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ISOMETRIES OF NAKANO SPACE OF VECTOR VALUED FUNCTIONS

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ABSTRACT. The Nakano space $L^{p(t)}(\mu)$ associated with $p(t)$ is defined to be the Musielak-Orlicz space $L_\Phi(\mu)$ such that $\Phi(u,t) = u^{p(t)}/p(t)$. We are going to consider the space $N = L^{p(t)}(\mu, \mathcal{H})$, where $\mathcal{H}$ is a separable complex Hilbert space with inner product $\langle\, , \,\rangle$ and norm $\|\cdot\|_2$. For any $f \in N$, let

$$M(f) = \int_0^1 \frac{\|f(t)\|_2^{p(t)}}{p(t)}\, d\mu(t),$$

where $1 < p_0 \le p(t) \le p_\infty < \infty$. For every $f \in N$, the norm of f on this space is

$$\|f\|_N = \inf\left\{\varepsilon > 0 : M\left(\frac{f}{\varepsilon}\right) \le 1\right\}.$$

We are interested in the form of the Hermitian operators and the form of the surjective isometries on this space N.

1. Introduction. Let $([0,1], \Sigma, \mu)$ be a nonatomic measure space and $\mathcal{H}$ a separable complex Hilbert space with inner product $\langle\, , \,\rangle$ and norm $\|\cdot\|_2$.

In the following we are going to consider the space $N = L^{p(t)}(\mu, \mathcal{H})$, where $p(t)$ is a measurable function from $[0,1]$ into $(1,\infty)$ such that $1 < p_0 \le p(t) \le p_\infty < \infty$. Also, for every vector $z \in \mathcal{H}$, we define the constant function $\mathbf{z}(t) = z$ for every $t \in [0,1]$.

We recall that, for a Young function Φ, $L_\Phi(\mu, \mathcal{H})$ denotes the space of all strongly measurable functions from $[0,1]$ to $\mathcal{H}$ for which

$$M_\Phi(\lambda f) = \int_0^1 \Phi(\lambda \|f(t)\|_2, t)\, d\mu(t) < \infty,$$

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for some $\lambda > 0$. The space $L_\Phi(\mu, \mathcal{H})$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi(\mu,\mathcal{H})} = \inf\left\{\varepsilon > 0 : M_\Phi\left(\frac{f}{\varepsilon}\right) \le 1\right\}.$$

We note that, for strongly measurable functions (defined as the μ-a.e. limit of simple functions of the form $\sum_{i=1}^n x_i \chi_{E_i}$, where $x_i \in \mathcal{H}$, $E_i \in \Sigma$ and χ_E is the characteristic function of the set E of a finite measure space (Ω, Σ, μ) (see [**10**, page 425])), a function $f \in L_\Phi(\mu, \mathcal{H})$ if and only if $\|f(\cdot)\|_2 \in L_\Phi(\mu)$. Also, the simple functions are dense in $L_\Phi(\mu, \mathcal{H})$ (see [**4**, page 363]).

The space $N = L^{p(t)}(\mu, \mathcal{H})$ is the vector-valued version of the Nakano space $L^{p(t)}(\mu)$ associated with $p(t)$ (see [**4**, page 76]), defined to be the Musielak-Orlicz space $L_\Phi(\mu)$ such that

$$\Phi(u, t) = \frac{u^{p(t)}}{p(t)}.$$

It can be shown that simple functions are dense in N, since Φ satisfies the Δ_2-condition (see [**7**, page 214] and [**5**, page 24]).

For any $f \in N$, let

$$M(f) = \int_0^1 \frac{\|f(t)\|_2^{p(t)}}{p(t)}\, d\mu(t),$$

where $1 < p_0 \le p(t) \le p_\infty < \infty$. Therefore,

$$N = \{f : M(\lambda f) < \infty,\ \text{for some } \lambda > 0\}.$$

For every $f \in N$, the norm of f on this space is

$$\|f\|_N = \inf\left\{\varepsilon > 0 : M\left(\frac{f}{\varepsilon}\right) \le 1\right\}.$$

If we let

$$N_0 = \{f : M(\lambda f) < \infty,\ \text{for all } \lambda > 0\},$$

we see that this subspace of N has nicer properties than the whole space N (see [**2**, page 140]), e.g., N_0 is separable if the measure space is separable, while N may be not separable. In addition, since $\Phi(u,t) = u^{p(t)}/p(t)$ satisfy the Δ_2 condition, we have $N_0 = N$.

On this space N, we are interested in the form of the Hermitian operators (see [**6**, page 39]) and the form of the surjective isometries.

2. Hermitian operators on N.

2.1. Preliminary results. To find the form of the Hermitian operators on N, first we need to determine a semi-inner product compatible with the norm on N.

Remark 1. For $f \in N$, it can be shown that $M(f/\|f\|_N) = 1$ since otherwise it must be $M(f/\|f\|_N) < 1$. In this case, we can assume that there is a positive scalar k_0 such that $M(k_0 f) = 1$, so $1/\|f\|_N < k_0$. It yields that $1 < \|k_0 f\|_N \leq M(k_0 f) = 1$, by [**1**, page 269]. This contradiction implies that $M(f/\|f\|_N) = 1$. Therefore, if $\|f\|_N = 1$, we must have $M(f) = 1$. The inverse implication follows directly. We then have $\|f\|_N = 1$ if and only if $M(f) = 1$.

Lemma 2. *A semi-inner product compatible with the norm on N is given by*

$$F_g(f) = C(g) \int_0^1 \frac{\langle f(t), g(t)\rangle}{\|g(t)\|_2} \left(\frac{\|g(t)\|_2}{\|g\|_N}\right)^{p(t)-1} d\mu(t),$$

where

$$C(g) = \frac{\|g\|_N^2}{\int_0^1 \|g(t)\|_2 \left(\|g(t)\|_2/\|g\|_N\right)^{p(t)-1} d\mu(t)}$$

and $g \in N_0$.

Proof. To show that $F_g(f)$ is a semi-inner product compatible with the norm on N, we need to check the conditions of the definition of a semi-inner product (see [**6**, page 31]). It is obvious that $F_g(g) = \|g\|_N^2$ and for a complex scalar α,

$$F_g(\alpha f + h) = \alpha F_g(f) + F_g(h).$$

To show the Cauchy-Schwartz inequality, let $f \in N$ be such that $\|f\|_N = 1$, which is equivalent to $M(f) = 1$ (see the remark above). We also have $M(g/\|g\|_N) = 1$.

$$\begin{aligned}|F_g(f)| &\leq \frac{\|g\|_N \int_0^1 |\langle f(t), g(t)\rangle| / \|g(t)\|_2 \left(\|g(t)\|_2/\|g\|_N\right)^{p(t)-1} d\mu(t)}{\int_0^1 \|g(t)\|_2/\|g\|_N \left(\|g(t)\|_2/\|g\|_N\right)^{p(t)-1} d\mu(t)} \\ &\leq \frac{\|g\|_N \int_0^1 \|f(t)\|_2 \left(\|g(t)\|_2/\|g\|_N\right)^{p(t)-1} d\mu(t)}{\int_0^1 \left(\|g(t)\|_2/\|g\|_N\right)^{p(t)} d\mu(t)}.\end{aligned}$$

By Young's inequality (see [**2**, page 142]), i.e., for any $u, v \geq 0$ and any $t \in [0,1]$,

$$uv \leq \Phi(u,t) + \Psi(v,t),$$

where $\Psi(v,t) = v^{q(t)}/q(t)$, we have:

$$\begin{aligned}|F_g(f)| &\leq \frac{\|g\|_N}{\int_0^1 \left(\|g(t)\|_2/\|g\|_N\right)^{p(t)} \left[(1/p(t)) + (1/q(t))\right] d\mu(t)} \\ &\quad \times \int_0^1 \left[\frac{\|f(t)\|_2^{p(t)}}{p(t)} + \left(\frac{\|g(t)\|_2}{\|g\|_N}\right)^{(p(t)-1)q(t)} \frac{1}{q(t)}\right] d\mu(t) \\ &= \frac{\|g\|_N \left[1 + \int_0^1 \left(\|g(t)\|_2/\|g\|_N\right)^{p(t)} d\mu(t)/q(t)\right]}{1 + \int_0^1 \left(\|g(t)\|_2/\|g\|_N\right)^{p(t)} (1/q(t))\, d\mu(t)} \\ &= \|g\|_N = \|g\|_N \|f\|_N.\end{aligned}$$

