonian operator is formally established. We also perform the numerical simulations on the obtained Hamiltonian system with the aim of investigating its soliton behavior.

1. Introduction. In 1994, Korsunsky [**6**] proposed a nonlinear two-mode dispersive wave equation which in scaled form is represented as [**7**]

$$u_{tt} - u_{xx} + (\partial_t - \alpha\partial_x)\, uu_x + (\partial_t - \beta\partial_x)\, u_{xxx} = 0, \tag{1}$$

where $u(x,t)$ is a field function, $-\infty < x,\ t < \infty$, $-1 \le \alpha,\ \beta \le 1$ and we have adopted different scalings from those in [**6**] in which the parameter s can be easily removed to obtain a much more neat scaled two-mode KdV (STMKdV) equation.

The STMKdV equation is assumed to govern the propagation in the same direction of two different wave modes simultaneously; on the other hand, the author in [**6**] has claimed that pure solitons might exist. But no one has ever figured out how to find or observe its solitons phenomenon, either numerically or theoretically. Soliton solution involves four basic ideas: (1) particle-like collisional stability of waves, (2) clean nonlinear interaction between the waves, (3) phase shift after the interaction and (4) conserved quantities such as Hamiltonians. The soliton equation often comes with a surprising Hamiltonian structure, which is now being recognized as an important aspect in soliton theory [**3, 8**]. For example, the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

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can be written as a Hamiltonian system as [**1**]

$$u_t = \frac{\partial}{\partial x}\frac{\delta \mathcal{H}}{\delta u}, \qquad \mathcal{H} = \int_{-\infty}^{\infty}\left(u^3 + \frac{1}{2}u_x^2\right)\mathrm{d}x.$$

Another example is the Boussinesq equation (BE) ([**9**, page 459])

$$u_{tt} = \frac{1}{3}u_{xxxx} + \frac{4}{3}\left(u^2\right)_{xx}, \tag{2}$$

which is a modified form of the general Boussinesq equation

$$\psi_{tt} + a_1\psi_{tt} + a_2\psi^2 + a_3\psi_{xxxx} = 0,$$

in which $\psi = u - 3/8$, $a_1 = -1$, $a_2 = -4/3$, and $a_3 = -1/3$. Accordingly, (2) can be converted into

$$u_t = v_x, \qquad v_t = \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x,$$

and its Hamiltonian system is represented as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \partial_x\cdot \\ \partial_x\cdot & 0 \end{pmatrix}\begin{pmatrix} \delta\mathcal{H}/\delta u \\ \delta\mathcal{H}/\delta v \end{pmatrix},$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & \partial_x\cdot \\ \partial_x\cdot & 0 \end{pmatrix},$$

and

$$\mathcal{H}\left[u, v\right] = \int\left(-\frac{1}{6}u_x^2 + \frac{4}{9}u^3 + \frac{1}{2}v^2\right)\mathrm{d}x.$$

However, the discussion of the Hamiltonian system for the STMKdV equation has been missing and subsequent research on the equation is so scarce that as far as we know no paper has been written to deal with its Hamiltonian formulation. In this article, we are going to present the Hamiltonian model of the STMKdV equation when $\alpha = 0$, which has been overlooked in [**6**]. We also perform small numerical simulations in order to observe the particle-like, nonlinear interaction between waves and phase shifts phenomenon, which has also been neglected in [**6**].

This paper is organized as follows. In Section 2, we briefly introduce the general description of Hamiltonian dynamics for differential equations. We set this general theory up so that in Section 3 we can apply it line by line to our STMKdV equation. Finally, we demonstrate the numerical results of solitary wave interaction by integrating the Hamiltonian system with the help of Fourier spectral method.

2. Hamiltonian dynamics. Hamiltonian systems of differential equations are often defined on a "manifold," a topological space that resembles Euclidean space locally. More precisely, a manifold is a group structure in mathematics carried by a Lie group [**4**], which has additional properties apart from the general group properties [**5**]. A differential manifold is a manifold that allows partial differentiation and all the features of differential calculus on it. We shall not go into the details of the manifold theory itself but only assume that our manifold is a C^∞ class endowed with a group structure in which the multiplication and inversion operations are also C^∞ operations.

Given a differential manifold $\mathcal{M}$, a Poisson bracket on $\mathcal{M}$ assigns each pair of differentiable, real-valued functions $F,\, H : \mathcal{M} \to \mathbf{R}$ to another differentiable, real-valued function on $\mathcal{M}$, which we denote by $\{F,\, H\}$, so as to satisfy the following properties:

1. Skew-symmetry:

$$\{F,\, G\} = -\{G,\, F\}\,. \tag{3}$$

2. Bilinearity:

$$\{\alpha F + \beta G,\, H\} = \alpha\,\{F,\, H\} + \beta\,\{G,\, H\}\,, \tag{4}$$

for any smooth real-valued functions F, G, H on $\mathcal{M}$.

