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ON JORDAN LEFT k-DERIVATIONS OF COMPLETELY PRIME Γ-RINGS

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ABSTRACT. With the notions of a left k-derivation and a Jordan left k-derivation of a Γ-ring we construct some important results relating to them in a concrete manner. In this article, we show that under a suitable condition every nonzero Jordan left k-derivation d of a 2-torsion free completely prime Γ-ring M induces the commutativity of M, and accordingly, d is a left k-derivation of M.

1. Introduction. The concept of Γ-ring is a generalization of classical ring. Nowadays, the study of Γ-rings is of great interest to the modern algebraists, especially for extending the significant results in classical ring theory to the topics in gamma ring theory. The notion of a Γ-ring was first introduced by Nobusawa [**7**] and then generalized by Barnes [**1**]. A number of important properties of Γ-rings were obtained by them as well as by Kyuno [**5**], Luh [**6**], and others. We begin with the following definition.

Let M and Γ be two additive abelian groups. If there exists a mapping $(a,\alpha,b)\mapsto a\alpha b$ of $M\times\Gamma\times M\to M$ which satisfies the conditions (a) $(a+b)\alpha c=a\alpha c+b\alpha c$, $a(\alpha+\beta)b=a\alpha b+a\beta b$, $a\alpha(b+c)=a\alpha b+a\alpha c$ and (b) $(a\alpha b)\beta c=a\alpha(b\beta c)$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$, then M is called a Γ-*ring* in the sense of Barnes [**1**].

For example, let R be a ring with identity 1 and $M_{m,n}(R)$ the set of all $m\times n$ matrices with entries in R. If we set $M=M_{m,n}(R)$ and $\Gamma=M_{n,m}(R)$, then M is a Γ-ring with respect to the matrix addition and multiplication. In particular, if we take $M=M_{1,2}(R)$ and $\Gamma=\{\begin{pmatrix} n.1 \\ 0 \end{pmatrix}: n \text{ is an integer}\}$, then M is also a Γ-ring.

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Note that the notions of a prime Γ-ring and a completely prime Γ-ring were introduced by Luh [**6**] and some analogous results corresponding to the prime rings were obtained by him and Kyuno [**5**], whereas the concept of a strongly completely prime Γ-ring was used and developed by Sapanci and Nakajima in [**8**].

Let M be a Γ-ring. Then M is called a *prime* Γ*-ring* if, for all $a, b \in M$, $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$. And, M is called *completely prime* if $a\Gamma b = 0$ (with $a, b \in M$) implies $a = 0$ or $b = 0$.

A Γ-ring M is said to be 2*-torsion free* if $2a = 0$ implies $a = 0$ for all $a \in M$. And, a Γ-ring M is said to be a *commutative* Γ*-ring* if $x\gamma y = y\gamma x$ holds for all $x, y \in M$ and $\gamma \in \Gamma$.

Sapanci and Nakajima [**8**] have introduced the notions of derivation and Jordan derivation of a Γ-ring. Afterwards, in the light of some significant results due to Jordan left derivation of a classical ring obtained by Jun and Kim in [**3**], some extensive results of left derivation and Jordan left derivation of a Γ-ring were determined by Ceven in [**2**]. But, the notion of k-derivation of a Γ-ring was introduced by Kandamar [**4**] and a number of important results on the k-derivation of a Γ-ring were obtained by him. Here we introduce the notions of left k-derivation and Jordan left k-derivation of a Γ-ring and then we build up some important results relating to them in a concrete manner.

Let M be a Γ-ring, and let $d : M \to M$ and $k : \Gamma \to \Gamma$ be additive mappings. If $d(a\alpha b) = a\alpha d(b) + b\alpha d(a)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$, then d is called a *left derivation* of M. For all $a, b \in M$ and $\alpha \in \Gamma$, if $d(a\alpha b) = a\alpha d(b) + ak(\alpha)b + b\alpha d(a)$ is satisfied, then d is called a *left* k*-derivation* of M. And, if $d(a\alpha a) = 2a\alpha d(a) + ak(\alpha)a$ holds for all $a \in M$ and $\alpha \in \Gamma$, then d is called a *Jordan left* k*-derivation* of M.