If we consider any $f \in N$, then $\|f/\|f\|_N\|_N = 1$; so by the previous calculations, we have $|F_g(f/\|f\|_N)| \leq \|g\|_N$, and therefore $|F_g(f)| \leq \|g\|_N \|f\|_N$. □

Remark 3. Using one of the well-known inequalities for Musielak-Orlicz functions (see [**2**, page 142]), which in this case is

$$u^{p(t)} \leq \frac{(2u)^{p(t)}}{p(t)},$$

we can write

$$\begin{aligned}\int_0^1 \|g(t)\|_2 \left(\frac{\|g(t)\|_2}{\|g\|_N}\right)^{p(t)-1} d\mu(t) &= \|g\|_N \int_0^1 \left(\frac{\|g(t)\|_2}{\|g\|_N}\right)^{p(t)} d\mu(t) \\ &\le \|g\|_N \int_0^1 \left(2\frac{\|g(t)\|_2}{\|g\|_N}\right)^{p(t)} \frac{d\mu(t)}{p(t)} \\ &= \|g\|_N\, M\left(2\frac{g}{\|g\|_N}\right) < \infty.\end{aligned}$$

Using the semi-inner product given above, we now have the following lemma.

Lemma 4. *Let H be an arbitrary Hermitian operator on N, and let $f_1,\ f_2 \in N_0$ with disjoint supports A_1 and A_2, respectively, where*

$$N_0 = \{f \in N : M(\lambda f) < \infty,\ \textit{for all } \lambda > 0\}.$$

Then

$$\begin{aligned}\int_{A_1} &\frac{\langle Hf_2(t), f_1(t)\rangle}{\|f_1(t)\|_2}\left(\frac{\|f_1(t)\|_2}{\|f_1 + e^{i\theta}f_2\|_N}\right)^{p(t)-1} d\mu(t) \\ &= \int_{A_2} \frac{\overline{\langle Hf_1(t), f_2(t)\rangle}}{\|f_2(t)\|_2}\left(\frac{\|f_2(t)\|_2}{\|f_1 + e^{i\theta}f_2\|_N}\right)^{p(t)-1} d\mu(t).\end{aligned}$$

Proof. The proof of this lemma is based on the fact that $F_{f_1+e^{i\theta}f_2}(H(f_1+e^{i\theta}f_2))$ is real for all real θ and $f_1,\ f_2 \in N$. (We follow the same steps as in [**2**, page 142].)

Let $f_1,\ f_2 \in N_0$, $\theta \in \mathbf{R}$ and $g = f_1 + e^{i\theta}f_2$. Then $g \in N_0$ and $Hg = Hf_1 + e^{i\theta}Hf_2$. By the previous lemma, we have

$$\begin{aligned}\frac{F_g(Hg)}{C(g)} &= \int_0^1 \frac{\langle Hg(t), g(t)\rangle}{\|g(t)\|_2}\left(\frac{\|g(t)\|_2}{\|g\|_N}\right)^{p(t)-1} d\mu(t) \\ &= \int_0^1 \frac{\langle Hf_1(t), g(t)\rangle}{\|g(t)\|_2}\left(\frac{\|g(t)\|_2}{\|g\|_N}\right)^{p(t)-1} d\mu(t)\end{aligned}$$

$$
\begin{aligned}
&+ e^{i\theta} \int_0^1 \frac{\langle Hf_2(t), g(t)\rangle}{\|g(t)\|_2} \left(\frac{\|g(t)\|_2}{\|g\|_N} \right)^{p(t)-1} d\mu(t) \\
&= \int_{A_1} \frac{\langle Hf_1(t), f_1(t)\rangle}{\|f_1(t)\|_2} \left(\frac{\|f_1(t)\|_2}{\|f_1 + e^{i\theta} f_2\|_N} \right)^{p(t)-1} d\mu(t) \\
&+ e^{-i\theta} \int_{A_2} \frac{\langle Hf_1(t), f_2(t)\rangle}{\|f_2(t)\|_2} \left(\frac{\|f_2(t)\|_2}{\|f_1 + e^{i\theta} f_2\|_N} \right)^{p(t)-1} d\mu(t) \\
&+ e^{i\theta} \int_{A_1} \frac{\langle Hf_2(t), f_1(t)\rangle}{\|f_1(t)\|_2} \left(\frac{\|f_1(t)\|_2}{\|f_1 + e^{i\theta} f_2\|_N} \right)^{p(t)-1} d\mu(t) \\
&+ \int_{A_2} \frac{\langle Hf_2(t), f_2(t)\rangle}{\|f_2(t)\|_2} \left(\frac{\|f_2(t)\|_2}{\|f_1 + e^{i\theta} f_2\|_N} \right)^{p(t)-1} d\mu(t).
\end{aligned}
$$

Also, since H is Hermitian, $F_g(Hg)$ is real. By Tam's lemma (see [**9**, page 236]), if $a + be^{i\theta} + ce^{-i\theta} \in \mathbf{R}$ then $b = \overline{c}$. Thus, if

$$
\begin{aligned}
b &= \int_{A_1} \frac{\langle Hf_2(t), f_1(t)\rangle}{\|f_1(t)\|_2} \left(\frac{\|f_1(t)\|_2}{\|f_1 + e^{i\theta} f_2\|_N} \right)^{p(t)-1} d\mu(t), \\
c &= \int_{A_2} \frac{\langle Hf_1(t), f_2(t)\rangle}{\|f_2(t)\|_2} \left(\frac{\|f_2(t)\|_2}{\|f_1 + e^{i\theta} f_2\|_N} \right)^{p(t)-1} d\mu(t),
\end{aligned}
$$

then

$$
\begin{aligned}
&\int_{A_1} \frac{\langle Hf_2(t), f_1(t)\rangle}{\|f_1(t)\|_2} \left(\frac{\|f_1(t)\|_2}{\|f_1 + e^{i\theta} f_2\|_N} \right)^{p(t)-1} d\mu(t) \\
&\qquad = \int_{A_2} \frac{\overline{\langle Hf_1(t), f_2(t)\rangle}}{\|f_2(t)\|_2} \left(\frac{\|f_2(t)\|_2}{\|f_1 + e^{i\theta} f_2\|_N} \right)^{p(t)-1} d\mu(t). \qquad \square
\end{aligned}
$$

Proposition 5. *Let H be an arbitrary Hermitian operator on N. Then, for any $z \in \mathcal{H}$ and any measurable set of positive measure σ, with its characteristic function χ_σ, we have*

$$\operatorname{supp} H(\chi_\sigma z) \subset \sigma$$

and

$$\chi_\sigma H(\mathbf{z}) = H(\chi_\sigma z),$$

where $\mathbf{z}(t) = z$, for every t.

