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AN IDENTITY OF THE SYMMETRY FOR THE FROBENIUS-EULER POLYNOMIALS ASSOCIATED WITH THE FERMIONIC p-ADIC INVARIANT q-INTEGRALS ON $\mathbf{Z}_p$

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ABSTRACT. The main purpose of this paper is to prove an identity of symmetry for the Frobenius-Euler polynomials. It turns out that the recurrence relation and multiplication theorem for the Frobenius-Euler polynomials which discussed in [**18**]. Finally we investigate several further interesting properties of symmetry for the Fermionic p-adic invariant q-integral on $\mathbf{Z}_p$ associated with the Frobenius-Euler polynomials and numbers.

1. Introduction. The nth Frobenius-Euler numbers $H_n(q)$ and the nth Frobenius-Euler polynomials $H_n(q,x)$ attached to an algebraic number $q(\neq 1)$ may be defined by the exponential generating functions

$$\begin{aligned}\sum_{n=1}^{\infty} H_n(q)\frac{t^n}{n!} &= \frac{1-q}{e^t-q}, \text{ see [6, 7]},\\ \sum_{n=0}^{\infty} H_n(q,x)\frac{t^n}{n!} &= \frac{1-q}{e^t-q}e^{xt}.\end{aligned} \tag{1}$$

It is easy to show that $H_n(q,x)=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}H_l(q)$. Let p be a fixed prime. Throughout this paper $\mathbf{Z}_p$, $\mathbf{Q}_p$, $\mathbf{C}$ and $\mathbf{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbf{Q}_p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex $q\in\mathbf{C}$, or a p-adic number $q\in\mathbf{C}_p$, see [**7–17, 19, 20, 21**]. If $q\in\mathbf{C}$, then we assume $|q|<1$. If $q\in c_p$, then we assume $|1-q|_p<1$. For $x\in\mathbf{Q}_p$, we use the notation $[x]_q=(1-q^x)/(1-q)$ and $[x]_{-q}=(1-(-q)^x)/(1+q)$, see [**5, 6**].

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The normalized valuation in $\mathbf{C}_p$ is denoted by $|\cdot|_p$ with $|p|_p = 1/p$. We say that f is a uniformly differentiable function at a point $a \in \mathbf{Z}_p$ and denote this property by $f \in UD(\mathbf{Z}_p)$, if the difference quotients $F_f(x, y) = f(x) - f(y)/x - y$ have a limit $l = f'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbf{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \le j < p^N} q^j f(j) = \sum_{0 \le j < p^N} f(j) \mu_q(j + p^N \mathbf{Z}_p),$$

representing a q-analogue of Riemann sums for f, see [**5, 6**]. The integral of f on $\mathbf{Z}_p$ will be defined as limit $(n \to \infty)$ of those sums, when it exists. The q-deformed bosonic p-adic integral of the function $f \in UD(\mathbf{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbf{Z}_p} f(x)\, d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{0 \le x < dp^N} f(x) q^x, \text{ see [5].}$$

Thus, we note that

$$qI_q(f_1) = I_q(f) + (q-1)f(0) + \frac{q-1}{\log q} f'(0),$$

where $f_1(x) = f(x+1)$, $f'(0) = df(0)/dx$.

The Fermionic p-adic invariant q-integral on $\mathbf{Z}_p$ is defined as
(2)

$$I_{-q}(f) = \int_{\mathbf{Z_p}} f(x)\, d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N - 1} f(x)(-q)^x, \text{ see [5].}$$

In [**22**], Tuenter provided a generalization of the Bernoulli number recurrence

$$B_m = \frac{1}{a(1-a^m)} \sum_{j=0}^{m-1} a^j \binom{m}{j} B_j \sum_{i=0}^{a-1} i^{m-j}, \text{ see [1, 3, 4],}$$

where $a, m \in \mathbf{Z}$ with $a > 1$, $m \ge 1$, attributed to Deeba and Rodriguez [**3**] and to Gessel [**1**]. Define $S_m(k) = 0^m + 1^m + \cdots + k^m$, where $a, m \in \mathbf{Z}$, with $a \ge 0$ and $m \ge 0$. Tuenter proved that the quantity

$$\sum_{j=0}^{m} \binom{m}{j} a^{j-1} B_j b^{m-j} S_{m-j}(a-1), \text{ see [22],}$$

is symmetric in a and b, provided $a, b, m \in \mathbf{Z}$, with $a > 0$,$b > 0$ and $m \geq 0$. In this paper we prove an identity of symmetry for the Frobenius-Euler polynomials. It turns out that the recurrence relation and multiplication theorem for the Frobenius-Euler polynomials was discussed in [**18**]. Finally we investigate the several further interesting properties of the symmetry for the Fermionic p-adic invariant q-integral on $\mathbf{Z}_p$ associated with the Frobenius-Euler polynomials and numbers.