For instance, let M be a Γ-ring and d a left k-derivation of M. Suppose $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. Define the operations of addition and multiplication on M_1 by $(x_1, x_1) + (x_2, x_2) = (x_1 + x_2, x_1 + x_2)$ and $(x_1, x_1)(\alpha, \alpha)(x_2, x_2) = (x_1\alpha x_2, x_1\alpha x_2)$ for every $x_1, x_2 \in M$ and $\alpha \in \Gamma$, respectively. Then it can easily be shown that M_1 is a Γ_1-ring under these operations of addition and multiplication. Let $d_1 : M_1 \to M_1$ and $k_1 : \Gamma_1 \to \Gamma_1$ be two additive maps defined by $d_1((x, x)) = (d(x), d(x))$ for all $x \in M$ and $k_1((\alpha, \alpha)) = (k(\alpha), k(\alpha))$

for all $\alpha \in \Gamma$, respectively. Then it follows that d_1 is a Jordan left k_1-derivation of M_1.

Now, let M be a Γ-ring, $d : M \to M$ and $k : \Gamma \to \Gamma$ two additive mappings, and let $a, b \in M$ and $\alpha \in \Gamma$. Then d is called (i) a *derivation* of M if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, (ii) a *$k$-derivation* of M if $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$, and (iii) a *Jordan k-derivation* of M if $d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$.

In this article we show that under a suitable condition every nonzero Jordan left k-derivation d of a 2-torsion free completely prime Γ-ring M induces the commutativity of M, and accordingly, d is then a left k-derivation of M.

2. Main results.

Lemma 2.1. *Let M be a Γ-ring and d a Jordan left k-derivation of M. Then, for all $a, b \in M$ and $\alpha \in \Gamma$, $d(a\alpha b + b\alpha a) = 2a\alpha d(b) + 2b\alpha d(a) + ak(\alpha)b + bk(\alpha)a$.*

Proof. Use the definition of a Jordan left k-derivation d of a Γ-ring M to compute $d((a+b)\alpha(a+b)) = 2(a+b)\alpha d(a+b) + (a+b)k(\alpha)(a+b)$, and then cancel the like terms from both sides to obtain the proof. □

Lemma 2.2. *Let d be a Jordan left k-derivation of a 2-torsion free Γ-ring M such that $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for every $a, b, c \in M$ and $\alpha \in \Gamma$. Then the following statements hold for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:*

(i) $d(a\alpha b\alpha a) = a\alpha a\alpha d(b) + (3a\alpha b - b\alpha a)\alpha d(a) + ak(\alpha)(2b\alpha a + a\alpha b) - bk(\alpha)a\alpha a$;

(ii) $d(a\alpha b\alpha c + c\alpha b\alpha a) = (a\alpha c + c\alpha a)\alpha d(b) + (3a\alpha b - b\alpha a)\alpha d(c) + (3c\alpha b - b\alpha c)\alpha d(a) + ak(\alpha)(2b\alpha c + c\alpha b) + ck(\alpha)(2b\alpha a + a\alpha b) - bk(\alpha)(a\alpha c + c\alpha a)$;

(iii) $(a\alpha b - b\alpha a)\alpha a\alpha d(a) = a\alpha(a\alpha b - b\alpha a)\alpha d(a) + ak(\alpha)(a\alpha b - b\alpha a)\alpha a - (a\alpha b - b\alpha a)\alpha ak(\alpha)a$;

(iv) $(a\alpha b - b\alpha a)\alpha(d(a\alpha b) - a\alpha d(b) - b\alpha d(a) - bk(\alpha)a) = 0$;

(v) $d(a\alpha a\alpha b) = a\alpha a\alpha d(b) + (a\alpha b + b\alpha a)\alpha d(a) + a\alpha d(a\alpha b - b\alpha a) + ak(\alpha)a\alpha b + bk(\alpha)a\alpha a$;

(vi) $d(b\alpha a\alpha a) = a\alpha a\alpha d(b) + (3b\alpha a - a\alpha b)\alpha d(a) - a\alpha d(a\alpha b - b\alpha a) + 2bk(\alpha)a\alpha a$;

(vii) $(a\alpha b - b\alpha a)\alpha(d(a\alpha b - b\alpha a) + ak(\alpha)b - bk(\alpha)a) = 0$;

(viii) $(a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a)\alpha d(b) + ak(\alpha)(a\alpha b\alpha b - 2b\alpha a\alpha b) + bk(\alpha)a\alpha a\alpha b = 0$;

(ix) $(b\alpha b\alpha a - 2b\alpha a\alpha b + a\alpha b\alpha b)\alpha d(a) + ak(\alpha)b\alpha b\alpha a + bk(\alpha)(b\alpha a\alpha a - 2a\alpha b\alpha a) = 0$.