3. Jacobi identity:

$$\{H,\, \{F,\, G\}\} + \{G,\, \{H,\, F\}\} + \{F,\, \{G,\, H\}\} = 0, \tag{5}$$

for any smooth (differentiable) real-valued functions F, G, H on $\mathcal{M}$.

4. Leibniz rule:

$$\{H,\, F\cdot G\} = \{H,\, F\}\cdot G + F\cdot\{H,\, G\}\,, \tag{6}$$

for any smooth real-valued functions F, G, H on $\mathcal{M}$. Here $\cdot$ denotes the ordinary point-wise multiplication of real-valued functions.

Theorem 1. *Let M be an even-dimensional Euclidean space $\mathbf{R}^{2n}$ with coordinates $(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$. If $F(q, p)$ and $G(q, p)$ are smooth (differentiable) functions, the Poisson bracket can be defined as*

$$\{F, G\} = \sum_{i=1}^{n} \left\{ \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right\}. \tag{7}$$

Proof. Direct substitution. □

The basic element of manifold theory is the *vector field*, which takes a central role in analyzing the symmetry of differential equations when M is a Poisson manifold, so that the Poisson bracket satisfies the basic requirements of (3)–(6). For a given smooth function H on M, the map $F \mapsto \{F, H\}$, defines a derivative space of F on $\mathcal{M}$, and hence the map $\cdot \mapsto \{\cdot, H\}$ determines a vector field on $\mathcal{M}$. This observation leads to the following results.

Definition 1. Let $\mathcal{M}$ be a Poisson manifold with Poisson bracket and $H : \mathcal{M} \to \mathbf{R}$ a smooth (differentiable) function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $\mathbf{v}_H$ on $\mathcal{M}$ that satisfies

$$\mathbf{v}_H(F) = \{F, H\} = -\{H, F\}, \tag{8}$$

for every smooth function F. The equations that govern the flow of $\mathbf{v}_H$ are referred to as *Hamilton's equations* and H is called the "Hamiltonian" function.

Therefore, if we consider a Poisson bracket (7) on $\mathbf{R}^{2n}$ with coordinates $(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$ and let $H : \mathbf{R}^{2n} \to \mathbf{R}$ be a smooth function, then the Hamiltonian vector field corresponding to $H(p, q)$ is

$$\mathbf{v}_H = \sum_{i=1}^{n} \left\{ \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \right\}, \tag{9}$$

and Hamilton's equations are

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n. \tag{10}$$

The flows can be obtained by integrating these equations to get

$$q_i = g_i\left(t; p_0, q_0\right), \qquad p_i = f_i\left(t; p_0, q_0\right), \tag{11}$$

where $q_0,\ p_0 \in \mathbf{R}^n$ are initial conditions, and $f_i,\ g_i$ are some smooth functions.

In general, let $H(x)$ be a smooth real-valued function with $x = (x_1, x_2, \ldots, x_m)$; then the general form of the Hamiltonian vector field associated with H is

$$\mathbf{v}_H = \sum_{i=1}^{m} \zeta_i\left(x\right) \frac{\partial}{\partial x_i}, \tag{12}$$

where $\zeta_i(x) = \mathbf{v}_H(x_i) = \{x_i,\ H\}$ and the Poisson bracket is

$$\{F,\ H\} = \sum_{i=1}^{m} \{x_i,\ H\} \frac{\partial F}{\partial x_i}. \tag{13}$$

Thus we have

$$\{F,\ H\} = \sum_{i=1}^{m} \sum_{j=1}^{m} \{x_i,\ x_j\} \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial x_j}, \quad i, j = 1, \ldots, m, \tag{14}$$

and

$$\mathbf{J}_{ij}\left(x\right) := \{x_i,\ x_j\}, \quad i, j = 1, \ldots, m. \tag{15}$$

Here $\mathbf{J}$ is called the *structural function* relative to the given local coordinate x and Hamiltonian $H(x)$. By using $\bigtriangledown H$ to denote by the gradient vector of H, we further have

$$\{F,\ H\} = \bigtriangledown F \cdot \mathbf{J} \bigtriangledown H, \tag{16}$$

where the structure matrix is

$$\mathbf{J} = \begin{bmatrix} 0 & -\mathbf{I}_{m\times m} \\ \mathbf{I}_{m\times m} & 0 \end{bmatrix},$$

and the general Hamilton's equations take the form

$$\frac{dx}{dt} = \mathbf{J}(x) \bigtriangledown H(x)\,. \tag{17}$$

When we are concerned with the infinite-dimensional Hamiltonian system, we must regard it as a generalization of the finite-dimensional Hamiltonian system as follows.

1. Replace the Hamiltonian function $H(x)$ by a Hamiltonian functional $\mathcal{H}[u]$.

2. Replace the gradient $\bigtriangledown H$ by the variational derivative $\delta\mathcal{H}/\delta u$,

$$\frac{\delta \mathcal{H}}{\delta u} := \left(\sum_{n=0}^{\infty} (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{\partial}{\partial u_{(n)}} \right) \mathcal{H},$$

where $\partial/\partial u_{(n)}$ is the nth derivative of u with respect to x.