2. An identity of symmetry for the Frobenius-Euler polynomials. From (2) we can derive

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ where } f_1(x) = f(x+1). \tag{3}$$

By continuing this process, we see that

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l),$$
$$\text{where } f_n(x) = f(x+n).$$

When n is an odd positive integer, we obtain

$$q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l f(l) q^l. \tag{4}$$

If $n \in \mathbf{N}$ with $n \equiv 0 \pmod 2$, then we have

$$q^n I_{-q}(f_n) - I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} f(l) q^l. \tag{5}$$

From (1) and (3) we derive

$$\int_{\mathbf{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{1-(-q)^{-1}}{e^t - (-q)^{-1}} = \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}. \tag{6}$$

Thus, we note that

$$\int_{\mathbf{Z}_p} x^n d\mu_{-q}(x) = H_n(-q^{-1}),$$

and

$$\int_{\mathbf{Z}_p} (y+x)^n d\mu_{-q}(x) = H_n(-q^{-1}, x).$$

Let $n \in \mathbf{N}$ with $n \equiv 1 \pmod 2$. Then we obtain

$$[2]_q \sum_{l=0}^{n-1} (-1)^l q^l l^m = q^n H_m(-q^{-1}, n) + H_m(-q^{-1}).$$

For $n \in \mathbf{N}$ with $n \equiv 0 \pmod 2$, we have

$$q^n H_m(-q^{-1}, n) - H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l l^m.$$

By substituting $f(x) = e^{xt}$ into (4), we can easily see that

$$\begin{aligned}\int_{\mathbf{Z}_p} q^n e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbf{Z}_p} e^{xt} d\mu_{-q}(x) &= [2]_q \frac{q^n e^{nt}+1}{qe^t+1} \\ &= [2]_q \sum_{l=0}^{n-1} (-1)^l q^l e^{lt}.\end{aligned} \tag{7}$$

Let $S_{k,q}(n) = \sum_{l=0}^{n} (-1)^l l^k q^l$. Then $S_{k,q}(n)$ is called the alternating sums of powers of consecutive q-integers. From the definition of the Fermionic p-adic invariant q-integral on $\mathbf{Z}_p$, we can derive

$$\int_{\mathbf{Z}_p} q^n e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbf{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q \int_{\mathbf{Z}_p} e^{xt} d\mu_{-q}(x)}{\int_{\mathbf{Z}_p} e^{nxt} q^{(n-1)x} d\mu_{-q}(x)}. \tag{8}$$

By (8), we easily see that

$$\int_{\mathbf{Z}_p} q^{(n-1)x} e^{nxt} d\mu_{-q}(x) = \frac{1+q}{q^n e^{nt}+1}.$$

Let $w_1, w_2 (\in \mathbf{N})$ be odd. By using double Fermionic p-adic invariant q-integral on $\mathbf{Z}_p$, we obtain

$$\frac{\int_{\mathbf{Z}_p}\int_{\mathbf{Z}_p} e^{(w_1x_1+w_2x_2)t}\, d\mu_{-q}(x_1)\, d\mu_{-q}(x_2)}{\int_{\mathbf{Z}_p} e^{w_1w_2xt} q^{(w_1w_2-1)x}\, d\mu_{-q}(x)} = \frac{[2]_q (q^{w_1w_2} e^{w_1w_2t}+1)}{(qe^{w_1t}+1)(qe^{w_2t}+1)}.$$

Now we also consider the following Fermionic p-adic invariant q-integral on $\mathbf{Z}_p$ associated with Frobenius-Euler polynomials.

$$
\begin{aligned}
I &= \frac{\int_{\mathbf{Z}_p}\int_{\mathbf{Z}_p} e^{(w_1x_1+w_2x_2+w_1w_2x)t}\, d\mu_{-q}(x_1)\, d\mu_{-q}(x_2)}{\int_{\mathbf{Z}_p} e^{w_1w_2xt}q^{(w_1w_2-1)x}\, d\mu_{-q}(x)} \\
&= \frac{[2]_q e^{w_1w_2xt}(q^{w_1w_2}e^{w_1w_2t}+1)}{(qe^{w_1t}+1)(qe^{w_2t}+1)}.
\end{aligned} \tag{9}
$$