Proof. (i) From Lemma 2.1, for all $a, b \in M$ and $\alpha \in \Gamma$, we have

$$d(a\alpha b + b\alpha a) = 2a\alpha d(b) + 2b\alpha d(a) + ak(\alpha)b + bk(\alpha)a. \tag{1}$$

Replacing $a\alpha b + b\alpha a$ for b, we get

$$\begin{aligned}2d(a\alpha b\alpha a) &+ d((a\alpha a)\alpha b + b\alpha(a\alpha a))\\ &= 2a\alpha d(a\alpha b + b\alpha a) + 2(a\alpha b + b\alpha a)\alpha d(a)\\ &\quad + ak(\alpha)(a\alpha b + b\alpha a) + (a\alpha b + b\alpha a)k(\alpha)a.\end{aligned}$$

By using Lemma 2.1 and the hypothesis that $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for every $a, b, c \in M$ and $\alpha \in \Gamma$, this yields

$$\begin{aligned}2d(a\alpha b\alpha a) &= 2a\alpha(2a\alpha d(b) + 2b\alpha d(a) + ak(\alpha)b + bk(\alpha)a)\\ &\quad + 2(a\alpha b + b\alpha a)\alpha d(a) + ak(\alpha)(a\alpha b + b\alpha a)\\ &\quad + (a\alpha b + b\alpha a)k(\alpha)a\\ &\quad - (2a\alpha a\alpha d(b) + 2b\alpha d(a\alpha a) + a\alpha ak(\alpha)b + bk(\alpha)a\alpha a)\\ &= 2a\alpha a\alpha d(b) + 6a\alpha b\alpha d(a) - 2b\alpha a\alpha d(a)\\ &\quad + 4ak(\alpha)b\alpha a + 2ak(\alpha)a\alpha b - 2bk(\alpha)a\alpha a.\end{aligned}$$

Since M is 2-torsion free, we have

$$\begin{aligned}d(a\alpha b\alpha a) &= a\alpha a\alpha d(b) + (3a\alpha b - b\alpha a)\alpha d(a)\\ &\quad + ak(\alpha)(2b\alpha a + a\alpha b) - bk(\alpha)a\alpha a.\end{aligned} \tag{2}$$

(ii) Putting $a + c$ for a in (2), we obtain

$$\begin{aligned}d((a+c)\alpha b\alpha(a+c)) &= (a+c)\alpha(a+c)\alpha d(b)\\ &\quad + (3(a+c)\alpha b - b\alpha(a+c))\alpha d(a+c)\\ &\quad + (a+c)k(\alpha)(2b\alpha(a+c) + (a+c)\alpha b)\\ &\quad - bk(\alpha)(a+c)\alpha(a+c).\end{aligned}$$

Here, the LHS $= d(a\alpha b\alpha c + c\alpha b\alpha a) + d(a\alpha b\alpha a) + d(c\alpha b\alpha c) = d(a\alpha b\alpha c + c\alpha b\alpha a) + a\alpha a\alpha d(b) + 3a\alpha b\alpha d(a) - b\alpha a\alpha d(a) + 2ak(\alpha)b\alpha a + ak(\alpha)a\alpha b - bk(\alpha)a\alpha a + c\alpha c\alpha d(b) + 3c\alpha b\alpha d(c) - b\alpha c\alpha d(c) + 2ck(\alpha)b\alpha c + ck(\alpha)c\alpha b - bk(\alpha)c\alpha c$;