3. Replace the skew-symmetric matrix $\mathbf{J}$ in (16) by a skew-adjoint operator $\mathcal{D}$ which may depend on u.

Hence the resulting infinite-dimensional Hamiltonian system takes the form

$$\frac{\partial u}{\partial t} = \mathcal{D} \frac{\delta \mathcal{H}}{\delta u},$$

and the Poisson bracket is defined as

$$\{\mathcal{F},\, \mathcal{G}\} = \int_{-\infty}^{\infty} \frac{\delta \mathcal{F}}{\delta u} \mathcal{D} \frac{\delta \mathcal{G}}{\delta u}\, \mathrm{d}x, \tag{18}$$

for any two smooth functionals $\mathcal{F}$, $\mathcal{G}$. Then for infinite-dimensional systems, Olver [**9**] gives the following results.

Definition 2. A differential operator $\mathcal{D}$ is called *Hamiltonian* if its Poisson bracket (18) satisfies the "*skew-symmetry*" property

$$\{\mathcal{P},\, \mathcal{Q}\} = -\{\mathcal{Q},\, \mathcal{P}\}\,, \tag{19}$$

and the "*Jacobi identity*"

$$\{\mathcal{P},\, \{\mathcal{Q},\, \mathcal{R}\}\} + \{\{\mathcal{R},\, \mathcal{P}\}\,,\, \mathcal{Q}\} + \{\{\mathcal{Q},\, \mathcal{R}\}\,,\, \mathcal{P}\} = 0, \tag{20}$$

for any smooth functionals $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$.

A radical simplification for determining a Hamiltonian operator is brought up by Olver [**9**] using the theory of "bi-vector" as follows.

Definition 3. For any skew-adjoint differential operator $\mathcal{D}$, the bi-vector of $\mathcal{D}$ is a functional of the following form

$$\Theta_{\mathcal{D}} \equiv \frac{1}{2}\int \{\theta \wedge \mathcal{D}\theta\}\ \mathrm{d}x = \frac{1}{2}\int \left(\sum_{\alpha,\beta=1}^{q} \theta^{\alpha} \wedge \mathcal{D}_{\alpha\beta}\theta^{\beta}\right) \mathrm{d}x,$$

where $\theta = (\theta^{\alpha},\, \theta^{\beta})$ is called a unit-vector and $\mathcal{D} = (\mathcal{D}_{\alpha\beta})$ is a $q \times q$ dimensional differential operator.

The significance of the bi-vector is that it defines a "bilinear" map as

$$\langle\Theta;\, P,\, Q\rangle = \frac{1}{2}\int (P \cdot \mathcal{D}Q - Q \cdot \mathcal{D}P)\ \mathrm{d}x = \int (P \cdot \mathcal{D}Q)\ \mathrm{d}x, \tag{21}$$

for any $P,\, Q \in \mathcal{A}^q$, space of q-tuples of differential functions. Therefore if $\mathcal{P}$ and $\mathcal{Q}$ are variational derivatives, i.e., $\delta\mathcal{P}/\delta u = P$, $\delta\mathcal{Q}/\delta u = Q$, then

$$\langle\Theta;\, \mathcal{P},\, \mathcal{Q}\rangle = \int \left(\frac{\delta\mathcal{P}}{\delta u} \cdot \mathcal{D}\frac{\delta\mathcal{Q}}{\delta u}\right) \mathrm{d}x, \tag{22}$$

produces the bracket $\{\mathcal{P},\, \mathcal{Q}\}$ determined by $\mathcal{D}$. Thus, studying the Hamiltonian operator is equally as important as studying the bi-vector.

Example 1. The KdV equation

$$u_t = uu_x + u_{xxx} = \partial_x\left(u_{xx} + \frac{1}{2}u^2\right),$$

has $\mathcal{D} = \partial_x \cdot$, and its bi-vector is

$$\Theta_{\mathcal{D}} = \frac{1}{2}\int (\theta \wedge \mathcal{D}\,(\theta))\ \mathrm{d}x = \frac{1}{2}\int (\theta \wedge \theta_x)\ \mathrm{d}x.$$

Example 2. The Boussinesq equation

$$u_t = v_x, \qquad v_t = \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x.$$

has the operator form as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \cdot \\ \partial_x \cdot & 0 \end{pmatrix}\begin{pmatrix} \delta\mathcal{H}/\delta u \\ \delta\mathcal{H}/\delta v \end{pmatrix},$$

and its bi-vector is

$$\begin{aligned} \mathbf{\Theta}_{\mathcal{D}} &= \frac{1}{2}\int \left((\theta, \zeta) \wedge \begin{pmatrix} 0 & \partial_x \cdot \\ \partial_x \cdot & 0 \end{pmatrix}\begin{pmatrix} \theta \\ \zeta \end{pmatrix}\right) \mathrm{d}x \\ &= \frac{1}{2}\int (\theta \wedge \zeta_x + \zeta \wedge \theta_x)\, \mathrm{d}x, \end{aligned}$$

where $\overline{\boldsymbol{\theta}} \equiv (\theta,\, \zeta)$ are unit vectors of u and v respectively.