Proof. Let σ be a measurable set of positive measure, and assume that $\operatorname{supp} H(\chi_\sigma z) \subset [0,1]\setminus\sigma$, for any $z \in \mathcal{H}$. We are going to apply the previous lemma for $f_1 = \alpha z \chi_\sigma$ and $f_2 = \beta f$, where $\alpha, \beta > 0$, $z \in \mathcal{H}$, $\|z\|_2 = 1$ and $f \in N_0$ with support $\rho \subset [0,1]\setminus\sigma$. We have

$$\int_\sigma \langle Hf(t), z\rangle \frac{\alpha^{p(t)-2}}{\|\alpha z\chi_\sigma + e^{i\theta}\beta f\|_N^{p(t)-1}}\, d\mu(t) = \int_\rho \overline{\langle H\chi_\sigma z(t), f(t)\rangle}\, \frac{\beta^{p(t)-2}\, \|f(t)\|_2^{p(t)-2}}{\|\alpha z\chi_\sigma + e^{i\theta}\beta f\|_N^{p(t)-1}}\, d\mu(t).$$

This is true for any $f \in N_0$ with support in $[0,1]\setminus\sigma$ and any $\alpha > 0$. If we let $\alpha \to 0^+$, the last equality becomes

$$0 = \int_\rho \overline{\langle H\chi_\sigma z(t), f(t)\rangle} \frac{\|f(t)\|_2^{p(t)-2}}{\beta\, \|f\|_N^{p(t)-1}}\, d\mu(t). \tag{2.1}$$

Let B be a set of positive measure in ρ such that $\|H(\chi_\sigma z)(t)\|_2 > 0$, for every $t \in B$. This implies that $B \subset \operatorname{supp} H(\chi_\sigma z)$. If we let

$$f(t) = \frac{\chi_B(t)\, H(\chi_\sigma z)(t)}{\|H(\chi_\sigma z)(t)\|_2},$$

then

$$\|f(t)\|_2 = \chi_B(t),\ M(f) = M(\chi_B) \text{ and } M(\lambda f) = M(\lambda\chi_B).$$

This implies that $\|f\|_N = \|\chi_B\|_N$. Therefore, (2.1) becomes:

$$\begin{aligned} 0 &= \int_\rho \overline{\left\langle H\chi_\sigma z(t), \frac{\chi_B(t)\, H(\chi_\sigma z)(t)}{\|H(\chi_\sigma z)(t)\|_2}\right\rangle} \frac{(\chi_B(t))^{p(t)-2}\, d\mu(t)}{\beta\, \|\chi_B\|_N^{p(t)-1}} \\ &= \int_B \frac{\|H(\chi_\sigma z)(t)\|_2}{\beta}\, d\mu(t). \end{aligned}$$

Now, for any non-negative real-valued function g such that $\int_B g(t)\, d\mu(t) = 0$, g must be zero, otherwise we can see that, for $\varepsilon > 0$, there is a set of positive measure $B_\varepsilon \subset B$, such that $g > \varepsilon$ on B_ε. The following

$$0 = \int_B g(t)\, d\mu(t) \geq \int_{B_\varepsilon} g(t)\ d\mu(t) > \int_{B_\varepsilon} \varepsilon\, d\mu(t)$$
$$= \varepsilon\mu(B_\varepsilon),$$

leads to a contradiction.

We must have then $\|H(\chi_\sigma z)(t)\|_2 = 0$ on any set of positive measure $B \subset [0,1]\backslash\sigma$. We conclude that $\operatorname{supp} H(\chi_\sigma z)$ must be in σ, and since for any $z \in \mathcal{H}$

$$z = \chi_\sigma z + \chi_{[0,1]\backslash\sigma} z$$

and

$$H(\mathbf{z}) = H(\chi_\sigma z) + H\left(\chi_{[0,1]\backslash\sigma} z\right),$$

where $\mathbf{z}(t) = z$ for every $t \in [0,1]$, we have

$$\chi_\sigma H(\mathbf{z}) = H(\chi_\sigma z). \qquad \square$$

Proposition 6. *If H is a Hermitian operator on N, then, for every $z \in \mathcal{H}$, we have*

$$\|H(\mathbf{z})(t)\|_2 \leq \|H\|\,\|z\|_2 \textit{ almost everywhere.}$$

Proof. Let H be a Hermitian operator on N, σ a set of positive measure and $z \in \mathcal{H}$. The Hermitian operator H is assumed to be bounded on N; therefore, for any positive measure σ, we have

$$\|H(\chi_\sigma z)\|_N \leq \|H\|\,\|\chi_\sigma z\|_N,$$
$$\left\| \frac{H(\chi_\sigma z)}{\|H\|\,\|\chi_\sigma z\|_N} \right\|_N \leq 1,$$

and so

$$\int_\sigma \left(\frac{\|H(\mathbf{z})(t)\|_2}{\|H\|\,\|\chi_\sigma z\|_N} \right)^{p(t)} \frac{d\mu(t)}{p(t)} = M\left(\frac{H(\chi_\sigma z)}{\|H\|\,\|\chi_\sigma z\|_N} \right) \leq 1, \tag{2.2}$$

by the previous proposition.

We claim that

$$\|H(\mathbf{z})(t)\|_2 \leq \|H\|\,\|z\|_2 \ \text{ almost everywhere.}$$

Let $\varepsilon > 0$ and $\sigma = \{t \in [0,1] : (1+\varepsilon)\|H\|\|z\|_2 < \|H(\mathbf{z})(t)\|_2\}$. Assume that σ has positive measure. On σ, we have

$$\frac{(1+\varepsilon)\,\|z\|_2}{\|\chi_\sigma z\|_N} < \frac{\|H\,(\mathbf{z})\,(t)\|_2}{\|H\|\,\|\chi_\sigma z\|_N},$$

and therefore,

$$\int_\sigma \left(\frac{(1+\varepsilon)\,\|z\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)} \frac{d\mu\,(t)}{p\,(t)} < \int_\sigma \left(\frac{\|H\,(\mathbf{z})\,(t)\|_2}{\|H\|\,\|\chi_\sigma z\|_N}\right)^{p(t)} \frac{d\mu\,(t)}{p\,(t)} \le 1 \quad \text{by (2.2).} \tag{2.3}$$

We have

$$M\left(\frac{\chi_\sigma z}{\|\chi_\sigma z\|_N}\right) = \int_\sigma \left(\frac{\|z\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)} \frac{d\mu\,(t)}{p\,(t)} < \int_\sigma \left(\frac{(1+\varepsilon)\,\|z\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)} \frac{d\mu\,(t)}{p\,(t)} < 1.$$

But, we know that

$$\left\|\frac{\chi_\sigma z}{\|\chi_\sigma z\|_N}\right\|_N = 1,$$

and therefore we must have

$$M\left(\frac{\chi_\sigma z}{\|\chi_\sigma z\|_N}\right) = 1,$$

which is a contradiction with the previous relation. Therefore $\mu(\sigma) = 0$. Since on σ

$$\|H\,(\mathbf{z})\,(t)\|_2 > (1+\varepsilon)\,\|H\|\,\|z\|_2 > \|H\|\,\|z\|_2\,,$$

then

$$\|H\,(\mathbf{z})\,(t)\|_2 \le \|H\|\,\|z\|_2 \text{ almost everywhere.} \qquad \square$$

2.2. Main theorem. The next theorem gives us a characterization of the Hermitian operators on N.

Theorem 7. *The operator H is a Hermitian operator on N if and only if there is a strongly measurable map A of $[0,1]$ such that $A(\cdot)$ is a Hermitian $B(\mathcal{H})$-valued function, $A(\cdot)z \in N$, $\|A(t)\| \leq \|H\|$ and for every f in N*

$$(Hf)(t) = A(t)\, f(t) \ \textit{almost everywhere.}$$

Proof. The sufficiency of the theorem follows directly. To prove the necessity, let H to be a Hermitian operator on N and (e_n) an orthonormal basis of $\mathcal{H}$. For every n and every $t \in [0,1]$, we define

$$\mathbf{e}_n(t) = e_n.$$

Also, let $\mathcal{D}_0$ be the set of all finite linear combinations of (e_n) with rational coefficients. Then $\mathcal{D}_0$ is dense in $\mathcal{H}$.