and the RHS $= a\alpha a\alpha d(b) + a\alpha c\alpha d(b) + c\alpha a\alpha d(b) + c\alpha c\alpha d(b) + 3a\alpha b\alpha d(a) + 3a\alpha b\alpha d(c) + 3c\alpha b\alpha d(a) + 3c\alpha b\alpha d(c) - b\alpha a\alpha d(a) - b\alpha a\alpha d(c) - b\alpha c\alpha d(a) - b\alpha c\alpha d(c) + 2ak(\alpha)b\alpha a + 2ak(\alpha)b\alpha c + 2ck(\alpha)b\alpha a + 2ck(\alpha)b\alpha c + ak(\alpha)a\alpha b + ak(\alpha)c\alpha b + ck(\alpha)a\alpha b + ck(\alpha)c\alpha b - bk(\alpha)a\alpha a - bk(\alpha)a\alpha c - bk(\alpha)c\alpha a - bk(\alpha)c\alpha c$.

Upon canceling the like terms from both sides, we get

$$\begin{aligned} d(a\alpha b\alpha c + c\alpha b\alpha a) = {} & (a\alpha c + c\alpha a)\alpha d(b) + (3a\alpha b - b\alpha a)\alpha d(c) \\ & + (3c\alpha b - b\alpha c)\alpha d(a) + ak(\alpha)(2b\alpha c + c\alpha b) \\ & + ck(\alpha)(2b\alpha a + a\alpha b) - bk(\alpha)(a\alpha c + c\alpha a). \end{aligned} \tag{3}$$

(iii) Let $A = d(a\alpha b\alpha a\alpha b + a\alpha b\alpha b\alpha a)$. First, using (3), we obtain

$$\begin{aligned} A &= d(a\alpha b\alpha(a\alpha b) + (a\alpha b)\alpha b\alpha a) \\ &= (a\alpha a\alpha b + a\alpha b\alpha a)\alpha d(b) + (3a\alpha b - b\alpha a)\alpha d(a\alpha b) \\ &\quad + (3a\alpha b\alpha b - b\alpha a\alpha b)\alpha d(a) + ak(\alpha)(2b\alpha a\alpha b + a\alpha b\alpha b) \\ &\quad + a\alpha bk(\alpha)(2b\alpha a + a\alpha b) - bk(\alpha)(a\alpha a\alpha b + a\alpha b\alpha a). \end{aligned} \tag{4}$$

Again, using the definition of d and by (2), we also get

$$\begin{aligned} A &= d((a\alpha b)\alpha(a\alpha b)) + d(a\alpha(b\alpha b)\alpha a) \\ &= 2a\alpha b\alpha d(a\alpha b) + a\alpha bk(\alpha)a\alpha b + a\alpha a\alpha d(b\alpha b) \\ &\quad + (3a\alpha b\alpha b - b\alpha b\alpha a)\alpha d(a) + ak(\alpha)(2b\alpha b\alpha a + a\alpha b\alpha b) \\ &\quad - b\alpha bk(\alpha)a\alpha a. \end{aligned} \tag{5}$$

Equating these two expressions for A from (4) and (5), simplify the obtained equation by canceling the like terms with the appropriate use of the hypothesis $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for every $a, b, c \in M$ and $\alpha \in \Gamma$ to obtain

$$\begin{aligned} (a\alpha b - b\alpha a)\alpha d(a\alpha b) = {} & a\alpha(a\alpha b - b\alpha a)\alpha d(b) + b\alpha(a\alpha b - b\alpha a)\alpha d(a) \\ & + ak(\alpha)(a\alpha b - b\alpha a)\alpha b + bk(\alpha)(a\alpha b - b\alpha a)\alpha a \\ & - (a\alpha b - b\alpha a)\alpha ak(\alpha)b. \end{aligned} \tag{6}$$

Replacing $a+b$ for b (which keeps $a\alpha b - b\alpha a$ unaltered), we have

$$\begin{aligned}(a\alpha b - b\alpha a)\alpha d(a\alpha(a+b)) = a\alpha(a\alpha b - b\alpha a)\alpha d(a+b)\\ + (a+b)\alpha(a\alpha b - b\alpha a)\alpha d(a)\\ + ak(\alpha)(a\alpha b - b\alpha a)\alpha(a+b)\\ + (a+b)k(\alpha)(a\alpha b - b\alpha a)\alpha a\\ - (a\alpha b - b\alpha a)\alpha ak(\alpha)(a+b).\end{aligned}$$