Now our key tool for determining the Hamiltonian operator was the following result proved by Olver [**9**]:

Theorem 2. *Let $\mathcal{D}$ be a skew-adjoint operator, and let*

$$\Theta_{\mathcal{D}} = \frac{1}{2}\int (\theta \wedge \mathcal{D}(\theta))\, \mathrm{d}x, \tag{23}$$

be the corresponding functional bi-vector. Then $\mathcal{D}$ is Hamiltonian if and only if

$$\mathrm{pr}\, \mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = 0. \tag{24}$$

3. Hamiltonian structure of the STMKdV equation. The STMKdV equation (1) can be cast as a system of equations as follows:

$$\begin{cases} u_t = q_x, \\ q_t = u_x + \alpha\left(\frac{1}{2}u^2\right)_x - uq_x + \beta u_{xxx} - q_{xxx}, \end{cases} \tag{25}$$

where we have introduced a new variable $q(x,t)$ and

$$\int_{-\infty}^{\infty} u_t\, \mathrm{d}x = 0, \tag{26}$$

together with boundary conditions such as $u(x,t)$, $q(x,t)$ their products and their derivatives vanishing as $x \to \pm\infty$.

There is a conserved quantity that makes a big contribution, namely that [**7**]

$$\mathcal{H} = \int_{-\infty}^{\infty} \left(uq + \frac{1}{3}u^3 - \frac{1}{2}u_x^2 \right) \mathrm{d}x, \tag{27}$$

and (25) can be written in a conservation law as

$$\begin{aligned}\frac{\partial}{\partial t}&\left(uq + \frac{1}{3}u^3 - \frac{1}{2}u_x^2 \right) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{1}{2}q^2 + \frac{\alpha}{3}u^3 + \beta\,(uu_{xx}) - \frac{\beta}{2}u_x^2 - uq_{xx} \right).\end{aligned}$$

To show this is indeed independent of time, we compute

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(uq + \frac{1}{3}u^3 - \frac{1}{2}u_x^2 \right) \mathrm{d}x, \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{1}{2}q^2 + \frac{\alpha}{3}u^3 + \beta\,(uu_{xx}) - \beta\left(\frac{1}{2}u_x^2\right) - uq_{xx} \right) \mathrm{d}x, \\ &= \left[\frac{1}{2}u^2 + \frac{1}{2}q^2 + \frac{\alpha}{3}u^3 + \beta\,(uu_{xx}) - \beta\left(\frac{1}{2}u_x^2\right) - uq_{xx} \right]_{-\infty}^{\infty}, \\ &= 0,\end{aligned}$$

using boundary conditions u^2, q^2, u^3, uu_{xx}, u_x^2, $uq_{xx} \to 0$ as $x \to \pm\infty$. Therefore, $\mathrm{d}\mathcal{H}/\mathrm{d}t = 0$ meaning that $\mathcal{H} =$ constant.

Furthermore (25) can be written in a matrix form as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ q \end{pmatrix} = \mathcal{D} \cdot \begin{pmatrix} \delta\mathcal{H}/\delta u \\ \delta\mathcal{H}/\delta q \end{pmatrix}, \tag{28}$$

where

$$\mathcal{D} = \begin{pmatrix} \partial_x \cdot & -\partial_x^3 \cdot - \partial_x u \cdot \\ -\partial_x^3 \cdot - u\partial_x \cdot & \begin{pmatrix} \partial_x \cdot + (\alpha/3)\,(u\partial_x \cdot + \partial_x u \cdot) \\ +\beta\partial_{xxx} \cdot + u\partial_x u \cdot \\ +\,(u\partial_{xxx} \cdot + \partial_{xxx} u \cdot) \\ +\partial_{xxxxx} \cdot \end{pmatrix} \end{pmatrix},$$

is an operator and its variational derivatives are

$$\frac{\delta \mathcal{H}_1}{\delta u} = q + u^2 + u_{xx}, \qquad \frac{\delta \mathcal{H}_1}{\delta q} = u.$$