For every n, let's define

$$f_n(t) = H(\mathbf{e}_n)(t) \ \text{almost everywhere.}$$

We will assume that a specific function rather than an equivalence class has been chosen (see [**8**, page 279]). We can see that, for scalars $\alpha_1, \dots, \alpha_n$, we have

$$H\left(\sum_{i=1}^{n} \alpha_i \mathbf{e}_i\right)(t) = \sum_{i=1}^{n} \alpha_i H(\mathbf{e}_i)(t) = \sum_{i=1}^{n} \alpha_i f_i(t),$$

outside of a set $E_{\alpha_1 \dots \alpha_n}$ of measure zero. If we let $E = \bigcup_{\alpha_i \in \mathbf{Q}} E_{\alpha_1 \dots \alpha_n}$, then E has measure zero. For every $t \in [0,1] \backslash E$ we define

$$A(t)\, e_n = f_n(t),$$

and we extend $A(t)$ linearly on $\mathcal{D}_0$

$$A(t)\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i f_i(t) = H\left(\sum_{i=1}^{n} \alpha_i \mathbf{e}_i\right)(t).$$

Hence for every $v \in \mathcal{D}_0$

$$A(t)\, v = H(\mathbf{v})(t) \ \text{ for } t \in [0,1] \backslash E.$$

We will extend $A(t)$ to a bounded operator on $\mathcal{H}$. Given $z \in \mathcal{H}$, there is a Cauchy sequence $(z_n) \in \mathcal{D}_0$ converging to z, and for $t \in [0,1]\backslash E$, we have

$$\begin{aligned}\|A(t)\,z_n - A(t)\,z_m\|_2 &= \|H(\mathbf{z}_n)(t) - H(\mathbf{z}_m)(t)\|_2 \\ &= \|H(\mathbf{z}_n - \mathbf{z}_m)(t)\|_2 \\ &\le \|H\|\,\|z_n - z_m\|_2 \to 0,\end{aligned}$$

by the previous proposition. This implies that $(A(t)z_n)$ is Cauchy in $\mathcal{H}$, so it must have a limit in $\mathcal{H}$. Then, for $z \in \mathcal{H}$, let

$$A(t)\,z = \lim_{n\to\infty} A(t)\,z_n = \lim_{n\to\infty} H(\mathbf{z}_n)(t) \quad \text{for every } t \text{ in } [0,1]\backslash E.$$

It can be seen that $A(t)$ is well defined. Also, $A(t)z$ is bounded since,

$$\begin{aligned}\|A(t)\,z\|_2 &= \lim_{n\to\infty} \|H(\mathbf{z}_n)(t)\|_2 \\ &\le \lim_{n\to\infty} \sup \|H(\mathbf{z}_n)(t)\|_2 \\ &\le \|H\|\,\|z\|_2 \ \text{ almost everywhere,}\end{aligned}$$

and, therefore,

$$\|A(t)\| \le \|H\| \ \text{ for every } t \text{ in } [0,1]\backslash E.$$

In addition,

$$M\left(\frac{A(\cdot)\,z}{\|H\|\,\|z\|_2}\right) = \int_0^1 \left(\frac{\|A(t)\,z\|_2}{\|H\|\,\|z\|_2}\right)^{p(t)} \frac{dt}{p(t)} \le 1.$$

Therefore $A(\cdot)z \in N$. We also have

$$M\left(\frac{A(\cdot)\,z - H(\mathbf{z}_n)(\cdot)}{\varepsilon}\right) = \int_0^1 \left(\frac{\|A(t)\,z - H(\mathbf{z}_n)(t)\|_2}{\varepsilon}\right)^{p(t)} \frac{d\mu(t)}{p(t)} \le 1$$

and

$$\|A(\cdot)\,z - H(\mathbf{z}_n)(\cdot)\|_N \le \varepsilon. \tag{2.4}$$

To prove that $A(\cdot)z$ is a strongly measurable function from $[0,1]$ to $\mathcal{H}$, we will fix $z \in \mathcal{H}$. Recall that a function $f : [0,1] \to \mathcal{H}$ is strongly measurable if it is the $\mu-$a.e. limit of a sequence of simple functions of the form $\sum_{i=1}^{n} x_i \chi_{E_i}$, where $x_i \in \mathcal{H}$ and $E_i \in \Sigma$. Let $(\varphi_m) = (\sum\limits_{i=1}^{k_m} x_i \chi_{E_i})$ to be a sequence of simple functions in N converging to z. Then, by Proposition 5, for each m,

$$H(\varphi_m)(t) = \sum_{i=1}^{k_m} H(x_i \chi_{E_i})(t) = \sum_{i=1}^{k_m} \chi_{E_i}(t) H(\mathbf{x}_i)(t)$$

is also a simple function, and using Proposition 6, we have

$$\begin{aligned} \|H(\mathbf{z})(t) - H(\varphi_m)(t)\|_2 &= \|H(\mathbf{z} - \varphi_m)(t)\|_2 \\ &\le \|H\| \, \|\mathbf{z}(t) - \varphi_m(t)\|_2 \longrightarrow 0. \end{aligned}$$

By the definition of $A(\cdot)z$, for every t in $[0,1]\backslash E$, $A(t)z = \lim_{m\to\infty} H(\varphi_m)(t)$, so we must have

$$\begin{aligned} &\|A(t)z - H(\mathbf{z})(t)\|_2 \\ &\qquad \le \|A(t)z - H(\varphi_m)(t)\|_2 + \|H(\mathbf{z})t - H(\varphi_m)(t)\|_2 \longrightarrow 0. \end{aligned}$$

Thus $A(t)z$ is strongly measurable for each $z \in \mathcal{H}$.

We claim now that $A(t) = A^*(t)$ for every t in $[0,1]\backslash E$. Since H is a Hermitian operator on N, the s.i.p $F_{\mathbf{z}}(H\mathbf{z}) \in \mathbf{R}$ for any $\mathbf{z} \in N$. In particular, $F_{\chi_\sigma z}(H\chi_\sigma z) \in \mathbf{R}$ for any set of positive measure $\sigma \subset [0,1]$:

$$\begin{aligned} F_{\chi_\sigma z}(H\chi_\sigma z) &= \frac{\|\chi_\sigma z\|_N^2 \int_0^1 \frac{\langle H(\chi_\sigma z)(t), \chi_\sigma z(t)\rangle}{\|\chi_\sigma z(t)\|_2} \left(\frac{\|\chi_\sigma z(t)\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)-1} d\mu(t)}{\int_0^1 \|\chi_\sigma z(t)\|_2 \left(\frac{\|\chi_\sigma z(t)\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)-1} d\mu(t)} \\ &= \frac{\|\chi_\sigma z\|_N^2 \int_0^1 \frac{\langle \chi_\sigma H(\mathbf{z})(t), (\chi_\sigma z)(t)\rangle}{\|\chi_\sigma z(t)\|_2} \left(\frac{\|\chi_\sigma z(t)\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)-1} d\mu(t)}{\int_0^1 \|\chi_\sigma z(t)\|_2 \left(\frac{\|\chi_\sigma z(t)\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)-1} d\mu(t)} \\ &= \frac{\|\chi_\sigma z\|_N^2 \int_\sigma \frac{\langle A(t)z, z\rangle}{\|\chi_\sigma z(t)\|_2} \left(\frac{\|\chi_\sigma z(t)\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)-1} d\mu(t)}{\int_0^1 \|\chi_\sigma z(t)\|_2 \left(\frac{\|\chi_\sigma z(t)\|_2}{\|\chi_\sigma z\|_N}\right)^{p(t)-1} d\mu(t)} \in \mathbf{R}. \end{aligned}$$

Recall that, if $\int_\sigma f\, d\mu = \int_\sigma \mathrm{Re}\,(f)\, d\mu + i \int_\sigma \mathrm{Im}\,(f)\, d\mu \in \mathbf{R}$, then

$$\int_\sigma \mathrm{Im}\ (f)\ d\mu = 0,$$

for any set of positive measure $\sigma \subset [0,1]$. Therefore $\mathrm{Im}\,(f) = 0$ almost everywhere, so $f \in \mathbf{R}$ almost everywhere. We must have then

$$\langle A\,(t)\, z, z\rangle \in \mathbf{R}$$

outside of a set of measure zero E_z. Since $\mathcal{D}_0$ is countable, there is a set $E_0 \in \Sigma$ of measure zero such that, for every $z \in \mathcal{D}_0$ and $t \in [0,1] \setminus (E_0 \bigcup E)$ we have $\langle A(t)z, z\rangle \in \mathbf{R}$. Also $\mathcal{D}_0$ is dense in $\mathcal{H}$ and the inner product $\langle \cdot, \cdot \rangle$ is continuous in both variables, so $\langle A(t)z, z\rangle \in \mathbf{R}$ for $t \in [0,1] \setminus (E_0 \bigcup E)$ and each $z \in \mathcal{H}$, hence $A(t) = A^*(t)$ almost everywhere.