By using (6) and the hypothesis $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for every $a, b, c \in M$ and $\alpha \in \Gamma$, after simplification it becomes

$$\begin{aligned}2(a\alpha b - b\alpha a)\alpha a\alpha d(a) = 2a\alpha(a\alpha b - b\alpha a)\alpha d(a)\\ + 2ak(\alpha)(a\alpha b - b\alpha a)\alpha a - 2(a\alpha b - b\alpha a)\alpha ak(\alpha)a.\end{aligned}$$

But, since M is 2-torsion free, it follows that

(7)

$$\begin{aligned}(a\alpha b - b\alpha a)\alpha a\alpha d(a) = a\alpha(a\alpha b - b\alpha a)\alpha d(a) + ak(\alpha)(a\alpha b - b\alpha a)\alpha a\\ - (a\alpha b - b\alpha a)\alpha ak(\alpha)a.\end{aligned}$$

(iv) Replacing $a+b$ for a (which keeps $a\alpha b - b\alpha a$ unaltered) in (7), we obtain

$$\begin{aligned}(a\alpha b - b\alpha a)\alpha(a+b)\alpha d(a+b) = (a+b)\alpha(a\alpha b - b\alpha a)\alpha d(a+b)\\ + (a+b)k(\alpha)(a\alpha b - b\alpha a)\alpha(a+b)\\ - (a\alpha b - b\alpha a)\alpha(a+b)k(\alpha)(a+b).\end{aligned}$$

Simplifying it by (7) using $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for all $a, b, c \in M$ and $\alpha \in \Gamma$ and then canceling the like terms from both sides, we get

$$\begin{aligned}(a\alpha b - b\alpha a)\alpha a\alpha d(b) + (a\alpha b - b\alpha a)\alpha b\alpha d(a)\\ = a\alpha(a\alpha b - b\alpha a)\alpha d(b) + b\alpha(a\alpha b - b\alpha a)\alpha d(a)\\ + ak(\alpha)(a\alpha b - b\alpha a)\alpha b + bk(\alpha)(a\alpha b - b\alpha a)\alpha a\\ - (a\alpha b - b\alpha a)\alpha ak(\alpha)b - (a\alpha b - b\alpha a)\alpha bk(\alpha)a.\end{aligned}$$

Hence, by using (6), it reduces to

(8) $$(a\alpha b - b\alpha a)\alpha(d(a\alpha b) - a\alpha d(b) - b\alpha d(a) - bk(\alpha)a) = 0.$$

(v) Using Lemma 2.1, we have

$$d(a\alpha(b\alpha a)+(b\alpha a)\alpha a) = 2a\alpha d(b\alpha a)+2b\alpha a\alpha d(a)+ak(\alpha)b\alpha a+bk(\alpha)a\alpha a \tag{9}$$

and

$$d(a\alpha(a\alpha b)+(a\alpha b)\alpha a) = 2a\alpha d(a\alpha b)+2a\alpha b\alpha d(a)+ak(\alpha)a\alpha b+ak(\alpha)b\alpha a. \tag{10}$$

Taking (9)–(10), we get

$$\begin{aligned} d(a\alpha a\alpha b - b\alpha a\alpha a) &= 2a\alpha d(a\alpha b - b\alpha a) + 2(a\alpha b - b\alpha a)\alpha d(a) \\ &\quad + ak(\alpha)a\alpha b - bk(\alpha)a\alpha a. \end{aligned} \tag{11}$$

Replacing $a\alpha a$ for a in (1), we obtain

$$\begin{aligned} &d(a\alpha a\alpha b + b\alpha a\alpha a) \\ &\quad = 2a\alpha a\alpha d(b) + 4b\alpha a\alpha d(a) + 2bk(\alpha)a\alpha a + ak(\alpha)a\alpha b + bk(\alpha)a\alpha a. \end{aligned}$$

Thus, we get

$$d(a\alpha a\alpha b + b\alpha a\alpha a) = 2a\alpha a\alpha d(b) + 4b\alpha a\alpha d(a) + ak(\alpha)a\alpha b + 3bk(\alpha)a\alpha a. \tag{12}$$