To obtain a genuine Hamiltonian structure of the equation, we need to check the skew-adjoint and Jacobi identity. First of all, the operator $\mathcal{D}$ is skew-adjoint for any α, β:

$$\begin{aligned}
\mathcal{D}^* &= \begin{pmatrix} \partial_x \cdot^* & \left(-\partial_x^3 \cdot - \partial_x u \cdot\right)^* \\ \left(-\partial_x^3 \cdot - u\partial_x \cdot\right)^* & \begin{pmatrix} \partial_x \cdot + (\alpha/3)\left(u\partial_x \cdot + \partial_x u \cdot\right) \\ +\beta \partial_{xxx} \cdot + u\partial_x u \cdot \\ + \left(u\partial_{xxx} \cdot + \partial_{xxx} u \cdot\right) \\ +\partial_{xxxxx} \cdot \end{pmatrix}^* \end{pmatrix}, \\
&= \begin{pmatrix} -\partial_x \cdot & -\left(-\partial_x^3 \cdot - u\partial_x \cdot\right) \\ -\left(-\partial_x^3 \cdot - u\partial_x \cdot\right) & -\begin{pmatrix} \partial_x \cdot + (\alpha/3)\left(u\partial_x \cdot + \partial_x u \cdot\right) \\ +\beta \partial_{xxx} \cdot + u\partial_x u \cdot \\ + \left(u\partial_{xxx} \cdot + \partial_{xxx} u \cdot\right) \\ +\partial_{xxxxx} \cdot \end{pmatrix} \end{pmatrix}, \\
&= -\mathcal{D}.
\end{aligned}$$

To check the Jacobi identity, we consider the unit vector for u and q as $\overline{\boldsymbol{\theta}} = (\theta, \zeta)$ and its bi-vector as

$$\boldsymbol{\Theta}_{\mathcal{D}} = \frac{1}{2} \int \begin{pmatrix} \theta \wedge \theta_x - \theta \wedge \zeta_{xxx} - u_x \theta \wedge \zeta - u\theta \wedge \zeta_x \\ -\zeta \wedge \theta_{xxx} - u\zeta \wedge \theta_x + \zeta \wedge \zeta_x \\ +\beta \zeta \wedge \zeta_{xxx} + u^2 \zeta \wedge \zeta_x + uu_x \zeta \wedge \zeta \\ +3u_{xx} \zeta \wedge \zeta_x + 3u_x \zeta \wedge \zeta_{xx} \\ +2u\zeta \wedge \zeta_{xxx} + \zeta \wedge \zeta_{xxxxx} + \frac{2}{3}\alpha u \zeta \wedge \zeta_x \end{pmatrix} \mathrm{d}x.$$

But first of all we need to apply the integration by parts and boundary conditions such as

$$\begin{aligned}
\int -u_x \theta \wedge \zeta \,\mathrm{d}x &= \int \left(u \cdot \partial_x \left(\theta \wedge \zeta\right)\right) \mathrm{d}x \\
&= \int \left(u\theta_x \wedge \zeta + u\theta \wedge \zeta_x\right) \mathrm{d}x, \\
\int 3u_{xx} \zeta \wedge \zeta_x \,\mathrm{d}x &= \int \left(3u\zeta_x \wedge \zeta_{xx} + 3u\zeta \wedge \zeta_{xxx}\right) \mathrm{d}x, \\
\int 3u_x \zeta \wedge \zeta_{xx} \,\mathrm{d}x &= \int \left(-3u\zeta_x \wedge \zeta_{xx} - 3u\zeta \wedge \zeta_{xxx}\right) \mathrm{d}x,
\end{aligned}$$

we then arrive at

$$\boldsymbol{\Theta}_{\mathcal{D}} = \frac{1}{2}\int \begin{pmatrix} \theta\wedge\theta_x - \theta\wedge\zeta_{xxx} + u\theta_x\wedge\zeta \\ -u\theta\wedge\zeta_x - \zeta\wedge\theta_{xxx} - u\zeta\wedge\theta_x \\ +\zeta\wedge\zeta_x + \frac{2}{3}\alpha u\zeta\wedge\zeta_x + \beta\zeta\wedge\zeta_{xxx} \\ +u^2\zeta\wedge\zeta_x + 3u\zeta_x\wedge\zeta_{xx} + 3u\zeta\wedge\zeta_{xxx} \\ -3u\zeta_x\wedge\zeta_{xx} - 3u\zeta\wedge\zeta_{xxx} \\ +\zeta\wedge\zeta_{xxxxx} + u\theta\wedge\zeta_x + 2u\zeta\wedge\zeta_{xxx} \end{pmatrix} \mathrm{d}x. \tag{29}$$

Furthermore, by using the wedge interchanging rule such as

$$u\theta_x\wedge\zeta = -u\zeta\wedge\theta_x,$$

the bi-vector (29) is simplified to

$$\boldsymbol{\Theta}_{\mathcal{D}} = \frac{1}{2}\int \begin{pmatrix} \theta\wedge\theta_x - \theta\wedge\zeta_{xxx} + 2u\theta_x\wedge\zeta \\ -\zeta\wedge\theta_{xxx} + \zeta\wedge\zeta_x + \frac{2}{3}\alpha u\zeta\wedge\zeta_x \\ +\beta\zeta\wedge\zeta_{xxx} + u^2\zeta\wedge\zeta_x \\ +\zeta\wedge\zeta_{xxxxx} + 2u\zeta\wedge\zeta_{xxx} \end{pmatrix} \mathrm{d}x.$$