We are left to prove that $(Hf)(t) = A(t)f(t)$ for f in N. Let's define a bounded linear operator M_A on N by

$$(M_A f)\,(t) = A\,(t)\, f\,(t) \ \text{ for almost every } t$$

(see [**4**, page 368]). We claim that M_A and H are the same for simple functions, and therefore for all functions on N. From (2.4) we have

$$H\,(\mathbf{z})\,(\cdot) = A\,(\cdot)\, z = M_A \mathbf{z}\,(\cdot)\,.$$

With this, for a simple function $\varphi = \sum_{i=1}^k x_i \chi_{E_i}$ we have

$$\begin{aligned}(M_A \varphi)\,(t) &= \left(M_A\left(\sum_{i=1}^k x_i \chi_{E_i}\right)\right)(t)\\ &= \sum_{i=1}^k \chi_{E_i}\,(t)\,(H\mathbf{x}_i)\,(t)\\ &= H\left(\sum_{i=1}^k x_i \chi_{E_i}\right)(t) = H\,(\varphi)\,(t)\,.\end{aligned}$$

Since the simple functions are dense in N, it follows that $M_A f = Hf$ for every $f \in N$, so we have

$$Hf\,(\cdot) = A\,(\cdot)\, f\,(\cdot)\,, \ \text{ for every } f \in N. \qquad \square$$

3. Isometries on N.

3.1. Preliminary results. Let $f \in N$ such that $\|f\|_N = 1$ and U is a surjective isometry of N. In what follows, we are interested in finding the form of the surjective isometries on N and for that we need the following results.

For $\sigma \in \Sigma$, define the operator C_σ on N by

$$(C_\sigma f)(t) = \chi_\sigma(t) f(t), \quad \text{for } f \in N.$$

If we consider the map M_A on N, defined as in the proof of Theorem 7 by $(M_A f)(t) = A(t)f(t)$ for each $A \in \Sigma$, we have

$$C_\sigma M_A = M_A C_\sigma. \tag{3.1}$$

In addition, the operator C_σ is a Hermitian projection on N. Since U is an isometry on N, it follows that the operator $UC_\sigma U^{-1}$ is a Hermitian projection on N (see [**4**, page 364]), and the previous theorem implies that, for $f \in N$

$$UC_\sigma U^{-1} f(\cdot) = P_\sigma(\cdot) f(\cdot),$$

where $P_\sigma(t)$ is a Hermitian projection for almost all $t \in [0,1]$. So

$$UC_\sigma U^{-1} = M_{P_\sigma}.$$

We will prove that

$$UC_\sigma U^{-1} = C_{\varphi^{-1}(\sigma)},$$

where φ^{-1} is a regular set isomorphism of Σ, using the following results.

Lemma 8. *If $A(\cdot)$ is strongly measurable, uniformly bounded Hermitian operator-valued function on $\mathcal{H}$, then*

$$M_{P_\sigma} M_A = M_A M_{P_\sigma} \text{ almost everywhere,}$$

where $M_{P_\sigma} = UC_\sigma U^{-1}$, $(C_\sigma f)(t) = \chi_\sigma(t) f(t)$ for $f \in N$ and U is a surjective isometry on N.

Proof. For any $f \in N$ with $\|f\|_N = 1$, we have

$$\begin{aligned} M(C_\sigma f) &= \int_0^1 \frac{\|C_\sigma f(t)\|_2^{p(t)}}{p(t)}\, d\mu(t) \\ &= \int_\sigma \frac{\|f(t)\|_2^{p(t)}}{p(t)}\, d\mu(t) \\ &\leq M(f) \leq 1, \end{aligned}$$

which is equivalent to $\|C_\sigma f\|_N \leq 1$. Since U is an isometry of N and $\|C_\sigma\| \leq 1$, we have

$$\begin{aligned} \|M_{P_\sigma} f\|_N &= \left\|U C_\sigma U^{-1} f\right\|_N = \left\|C_\sigma U^{-1} f\right\|_N \\ &\leq \|C_\sigma\| \left\|U^{-1} f\right\|_N = \|C_\sigma\| \|f\|_N \\ &\leq \|f\|_N = 1. \end{aligned}$$

By our assumptions, $A(\cdot)$ is a strongly measurable uniformly bounded Hermitian operator-valued function on $\mathcal{H}$. Then $U^{-1} M_A U$ defines a Hermitian operator on N, and therefore it must be of the form $M_{\tilde{A}} f(t) = \tilde{A}(t) f(t)$ almost everywhere, where $\tilde{A}(\cdot)$ is a Hermitian operator-valued on $\mathcal{H}$. By (3.1) we have

$$C_\sigma M_{\tilde{A}} = M_{\tilde{A}} C_\sigma \text{ almost everywhere,}$$

and so

$$\begin{aligned} M_{P_\sigma} M_A &= U C_\sigma U^{-1} M_A = U C_\sigma M_{\tilde{A}} U^{-1} \\ &= U M_{\tilde{A}} C_\sigma U^{-1} = M_A U C_\sigma U^{-1} \\ &= M_A M_{P_\sigma} \text{ almost everywhere.} \qquad \square \end{aligned}$$

Corollary 9. *If $T = T^*$, where $T(\cdot) \in \mathcal{B}(\mathcal{H})$ then $M_{P_\sigma} M_T = M_T M_{P_\sigma}$ almost everywhere.*

Corollary 10. *If $K(\cdot) \in \mathcal{B}(\mathcal{H})$, then $M_{P_\sigma} M_K = M_K M_{P_\sigma}$ almost everywhere.*

Proof. If $K(\cdot) \in \mathcal{B}(\mathcal{H})$ we can write

$$K = \frac{K + K^*}{2} + i \frac{K - K^*}{2i},$$

where $(K+K^*)/2$ and $(K-K^*)/2i$ are Hermitian operators. Applying the previous corollary to each $(K+K^*)/2$, $(K-K^*)/2i$, we have

$$M_{P_\sigma} M_{(K+K^*)/2} = M_{(K+K^*)/2} M_{P_\sigma} \text{ almost everywhere,}$$

and

$$M_{P_\sigma} M_{(K-K^*)/2i} = M_{(K-K^*)/2i} M_{P_\sigma} \text{ almost everywhere.}$$

It is easy to see that $M_K M_{P_\sigma} = M_{P_\sigma} M_K$ almost everywhere, since

$$\begin{aligned} M_K f(t) &= K(t) f(t) \\ &= \frac{K+K^*}{2}(t) f(t) + i\frac{K-K^*}{2i}(t) f(t) \\ &= M_{(K+K^*)/2} f(t) + i M_{(K-K^*)/2i} f(t), \end{aligned}$$

and therefore

$$\begin{aligned} M_K M_{P_\sigma} &= M_{(K+K^*)/2} M_{P_\sigma} + i M_{(K-K^*)/2i} M_{P_\sigma} \\ &= M_{P_\sigma} M_{(K+K^*)/2} + i M_{P_\sigma} M_{(K-K^*)/2i} \\ &= M_{P_\sigma} M_K \text{ almost everywhere.} \qquad \square \end{aligned}$$

Lemma 11. *For each $\sigma \in \Sigma$, there exist a regular set isomorphism φ^{-1} of Σ such that*

$$U C_\sigma U^{-1} = C_{\varphi^{-1}(\sigma)}.$$

Proof. By the previous corollary, for any $z \in \mathcal{H}$ and for any $K(\cdot) \in \mathcal{B}(\mathcal{H})$, there is a set $E(z, K, \sigma)$ of measure zero, such that

$$P_\sigma(t) K(t) z = K(t) P_\sigma(t) z, \tag{3.2}$$

for every t outside of $E(z, K, \sigma)$.