By (11) + (12), we have

$$\begin{aligned} 2d(a\alpha a\alpha b) &= 2a\alpha a\alpha d(b) + 2(a\alpha b + b\alpha a)\alpha d(a) \\ &\quad + 2a\alpha d(a\alpha b - b\alpha a) + 2ak(\alpha)a\alpha b + 2bk(\alpha)a\alpha a. \end{aligned}$$

Since M is 2-torsion free, we obtain

$$\begin{aligned} d(a\alpha a\alpha b) &= a\alpha a\alpha d(b) + (a\alpha b + b\alpha a)\alpha d(a) \\ &\quad + a\alpha d(a\alpha b - b\alpha a) + ak(\alpha)a\alpha b + bk(\alpha)a\alpha a. \end{aligned} \tag{13}$$

(vi) Taking (11)–(12), we get

$$\begin{aligned} &2d(b\alpha a\alpha a) \\ &= 2a\alpha a\alpha d(b) + 2(3b\alpha a - a\alpha b)\alpha d(a) - 2a\alpha d(a\alpha b - b\alpha a) + 4bk(\alpha)a\alpha a. \end{aligned}$$

Since M is 2-torsion free, we obtain

$$d(b\alpha a\alpha a) = a\alpha a\alpha d(b)+(3b\alpha a-a\alpha b)\alpha d(a)-a\alpha d(a\alpha b-b\alpha a)+2bk(\alpha)a\alpha a. \tag{14}$$

(vii) From (1), we have

$$d(a\alpha b) = -d(b\alpha a) + 2a\alpha d(b) + 2b\alpha d(a) + ak(\alpha)b + bk(\alpha)a.$$

Substituting this into (8), we get

$$(a\alpha b - b\alpha a)\alpha(d(b\alpha a) - a\alpha d(b) - b\alpha d(a) - ak(\alpha)b) = 0. \tag{15}$$

By (8)–(15), we obtain

$$(a\alpha b - b\alpha a)\alpha(d(a\alpha b - b\alpha a) + ak(\alpha)b - bk(\alpha)a) = 0. \tag{16}$$

(viii) We have

$$\begin{aligned} d((a\alpha b - b\alpha a)\alpha(a\alpha b - b\alpha a)) = d(a\alpha(b\alpha a\alpha b) + (b\alpha a\alpha b)\alpha a) \\ - d(a\alpha(b\alpha b)\alpha a) - d(b\alpha(a\alpha a)\alpha b). \end{aligned}$$

Using the definition of d and applying (1) and (2), we get

$$\begin{aligned} &2(a\alpha b - b\alpha a)\alpha d(a\alpha b - b\alpha a) + (a\alpha b - b\alpha a)k(\alpha)(a\alpha b - b\alpha a) \\ &\quad = 2a\alpha d(b\alpha a\alpha b) + 2b\alpha a\alpha b\alpha d(a) + ak(\alpha)b\alpha a\alpha b + b\alpha a\alpha bk(\alpha)a \\ &\qquad - [a\alpha a\alpha d(b\alpha b) + (3a\alpha b\alpha b - b\alpha b\alpha a)\alpha d(a) \\ &\qquad + ak(\alpha)(2b\alpha b\alpha a + a\alpha b\alpha b) - b\alpha bk(\alpha)a\alpha a] \\ &\qquad - [b\alpha b\alpha d(a\alpha a) + (3b\alpha a\alpha a - a\alpha a\alpha b)\alpha d(b) \\ &\qquad + bk(\alpha)(2a\alpha a\alpha b + b\alpha a\alpha a) - a\alpha ak(\alpha)b\alpha b]. \end{aligned}$$

Simplifying it by using the definition of d, (16) and (2) with the hypothesis $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for every $a, b, c \in M$ and $\alpha \in \Gamma$, we obtain

(17)

$$\begin{aligned} &3(a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a)\alpha d(b) + (b\alpha b\alpha a - 2b\alpha a\alpha b + a\alpha b\alpha b)\alpha d(a) \\ &+ak(\alpha)(b\alpha b\alpha a - 6b\alpha a\alpha b + 3a\alpha b\alpha b) + bk(\alpha)(3a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a) = 0. \end{aligned}$$