Now we obtain

$$\begin{cases} \mathrm{pr}\,\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(u) = \theta_x - \zeta_{xxx} - u_x\zeta - u\zeta_x, \\ \mathrm{pr}\,\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(q) = \begin{pmatrix} -\theta_{xxx} - u\theta_x + \zeta_x + \frac{2}{3}\alpha u\zeta_x + \frac{1}{3}\alpha\zeta \\ +\beta\zeta_{xxx} + uu_x\zeta + u^2\zeta_x + 2u\zeta_{xxx} \\ +3u_{xx}\zeta_{xx} + 3u_x\zeta_{xx} + u_{xxx}\zeta + \zeta_{xxxxx} \end{pmatrix}, \\ \mathrm{pr}\,\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\left(u^2\right) = 2u\theta_x - \zeta_{xxx} - u_x\zeta - u\zeta_x. \end{cases} \tag{30}$$

Therefore,

$$\begin{aligned} &\mathrm{pr}\,\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(\boldsymbol{\Theta}_{\mathcal{D}}) \\ &\qquad = \frac{1}{2}\mathrm{pr}\,\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int \begin{pmatrix} \theta\wedge\theta_x - \theta\wedge\zeta_{xxx} + 2u\theta_x\wedge\zeta \\ -\zeta\wedge\theta_{xxx} + \zeta\wedge\zeta_x + \frac{2}{3}\alpha u\zeta\wedge\zeta_x \\ +\beta\zeta\wedge\zeta_{xxx} + u^2\zeta\wedge\zeta_x \\ +\zeta\wedge\zeta_{xxxxx} + 2u\zeta\wedge\zeta_{xxx} \end{pmatrix} \mathrm{d}x. \end{aligned} \tag{31}$$

Note further that

$$\begin{aligned}
\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(\theta\wedge\theta_x)\,\mathrm{d}x &= \int \operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(1)\,(\theta\wedge\theta_x)\,\mathrm{d}x = 0,\\
\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(-\theta\wedge\zeta_{xxx})\,\mathrm{d}x &= \int \operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(-1)\,(\theta\wedge\zeta_{xxx})\,\mathrm{d}x\\
&= 0,\\
\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(-\zeta\wedge\theta_{xxx})\,\mathrm{d}x &= \int \operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(-1)\,(\zeta\wedge\theta_{xxx})\,\mathrm{d}x\\
&= 0,\\
\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(\zeta\wedge\zeta_x)\,\mathrm{d}x &= \int \operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(1)\,(\zeta\wedge\zeta_x)\,\mathrm{d}x = 0,\\
\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(\beta\zeta\wedge\zeta_{xxx})\,\mathrm{d}x &= \int \operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(\beta)\,(\zeta\wedge\zeta_{xxx})\,\mathrm{d}x = 0,\\
\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(\zeta\wedge\zeta_{xxxxx})\,\mathrm{d}x &= \int \operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(1)\,(\zeta\wedge\zeta_{xxxxx})\,\mathrm{d}x = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(u\theta_x\wedge\zeta)\,\mathrm{d}x\\
&\qquad\qquad = \int((\theta_x-\zeta_{xxx}-u_x\zeta-u\zeta_x)\wedge\theta_x\wedge\zeta)\,\mathrm{d}x,\\
&\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(u\zeta\wedge\zeta_x)\,\mathrm{d}x\\
&\qquad\qquad = \int((\theta_x-\zeta_{xxx}-u_x\zeta-u\zeta_x)\wedge\zeta\wedge\zeta_x)\,\mathrm{d}x,\\
&\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int\left(u^2\zeta\wedge\zeta_x\right)\,\mathrm{d}x\\
&\qquad\qquad = \int((2u\theta_x-\zeta_{xxx}-u_x\zeta-u\zeta_x)\wedge\zeta\wedge\zeta_x)\,\mathrm{d}x,\\
&\operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}\int(u\zeta\wedge\zeta_{xxx})\,\mathrm{d}x\\
&\qquad\qquad = \int((\theta_x-\zeta_{xxx}-u_x\zeta-u\zeta_x)\wedge\zeta\wedge\zeta_{xxx})\,\mathrm{d}x,
\end{aligned}$$

we are left with

$$\text{(32)}\quad \operatorname{pr}\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(\boldsymbol{\Theta}_{\mathcal{D}}) = \frac{1}{2}\int \begin{pmatrix} -2\zeta_{xxx}\wedge\theta_x\wedge\zeta - 2u\zeta_x\wedge\theta_x\wedge\zeta \\ +\frac{2}{3}\alpha\theta_x\wedge\zeta\wedge\zeta_x - \frac{2}{3}\alpha\zeta_{xxx}\wedge\zeta\wedge\zeta_x \\ +2u\theta_x\wedge\zeta\wedge\zeta_x - 2u\zeta_{xxx}\wedge\zeta\wedge\zeta_x \\ +2\theta_x\wedge\zeta\wedge\zeta_{xxx} - 2u\zeta_x\wedge\zeta\wedge\zeta_{xxx} \end{pmatrix} \mathrm{d}x.$$