Let $u, v \in \mathcal{H}$, such that $\|u\|_2 = 1$, $\|v\|_2 = 1$, and for a vector $w \in \mathcal{H}$, we define the constant function $\mathbf{w}(t) = w$ for every $t \in [0,1]$, and

$$K(\cdot)\mathbf{w}(\cdot) = (u \otimes v) w = \langle w, v \rangle u.$$

Then, we have

$$\|K(t)\,\mathbf{w}(t)\|_2 \le \|w\|_2\,\|v\|_2\,\|u\|_2 = \|w\|_2\,.$$

Also, by separability of $\mathcal{H}$, there is a countable dense set $\mathcal{H}_0$ in $\mathcal{H}$ and two sequences (u_n), $(v_n) \in \mathcal{H}_0$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$. We define

$$K_n(t)\,\mathbf{w}(t) = \langle w, v_n\rangle u_n, \quad t \in [0,1]\,,$$

which converges in norm to K. To see that, let $\|w\|_2 = 1$; we compute

$$\begin{aligned} K(t)\,\mathbf{w}(t) - K_n(t)\,\mathbf{w}(t) &= \langle w, v\rangle u - \langle w, v_n\rangle u_n \\ &= \langle w, v\rangle u - \langle w, v_n\rangle u \\ &\quad + \langle w, v_n\rangle u - \langle w, v_n\rangle u_n \\ &= \langle w, v - v_n\rangle u + \langle w, v_n\rangle\,(u - u_n) \end{aligned}$$

and

$$\begin{aligned} &\|K(t)\,\mathbf{w}(t) - K_n(t)\,\mathbf{w}(t)\|_2 \\ &\qquad \le \|w\|_2\,\|v - v_n\|_2\,\|u\|_2 + \|w\|_2\,\|v_n\|_2\,\|u - u_n\|_2\,, \end{aligned}$$

which tends to zero as $n \to \infty$. Therefore

$$\|K - K_n\| \to 0, \quad \text{as } n \to \infty.$$

By (3.2), for $t \in [0,1] \setminus E(z, K_n, \sigma)$, we have

$$P_\sigma(t)\,K_n(t)\,z = K_n(t)\,P_\sigma(t)\,z.$$

Let $E(z,\sigma) = \cup_{n\ge1} E(z, K_n, \sigma)$; we can see that the measure of $E(z,\sigma)$ is zero and, for $t \in [0,1] \setminus E(z,\sigma)$,

$$\begin{aligned} 0 &\le \|P_\sigma(t)\,K(t)\,z - K(t)\,P_\sigma(t)\,z\|_2 \\ &\le \|P_\sigma(t)\,K(t)\,z - P_\sigma(t)\,K_n(t)\,z\|_2 \\ &\quad + \|K_n(t)\,P_\sigma(t)\,z - K(t)\,P_\sigma(t)\,z\|_2 \\ &\le \|P_\sigma(t)\|_2\,\|K - K_n\|\,\|z\|_2 \\ &\quad + \|K_n - K\|\,\|P_\sigma(t)\|_2\,\|z\|_2 \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore,

$$P_\sigma(t)\,K(t)\,z = K(t)\,P_\sigma(t)\,z, \text{ for } t \in [0,1] \setminus E(z,\sigma).$$

Also, we can find a countable dense set $\mathcal{H}_1$ in $\mathcal{H}$, such that for any $z \in \mathcal{H}$, there is a sequence $(z_n) \in \mathcal{H}_1$ such that $z_n \to z$ and

$$P_\sigma(t)\,K(t)\,z_n = K(t)\,P_\sigma(t)\,z_n,$$

for every $t \in [0,1] \setminus E(z_n,\sigma)$. Let $E(\sigma) = \cup_{n\geq 1} E(z_n,\sigma)$; we can see that the measure of $E(\sigma)$ is zero and for $t \in [0,1] \setminus E(\sigma)$,

$$\begin{aligned}
0 &\leq \|P_\sigma(t)\,K(t)\,z - K(t)\,P_\sigma(t)\,z\|_2 \\
&\leq \|P_\sigma(t)\,K(t)\,z - P_\sigma(t)\,K(t)\,z_n\|_2 \\
&\quad + \|K(t)\,P_\sigma(t)\,z_n - K(t)\,P_\sigma(t)\,z\|_2 \\
&\leq \|P_\sigma(t)\|_2\,\|K\|\,\|z - z_n\|_2 \\
&\quad + \|K\|\,\|P_\sigma(t)\|_2\,\|z - z_n\|_2 \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, we obtain a measure zero set $E(\sigma)$ outside of which

$$P_\sigma(t)\,K(t)\,z = K(t)\,P_\sigma(t)\,z. \tag{3.3}$$

Given $\sigma \in \Sigma$, we define

$$S_\sigma = \{t \in [0,1] : P_\sigma(t) \neq 0\} = \Big\{t \in [0,1] : \text{there exists } z_t \text{ such that } P_\sigma(t)\,z_t \neq 0\Big\}.$$

Suppose there is a subset of positive measure $\sigma_1 \subset S_\sigma \cap ([0,1] \setminus E(\sigma))$ such that $P_\sigma(t) \neq \mathrm{Id}_{\mathcal{H}}$, for every $t \in \sigma_1$. If we let $t \in \sigma_1$, there exist v_1 and $v_2 \in \mathcal{H}$ with $\|v_1\|_2 = \|v_2\|_2 = 1$, such that $P_\sigma(t)v_1 = v_1$ and $P_\sigma(t)v_2 = 0$. Let $K(\cdot)w = \langle w, v_1\rangle v_2$. Then

$$P_\sigma(t)\,K(t)\,v_1 = 0 \neq v_2 = K(t)\,P_\sigma(t)\,v_1,$$

which is a contradiction to (3.3). Therefore, on every set of positive measure $\sigma_1 \subset S_\sigma \cap ([0,1] \setminus E(\sigma))$, we have $P_\sigma(\cdot) = \mathrm{Id}_{\mathcal{H}}$. Define $\varphi^{-1}(\sigma) = \{t \in S_\sigma \cap ([0,1] \setminus E(\sigma)) : P_\sigma(t) = \mathrm{Id}_{\mathcal{H}}\}$. It follows that φ^{-1} is a regular set isomorphism of Σ ([**2**, page 141]),

$$P_\sigma(\cdot) = \chi_{\varphi^{-1}(\sigma)}(\cdot)\, Id_{\mathcal{H}}$$

and $UC_\sigma U^{-1} = C_{\varphi^{-1}(\sigma)}$. □

Remark 12. If φ is a regular set isomorphism of Σ, then $\mu \circ \varphi^{-1}$ is an absolutely continuous measure with respect to μ. If we let u be its Radon-Nikodym derivative, i.e., $\mu(\varphi^{-1}(\sigma)) = \int_\sigma u(t)\, d\mu(t)$ for any σ, then we have $\mu(\sigma) = \int_{\varphi(\sigma)} u(t)\, d\mu(t) = \int \chi_\sigma(\varphi^{-1}(t)) u(t)\, d\mu(t)$. It can easily be shown that

$$\int_\sigma f(t)\ d\mu(t) = \int_{\varphi(\sigma)} f\left(\varphi^{-1}(t)\right) u(t)\ d\mu(t) \tag{3.4}$$

and

$$\int_{\varphi(\sigma)} f(t)\ d\mu(t) = \int_\sigma f(\varphi(t)) \left[u(\varphi(t))\right]^{-1} d\mu(t)\,.$$

3.2. Main theorem. The next theorem gives us a characterization of the surjective isometries on N.

Theorem 13. *If U is a surjective isometry on N, then there is a regular set isomorphism φ^{-1} of Σ, a strongly measurable map V of $[0,1]$ into $B(\mathcal{H})$ such that $V(t)$ is an isometry of $\mathcal{H}$ onto itself for almost all $t \in [0,1]$, and u is a measurable function that satisfies $\mu(\sigma) = \int_{\varphi(\sigma)} u(t)\, d\mu(t)$ such that $p(t) = p(\varphi^{-1}(t))$ almost everywhere and*

$$Uf(t) = \left[u(\varphi(t))\right]^{-1/p(t)} V(\varphi(t))\, f(\varphi(t))\,.$$

Conversely, if there is a regular set isomorphism φ^{-1} of Σ such that $p(t) = p(\varphi^{-1}(t))$ almost everywhere, a strongly measurable map V of $[0,1]$ into $B(\mathcal{H})$ such that $V(t)$ is an isometry of $\mathcal{H}$ onto itself for almost all $t \in [0,1]$, and u a measurable function that satisfies $\mu(\sigma) = \int_{\varphi(\sigma)} u(t)\, d\mu(t)$ such that

$$Uf(t) = \left[u(\varphi(t))\right]^{-1/p(t)} V(\varphi(t))\, f(\varphi(t))\,.$$

then U is a surjective isometry on N.