From (8) and (9), we can derive

$$
\begin{aligned}
\frac{[2]_q\int_{\mathbf{Z}_p} e^{xt}d\mu_{-q}(x)}{\int_{\mathbf{Z}_p} e^{w_1xt}q^{(w_1-1)x}\, d\mu_{-q}(x)} &= [2]_q\sum_{l=0}^{w_1-1}(-1)^l q^l e^{lt} \\
&= \sum_{k=0}^{\infty}\left([2]_q\sum_{l=0}^{w_1-1}(-1)^l q^l l^k\right)\frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty}[2]_q S_{k,q}(w_1-1)\frac{t^k}{k!}.
\end{aligned} \tag{10}
$$

By (9) and (10), we easily see that
(11)

$$
\begin{aligned}
I &= \left(\frac{1}{[2]_q}\int_{\mathbf{Z}_p} e^{w_1(x_1+w_2x)t}d\mu_{-q}(x_1)\right)\left(\frac{[2]_q\int_{\mathbf{Z}_p} e^{w_2x_2t}d\mu_{-q}(x_2)}{\int_{\mathbf{Z}_p} e^{w_1w_2xt}q^{(w_1w_2-1)x}d\mu_{-q}(x)}\right) \\
&= \left(\frac{1}{[2]_q}\sum_{i=0}^{\infty}H_i(-q^{-1},w_2x)\frac{w_1^i}{i!}t^i\right)\left([2]_q\sum_{l=0}^{\infty}S_{l,q^{w_2}}(w_1-1)\frac{w_2^l}{l!}t^l\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}H_i(-q^{-1},w_2x)S_{n-i,q^{w_2}}(w_1-1)w_1^i w_2^{n-i}\right)\frac{t^n}{n!},
\end{aligned}
$$

where $H_n(-q^{-1},x)$ are the nth Frobenius-Euler polynomials.

On the other hand,

(12)

$$I=\left(\frac{1}{[2]_q}\int_{\mathbf{Z}_p}e^{w_2(x_2+w_1x)t}d\mu_{-q}(x_2)\right)\left(\frac{[2]_q\int_{\mathbf{Z}_p}e^{w_1x_1t}d\mu_{-q}(x_1)}{\int_{\mathbf{Z}_p}e^{w_1w_2xt}q^{(w_1w_2-1)x}d\mu_{-q}(x)}\right)$$
$$=\frac{1}{[2]_q}\left(\sum_{i=0}^{\infty}H_i(-q^{-1},w_1x)\frac{w_2^it^i}{i!}\right)\left([2]_q\sum_{l=0}^{\infty}S_{l,q^{w_1}}(w_2-1)\frac{w_1^lt^l}{l!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}H_i(-q^{-1},w_1x)S_{n-i,q^{w_1}}(w_2-1)w_2^iw_1^{n-i}\right)\frac{t^n}{n!}.$$

By comparing the coefficients $t^n/n!$ on the both sides of (11) and (12), we obtain the following theorem.

Theorem 1. *Let $w_1, w_2(\in \mathbf{N})$ be odd, and let $n(\geq 0)$ with $n\equiv 1 \pmod 2$. Then we have*

$$\sum_{i=0}^{n}\binom{n}{i}H_i(-q^{-1},w_2x)S_{n-i,q^{w_2}}(w_1-1)w_1^iw_2^{n-i}$$
$$=\sum_{i=0}^{n}\binom{n}{i}H_i(-q^{-1},w_1x)S_{n-i,q^{w_1}}(w_2-1)w_2^iw_1^{n-i}, \tag{13}$$

where $H_n(q,x)$ are the nth Frobenius-Euler polynomials.

Setting $x=0$ in (13), we obtain the following corollary.

Corollary 2. *Let $w_1, w_2(\in \mathbf{N})$ be odd, and let $n\in\mathbf{Z}_+$ be odd. Then we have*

$$\sum_{i=0}^{n}\binom{n}{i}H_i(-q^{-1})S_{n-i,q^{w_2}}(w_1-1)w_1^iw_2^{n-i}$$
$$=\sum_{i=0}^{n}\binom{n}{i}H_i(-q^{-1})S_{n-i,q^{w_1}}(w_2-1)w_2^iw_1^{n-i},$$

where $H_i(-q^{-1})$ are the nth Frobenius-Euler numbers.