Note that in (32),

$$\begin{aligned} \int (2\theta_x\wedge\zeta\wedge\zeta_{xxx})\ \mathrm{d}x - \int (2\zeta_{xxx}\wedge\theta_x\wedge\zeta)\ \mathrm{d}x \\ &= \int (2\theta_x\wedge\zeta\wedge\zeta_{xxx})\ \mathrm{d}x \\ &\quad - \int (2\theta_x\wedge\zeta\wedge\zeta_{xxx})\ \mathrm{d}x, \text{(interchange)} \\ &= 0, \end{aligned}$$

$$\begin{aligned} &\int (-2u\zeta_x\wedge\theta_x\wedge\zeta)\ \mathrm{d}x + \int (2u\theta_x\wedge\zeta\wedge\zeta_x)\ \mathrm{d}x \\ &\quad = \int (-2u\zeta_x\wedge\theta_x\wedge\zeta)\ \mathrm{d}x + \int (2u\zeta_x\wedge\theta_x\wedge\zeta)\ \mathrm{d}x,\ \text{(interchange)} \\ &\quad = 0, \end{aligned}$$

$$\begin{aligned} &\int (-2u\zeta_{xxx}\wedge\zeta\wedge\zeta_x)\ \mathrm{d}x - \int (2u\zeta_x\wedge\zeta\wedge\zeta_{xxx})\ \mathrm{d}x \\ &= \int (-2u\zeta_{xxx}\wedge\zeta\wedge\zeta_x)\ \mathrm{d}x - \int (2u\zeta_{xxx}\wedge\zeta_x\wedge\zeta)\ \mathrm{d}x,\ \text{(interchange)} \\ &= \int (-2u\zeta_{xxx}\wedge\zeta\wedge\zeta_x)\ \mathrm{d}x + \int (2u\zeta_{xxx}\wedge\zeta\wedge\zeta_x)\ \mathrm{d}x,\ \text{(interchange)} \\ &= 0, \end{aligned}$$

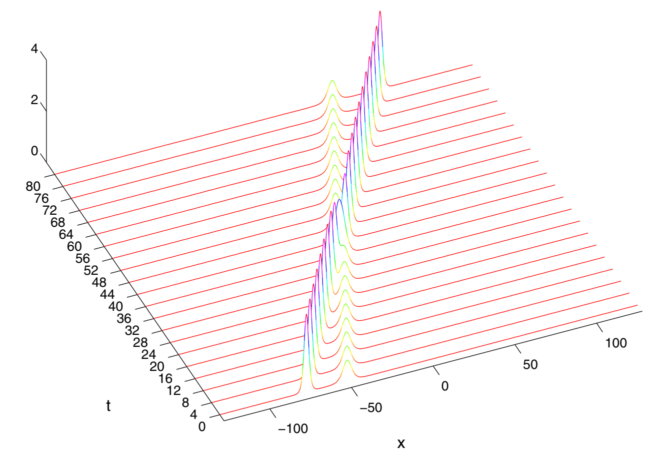

FIGURE 1. (a) STMKdV solution profiles solutions, the initial condition is set up with $\lambda_1 = 1.8$, $\lambda_2 = 1.2$, $x_1 = -75$ and $x_2 = -50$ for waves in (33).

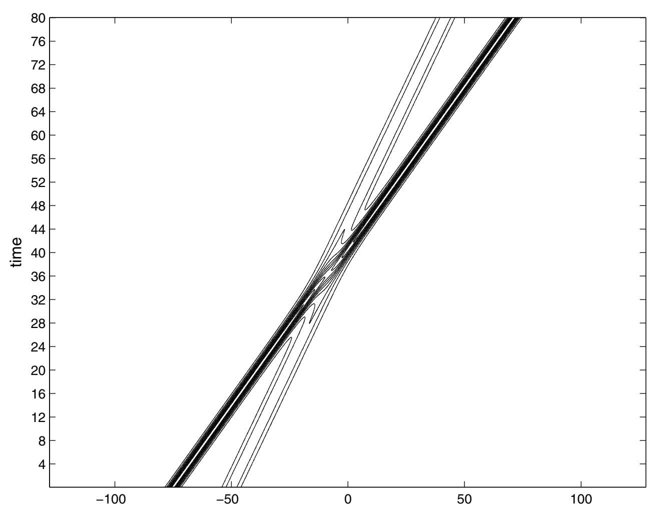

FIGURE 2. Contour plot of interacting solitary waves.