Proof. Let σ be a set of positive measures defined on Σ, and let U be a surjective isometry of the space N. Based on Lemma 11, for any $z \in \mathcal{H}$, we have

$$U(\chi_\sigma z) = U(C_\sigma \mathbf{z}) = U\left(C_\sigma U^{-1} U\mathbf{z}\right) = C_{\varphi^{-1}(\sigma)} U\mathbf{z} = \chi_{\varphi^{-1}(\sigma)} U\mathbf{z},$$

where φ is a regular set isomorphism of Σ and therefore, for a function $f \in N$,

$$U(\chi_\sigma f) = \chi_{\varphi^{-1}(\sigma)} Uf = (\chi_\sigma \circ \varphi) Uf.$$

We can extend this relation linearly to have

$$U\bigg(\sum_{i=1}^{n} \alpha_i \chi_{\sigma_i} f\bigg)(t) = \bigg(\sum_{i=1}^{n} \alpha_i \chi_{\sigma_i}(\varphi(t))\bigg) Uf(t)$$

and, therefore, for any scalar function h on $[0, 1]$, we have

$$U(hf)(t) = h(\varphi(t))\, Uf(t)\,. \tag{3.5}$$

If $f \in N$ such that $\|f(t)\|_2 = 1$ and h is a scalar function on $[0, 1]$ with

$$\int_0^1 \frac{|h(t)|^{p(t)}}{p(t)}\, d\mu(t) = 1, \tag{3.6}$$

we have $M(hf) = 1$, and therefore $\|hf\|_N = 1$. Since U is an isometry on N, we have $\|U(hf)\|_N = 1$ so, by (3.5), that gives us

$$1 = M(U(hf)) = \int_0^1 \frac{|h(\varphi(t))|^{p(t)}}{p(t)} \|g(t)\|_2^{p(t)}\, d\mu(t)\,,$$

where $g = U(f)$. If we make a change of variables, by (3.4), the previous relation changes to

$$\int_0^1 \frac{|h(t)|^{p\left(\varphi^{-1}(t)\right)}}{p(\varphi^{-1}(t))} \left\|g\left(\varphi^{-1}(t)\right)\right\|_2^{p\left(\varphi^{-1}(t)\right)} u(t)\; d\mu(t) = 1, \tag{3.7}$$

where u is the Radon-Nikodym derivation of $\mu \circ \varphi^{-1}$ with respect to μ. The relation (3.7) is true for any scalar function h on $[0, 1]$ that satisfies (3.6).

Next, we claim that $p(t) = p(\varphi^{-1}(t))$ almost everywhere. Let $A = \{t \in [0,1] : p(t) < \gamma < \beta < p(\varphi^{-1}(t))$, for some positive γ and $\beta\}$. If we assume that A has positive measure, we can find two disjoint positive measure subsets A_1 and A_2 of A such that $A_1 \cup A_2 = A$. We select two positive scalars α_1 and α_2 such that

$$\int_{A_1} \frac{\alpha_1^{p(t)}}{p(t)}\, d\mu(t) = 1 \quad \text{and} \quad \int_{A_2} \frac{\alpha_2^{p(t)}}{p(t)}\, d\mu(t) = 1. \tag{3.8}$$

By (3.7) we must have

$$\int_{A_1} \frac{\alpha_1^{p(\varphi^{-1}(t))}}{p(\varphi^{-1}(t))} \left\| g\left(\varphi^{-1}(t)\right)\right\|_2^{p(\varphi^{-1}(t))} u(t)\, d\mu(t) = 1 \tag{3.9}$$

and

$$\int_{A_2} \frac{\alpha_2^{p(\varphi^{-1}(t))}}{p(\varphi^{-1}(t))} \left\| g\left(\varphi^{-1}(t)\right)\right\|_2^{p(\varphi^{-1}(t))} u(t)\, d\mu(t) = 1. \tag{3.10}$$

Now, let c_1 and c_2 be two positive scalars such that $\|c_1\chi_{A_1}\alpha_1 + c_2\chi_{A_2}\alpha_2\|_N = 1$. We can see that $c_1, c_2 \leq 1$. If we let $h = c_1\chi_{A_1}\alpha_1 + c_2\chi_{A_2}\alpha_2$, by the previous relation we have $\|h\|_N = 1$, so

$$1 = M(h) = \int_0^1 \frac{|(c_1\chi_{A_1}\alpha_1 + c_2\chi_{A_2}\alpha_2)(t)|^{p(t)}}{p(t)}\, d\mu(t).$$

The sets A_1 and A_2 were chosen to be disjoint, and the scalars α_1 and α_2 were chosen to satisfy (3.8), so we have

$$1 = \int_{A_1} \frac{c_1^{p(t)}\alpha_1^{p(t)}}{p(t)}\, d\mu(t) + \int_{A_2} \frac{c_2^{p(t)}\alpha_2^{p(t)}}{p(t)}\, d\mu(t) > c_1^{\gamma} + c_2^{\gamma}.$$

On the other hand, we have $1 = M(U(h))$, so by (3.7) we have

$$1 = \int_0^1 \frac{|(c_1\chi_{A_1}\alpha_1 + c_2\chi_{A_2}\alpha_2)(t)|^{p(\varphi^{-1}(t))}}{p(\varphi^{-1}(t))} \left\| g\left(\varphi^{-1}(t)\right)\right\|_2^{p(\varphi^{-1}(t))} \times u(t)\, d\mu(t).$$

Similarly, it follows from (3.9) and (3.10) that

$$\begin{aligned} 1 &= \int_{A_1} \frac{c_1^{p\left(\varphi^{-1}(t)\right)} \alpha_1^{p\left(\varphi^{-1}(t)\right)}}{p\left(\varphi^{-1}(t)\right)} \left\| g\left(\varphi^{-1}(t)\right)\right\|_2^{p\left(\varphi^{-1}(t)\right)} u(t)\, d\mu(t) \\ &\quad + \int_{A_2} \frac{c_2^{p\left(\varphi^{-1}(t)\right)} \alpha_2^{p\left(\varphi^{-1}(t)\right)}}{p\left(\varphi^{-1}(t)\right)} \left\| g\left(\varphi^{-1}(t)\right)\right\|_2^{p\left(\varphi^{-1}(t)\right)} u(t)\, d\mu(t) \\ &< c_1^{\beta} + c_2^{\beta} < c_1^{\gamma} + c_2^{\gamma}, \end{aligned}$$

which contradicts the previous relation obtained. Therefore $\mu(A) = 0$, so, outside of a set A of measure zero, we have

$$p(t) \geq p\left(\varphi^{-1}(t)\right).$$

Using a similar argument, we can prove that, outside of a set A' of measure zero,

$$p(t) \leq p\left(\varphi^{-1}(t)\right).$$

Consequently,

$$p(t) = p\left(\varphi^{-1}(t)\right) \text{ almost everywhere.}$$

With this, (3.7) becomes
(3.11)

$$\int_0^1 \frac{|h(t)|^{p(t)}}{p(t)} \left\| g\left(\varphi^{-1}(t)\right)\right\|_2^{p(t)} u(t)\, d\mu(t) = 1 \text{ almost everywhere,}$$

whenever

$$\int_0^1 \frac{|h(t)|^{p(t)}}{p(t)} = 1.$$

Now, in a similar way, we prove that $\|g(\varphi^{-1}(t))\|_2^{p(t)} u(t) = 1$ almost everywhere. We assume that there is a set of positive measure $B = \{t \in [0,1] : \|g(\varphi^{-1}(t))\|_2^{p(t)} > 1/u(t)\}$. If we let h be a scalar function with support in B that satisfies (3.6), by (3.11) we have outside of a zero measure set A,

$$\begin{aligned} 1 &= \int_0^1 \frac{|h(t)|^{p(t)}}{p(t)} \left\| g\left(\varphi^{-1}(t)\right)\right\|_2^{p(t)} u(t)\, d\mu(t) \\ &> \int_B \frac{|h(t)|^{p(t)}}{p(t)}\, d\mu(t) = 1, \end{aligned}$$

which is a contradiction, so we must have $\mu(B) = 0$. In a similar fashion we can prove that the set B', on which $\|g(\varphi^{-1}(t))\|_2^{p(t)} < 1/u(t)$, must have measure zero; therefore, $\|g(\varphi^{-1}(t))\|_2^{p(t)} = 1/u(t)$ outside of the measure zero set $(A \cup A' \cup B \cup B')$. Replacing back $g = U(f)$, we have

$$\left\|U\left(f\right)\left(\varphi^{-1}\left(t\right)\right)\right\|_{2} = \frac{1}{u\left(t\right)^{1/p(t)}} \text{ almost everywhere,}$$

for a function $f \in N$ such that $\|f(t)\|_2 = 1$. Therefore, for any $f \in N$, we have

$$\left\|U\left(f\right)\left(\varphi^{-1}\left(t\right)\right)\right\|_{2} u\left(t\right)^{1/p(t)} = \left\|f\left(t\right)\right\|_{2} \text{ almost everywhere.} \tag{3.12}$$