If we take $w_2 = 1$ in (13), then we have

$$H_n(-q^{-1}, w_1 x) = \sum_{i=0}^{n} \binom{n}{i} H_i(-q^{-1}, x) S_{n-i,q}(w_1 - 1) w_1^i. \tag{14}$$

Setting $x = 0$ in (14), we obtain the following corollary.

Corollary 3. *Let $w_1(>1)$ be an odd integer, and let $n \in \mathbf{Z}_+$ with $n \equiv 1 \pmod 2$. Then we have*

$$H_n(-q^{-1}) = \sum_{i=0}^{n} \binom{n}{i} H_i(-q^{-1}) S_{n-i,q}(w_1 - 1) w_1^i.$$

From (7) and (8), we derive

$$\begin{aligned} I &= \left(\frac{e^{w_1 w_2 x t}}{[2]_q} \int_{\mathbf{Z}_p} e^{w_1 x_1 t} d\mu_{-q}(x_1) \right) \left(\frac{[2]_q \int_{\mathbf{Z}_p} e^{w_2 x_2 t} d\mu_{-q}(x_2)}{\int_{\mathbf{Z}_p} e^{w_1 w_2 x t} q^{(w_1 w_2 - 1)x} d\mu_{-q}(x)} \right) \\ &= \left(\frac{e^{w_1 w_2 x t}}{[2]_q} \int_{\mathbf{Z}_p} e^{w_1 x_1 t} d\mu_{-q}(x_1) \right) \left([2]_q \sum_{l=0}^{w_1 - 1} (-1)^l q^{w_2 l} e^{w_2 l t} \right) \\ &= \sum_{l=0}^{w_1 - 1} (-1)^l q^{w_2 l} \int_{\mathbf{Z}_p} e^{(x_1 + w_2 x + (w_2/w_1) l) t w_1} d\mu_{-q}(x_1) \\ &= \sum_{n=0}^{\infty} \left(w_1^n \sum_{l=0}^{w_1 - 1} (-1)^l q^{w_2 l} H_n(-q^{-1}, w_2 x + \frac{w_2}{w_1} l) \right) \frac{t^n}{n!}. \end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned} I &= \left(\frac{e^{w_1 w_2 x t}}{[2]_q} \int_{\mathbf{Z}_p} e^{w_2 x_2 t} d\mu_{-q}(x_2) \right) \left(\frac{[2]_q \int_{\mathbf{Z}_p} e^{w_1 x_1 t} d\mu_{-q}(x_1)}{\int_{\mathbf{Z}_p} e^{w_1 w_2 x t} q^{(w_1 w_2 - 1)x} d\mu_{-q}(x)} \right) \\ &= \left(\frac{1}{[2]_q} \int_{\mathbf{Z}_p} e^{w_2 x_2 t} d\mu_{-q}(x_2) \right) \left([2]_q \sum_{l=0}^{w_2 - 1} (-1)^l q^{w_1 l} e^{(w_1 l + w_1 w_2 x) t} \right) \end{aligned} \tag{16}$$

$$= \sum_{l=0}^{w_2-1} (-1)^l q^{w_1 l} \int_{\mathbf{Z}_p} e^{(x_2+w_1x+(w_1/w_2)l)tw_2} d\mu_{-q}(x_2)$$
$$= \sum_{n=0}^{\infty} \left(w_2^n \sum_{l=0}^{w_2-1} (-1)^l q^{w_1 l} H_n\left(-q^{-1}, w_1 x + \frac{w_1}{w_2} l \right) \right) \frac{t^n}{n!}.$$

By comparing the coefficients $t^n/n!$ on the both sides of (15) and (16), we obtain the following theorem.

Theorem 4. *Let $w_1, w_2 (\in \mathbf{N})$ be odd, and let $n \in \mathbf{Z}_+$ with $n \equiv 1 \pmod 2$. Then we have*

$$w_1^n \sum_{l=0}^{w_1-1} (-1)^l q^{w_2 l} H_n\left(-q^{-1}, w_2 x + \frac{w_2}{w_1} l \right) = w_2^n \sum_{l=0}^{w_2-1} (-1)^l q^{w_1 l} H_n\left(-q^{-1}, w_1 x + \frac{w_1}{w_2} l \right).$$

Setting $w_2 = 1$ in Theorem 4, we get the multiplication theorem for the Frobenius-Euler polynomials as follows:

$$H_n(-q^{-1}, w_1 x) = w_1^n \sum_{l=0}^{w_1-1} (-1)^l q^l H_n\left(-q^{-1}, x + \frac{l}{w_1} \right).$$

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