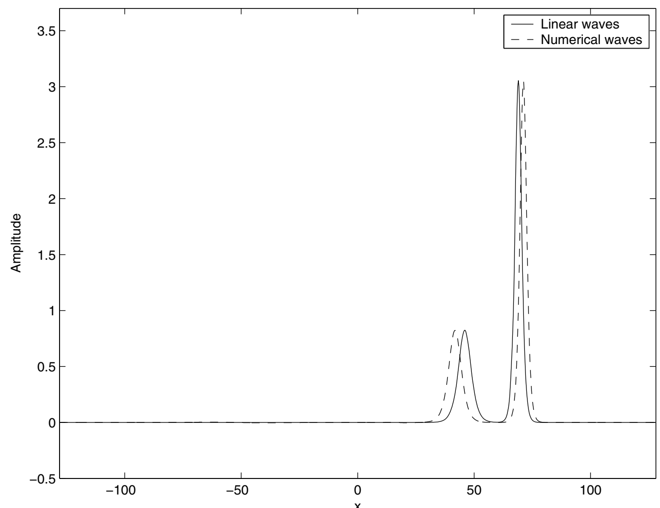

FIGURE 3. Comparison with linear waves at the final time of calculation $t = 80$, showing phase shifts.

and

$$\begin{aligned}\int\left(-\frac{2}{3}\alpha\zeta_{xxx}\wedge\zeta\wedge\zeta_x\right)\mathrm{d}x\\ &= \int\left(\frac{2}{3}\alpha\zeta_{xx}\wedge(\zeta\wedge\zeta_x)_x\right)\mathrm{d}x,\ \text{(boundary condition)}\\ &= \int\left(\frac{2}{3}\alpha\zeta_{xx}\wedge\zeta_x\wedge\zeta_x\right)\mathrm{d}x\\ &\quad+\int\left(\frac{2}{3}\alpha\zeta_{xx}\wedge\zeta\wedge\zeta_{xx}\right)\mathrm{d}x,\\ &= 0.\end{aligned}$$

We finally have, for general θ, ζ, the following result

$$\mathrm{pr}\,\mathbf{v}_{\mathcal{D}\overline{\boldsymbol{\theta}}}(\boldsymbol{\Theta}_{\mathcal{D}}) = \frac{1}{2}\int\left(\frac{2}{3}\alpha\theta_x\wedge\zeta\wedge\zeta_x\right)\mathrm{d}x.$$

Hence $\mathcal{D}$ will be a Hamiltonian operator when $\alpha = 0$ and STMKdV equation admits a Hamiltononian structure under this condition.

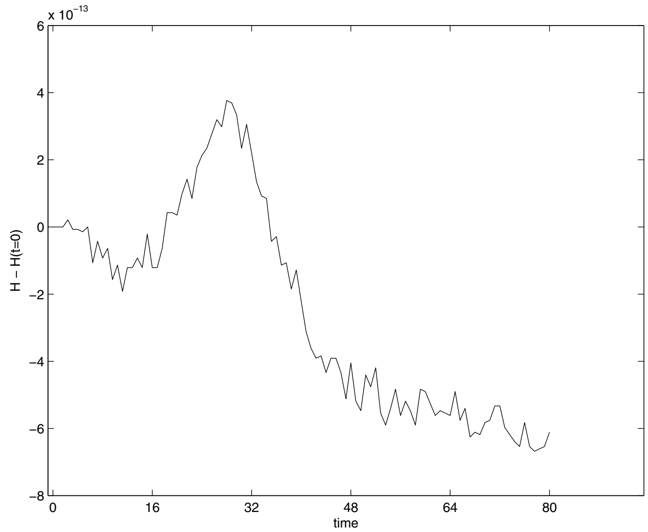

FIGURE 4. Hamiltonian in (27); the values are presented as differences from the initial value.

Equation (25) is numerically integrated in the interval $[-128, 128]$ with $\Delta x = 0.25$ and $\Delta t = 10^{-3}$. The time discretization for our scheme is implemented by the classical fourth-order Runge-Kutta (RK-4) method and spatial derivative is approximated "spectrally" [**2**]. The initial condition is given by a linear sum of two well-separated solitary waves of different amplitudes and velocities as

$$u(x,0) = A_1 \operatorname{sech}^2 \left(B_1 \left(x - x_1\right)\right) + A_2 \operatorname{sech}^2 \left(B_2 \left(x - x_2\right)\right), \tag{33}$$

where $A_i = 3(\lambda_i^2 - 1)/(\lambda_i + \alpha)$ and $B_i = (1/2)\sqrt{(\lambda_i^2 - 1)/(\lambda_i + \beta)}$, for $i = 1, 2$. We also compute the Hamiltonian in (27) to give full validation of the accuracy of our numerical scheme. The solution profiles for the STMKdV equation are presented in Figures 1–4. We note that by considering $\alpha = 0$, $\beta = 0.2$ for the STMKdV equation, we are dealing with both the Hamiltonian system and its soliton phenomenon at the same time. One can see that the STMKdV taller solitary wave catches the shorter, coalesces to form a single wave, then reappears in front of the shorter wave (to the right). The interaction is nonlinear, not simply the superposition of two individual waves. The Hamiltonian is as accurate as of $\mathcal{O}(10^{-13})$, showing that our results are genuine.

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