Since $\mathcal{H}$ is separable, there is a dense linear span $\mathcal{D}_0$ of all linear combinations with rational coefficients of an orthonormal basis of $\mathcal{H}$. For every element of (e_n), where $e_n \in \mathcal{D}_0$ for every $n \geq 1$, let's define the operator $V(t)$ by

$$V\left(t\right)e_n = U\left(\mathbf{e}_n\right)\left(\varphi^{-1}\left(t\right)\right)\left[u\left(t\right)\right]^{1/p(t)},$$

where t is in $[0,1]$ outside of a set of measure zero σ_n. By (3.12), we can see that $V(t)$ is a linear isometry almost everywhere on the subspace $\mathcal{D}_0$, and for any $t \in [0,1] \setminus (\cup\sigma_n)$ and any $w = \sum \lambda_j e_j \in \mathcal{D}_0$, we have

$$\begin{aligned}
V\left(t\right)w &= V\left(t\right)\left(\sum \lambda_j e_j\right) \\
&= \sum \lambda_j V\left(t\right)e_j \\
&= \sum \lambda_j U\left(\mathbf{e}_j\right)\left(\varphi^{-1}\left(t\right)\right)\left[u\left(t\right)\right]^{1/p(t)} \\
&= U\left(\sum \lambda_j \mathbf{e}_j\right)\left(\varphi^{-1}\left(t\right)\right)\left[u\left(t\right)\right]^{1/p(t)} \\
&= U\left(\mathbf{w}\right)\left(\varphi^{-1}\left(t\right)\right)\left[u\left(t\right)\right]^{1/p(t)}.
\end{aligned}$$

For a $z \in \mathcal{H}$, there is a sequence $(w_n) \in \mathcal{D}_0$ converging to z. Since (w_n) is Cauchy, it follows that $(V(t)w_n)$ is Cauchy for any $t \in [0,1] \setminus \sigma$ with

$\sigma = \cup \sigma_n$, since

$$\begin{aligned}
\|V(t)\, w_k - V(t)\, w_m\|_2 \\
&= \Big\| U(\mathbf{w}_k)\left(\varphi^{-1}(t)\right)[u(t)]^{1/p(t)} \\
&\quad - U(\mathbf{w}_m)\left(\varphi^{-1}(t)\right)[u(t)]_2^{1/p(t)} \Big\|_2 \\
&= \left\|[U(\mathbf{w}_k) - U(\mathbf{w}_m)]\left(\varphi^{-1}(t)\right)\right\|_2 [u(t)]^{1/p(t)} \\
&= \left\|U(\mathbf{w}_k - \mathbf{w}_m)\left(\varphi^{-1}(t)\right)\right\|_2 [u(t)]^{1/p(t)} \\
&= \|w_k - w_m\|_2 \to 0, \text{ a.e. by (3.12)}.
\end{aligned}$$

Hence, we can define

$$\lim_{n\to\infty} V(t)\, w_n = V(t)\, z.$$

To see that this is well defined, let (z_n) be another sequence of $\mathcal{D}_0$ converging to z. We have

$$\|z_n - w_n\|_2 \le \|z_n - z\|_2 + \|w_n - z\|_2 \longrightarrow 0$$

and, as before,

$$\|V(t)\, w_n - V(t)\, z_n\|_2 = \|w_n - z_n\|_2 \to 0 \text{ almost everywhere.}$$

Therefore,

$$\begin{aligned}
\|V(t)\, z_n - V(t)\, z\|_2 \\
&\le \|V(t)\, z_n - V(t)\, w_n\|_2 - \|V(t)\, w_n - V(t)\, z\|_2 \longrightarrow 0,
\end{aligned}$$

which says that

$$\lim_{n\to\infty} V(t)\, z_n = V(t)\, z.$$

From the fact that

$$\|z\|_2 = \lim_{n\to\infty} \|w_n\|_2 = \lim_{n\to\infty} \|V(t)\, w_n\|_2 = \|V(t)\, z\|_2\,,$$

it yields that $V(t)$ is an isometry almost everywhere on $\mathcal{H}$.

Let's now define

$$Wf(t) = [u(t)]^{-1/p(t)} V(t) f(t).$$

We claim that $U(f)(t)$ agrees with $Wf(\varphi(t))$ since, by $p(\varphi^{-1}(t)) = p(t)$, we have

$$\begin{aligned} W(\chi_\sigma z)(\varphi(t)) &= [u(\varphi(t))]^{-1/p(t)} V(\varphi(t)) (\chi_\sigma z)(\varphi(t)) \\ &= [u(\varphi(t))]^{-1/p(t)} U(\chi_\sigma z)\left(\varphi^{-1}(\varphi(t))\right) [u(\varphi(t))]^{1/p(t)} \\ &= U(\chi_\sigma z)(t). \end{aligned}$$

Since $U(f)(t)$ agrees with $Wf(\varphi(t))$ for simple functions f, they must agree for any function of the Nakano space N. Therefore

$$U(f)(t) = Wf(\varphi(t)) = [u(\varphi(t))]^{-1/p(t)} V(\varphi(t)) f(\varphi(t)).$$

Now, for the sufficiency, assume that there is a regular set isomorphism φ^{-1} of Σ such that $p(t) = p(\varphi^{-1}(t))$ almost everywhere, a strongly measurable map V of $[0,1]$ into $B(\mathcal{H})$ such that $V(t)$ is an isometry of $\mathcal{H}$ onto itself for almost all $t \in [0,1]$, and u a measurable function that satisfies $\mu(\sigma) = \int_{\varphi(\sigma)} u(t)\, d\mu(t)$ such that

$$Uf(t) = [u(\varphi(t))]^{-1/p(t)} V(\varphi(t)) f(\varphi(t)).$$

Let's compute

$$\begin{aligned} M(Uf) &= \int_0^1 \frac{\|Uf(t)\|_2^{p(t)}}{p(t)}\, d\mu(t) \\ &= \int_0^1 \frac{\left\|[u(\varphi(t))]^{-1/p(t)} V(\varphi(t)) f(\varphi(t))\right\|_2^{p(t)}}{p(t)}\, d\mu(t) \\ &= \int_{\varphi([0,1])} \frac{\left\|[u(\xi)]^{-1/p\left(\varphi^{-1}(\xi)\right)} V(\xi) f(\xi)\right\|_2^{p\left(\varphi^{-1}(\xi)\right)}}{p(\varphi^{-1}(\xi))} u(\xi)\, d\mu(\xi) \\ &= \int_0^1 \frac{\|V(\xi) f(\xi)\|_2^{p\left(\varphi^{-1}(\xi)\right)}}{p(\varphi^{-1}(\xi))}\, d\mu(\xi) \\ &= \int_0^1 \frac{\|f(\xi)\|_2^{p\left(\varphi^{-1}(\xi)\right)}}{p(\varphi^{-1}(\xi))}\, d\mu(\xi) \end{aligned}$$

since $V(\xi)$ is an isometry of $\mathcal{H}$ almost everywhere.

By the assumption that $p(\varphi^{-1}(\xi)) = p(\xi)$ almost everywhere, it follows that $M(Uf) = M(f)$ almost everywhere, and since the modular isometries are isometries, we have $\|Uf\|_N = \|f\|_N$ almost everywhere. This concludes our proof. $\square$

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