Hence, from (7), we have

$$\begin{aligned} (a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a)\alpha d(a) + ak(\alpha)(a\alpha b - b\alpha a)\alpha a \\ - (a\alpha b - b\alpha a)\alpha ak(\alpha)a = 0. \end{aligned} \tag{18}$$

Replace $a + b$ for a (which keeps $a\alpha b - b\alpha a$ unaltered) in (18) to get

$$\begin{aligned}(&a\alpha a\alpha b + a\alpha b\alpha b + b\alpha a\alpha b + b\alpha b\alpha b - 2a\alpha b\alpha a - 2a\alpha b\alpha b - 2b\alpha b\alpha a\\ &- 2b\alpha b\alpha b + b\alpha a\alpha a + b\alpha a\alpha b + b\alpha b\alpha a + b\alpha b\alpha b)\alpha[d(a) + d(b)]\\ &+ak(\alpha)(a\alpha b-b\alpha a)\alpha a+ak(\alpha)(a\alpha b-b\alpha a)\alpha b+bk(\alpha)(a\alpha b-b\alpha a)\alpha a\\ &+bk(\alpha)(a\alpha b - b\alpha a)\alpha b - (a\alpha b-b\alpha a)\alpha ak(\alpha)a-(a\alpha b-b\alpha a)\alpha ak(\alpha)b\\ &- (a\alpha b - b\alpha a)\alpha bk(\alpha)a - (a\alpha b - b\alpha a)\alpha bk(\alpha)b = 0.\end{aligned}$$

It gives

$$\begin{aligned}&[(a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a)\alpha d(a) + ak(\alpha)(a\alpha b - b\alpha a)\alpha a\\ &- (a\alpha b - b\alpha a)\alpha ak(\alpha)a] - [(b\alpha b\alpha a - 2b\alpha a\alpha b + a\alpha b\alpha b)\alpha d(b)\\ &+ bk(\alpha)(b\alpha a - a\alpha b)\alpha b - (b\alpha a - a\alpha b)\alpha bk(\alpha)b]\\ &+ (-a\alpha b\alpha b + 2b\alpha a\alpha b - b\alpha b\alpha a)\alpha d(a)\\ &+ (a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a)\alpha d(b)\\ &+ ak(\alpha)(a\alpha b\alpha b - b\alpha a\alpha b) + bk(\alpha)(a\alpha b\alpha a - b\alpha a\alpha a)\\ &- (a\alpha b\alpha ak(\alpha)b - b\alpha a\alpha ak(\alpha)b) - (a\alpha b\alpha bk(\alpha)a - b\alpha a\alpha bk(\alpha)a) = 0.\end{aligned}$$

By (18), the first two terms in the third brackets vanish, and it remains

$$\begin{aligned}&(a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a)\alpha d(b) - (a\alpha b\alpha b - 2b\alpha a\alpha b + b\alpha b\alpha a)\alpha d(a)\\ &+ ak(\alpha)(a\alpha b\alpha b - b\alpha a\alpha b) + bk(\alpha)(a\alpha b\alpha a - b\alpha a\alpha a) - ak(\alpha)b\alpha a\alpha b\\ &+ bk(\alpha)a\alpha a\alpha b - ak(\alpha)b\alpha b\alpha a + bk(\alpha)a\alpha b\alpha a = 0.\end{aligned}$$

Hence, it follows that

(19)
$$\begin{aligned}(a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a)\alpha d(b) - (b\alpha b\alpha a - 2b\alpha a\alpha b + a\alpha b\alpha b)\alpha d(a)&\\ - ak(\alpha)(b\alpha b\alpha a + 2b\alpha a\alpha b - a\alpha b\alpha b)&\\ + bk(\alpha)(a\alpha a\alpha b + 2a\alpha b\alpha a - b\alpha a\alpha a) = 0.&\end{aligned}$$

Taking (17) + (19), we obtain

$$\begin{aligned}4(a\alpha a\alpha b - 2a\alpha b\alpha a + b\alpha a\alpha a)\alpha d(b) + 4ak(\alpha)(a\alpha b\alpha b - 2b\alpha a\alpha b)&\\ + 4bk(\alpha)a\alpha a\alpha b = 0.&\end{aligned}$$

Since M is 2-torsion free, we have

$$(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\alpha d(b)+ak(\alpha)(a\alpha b\alpha b-2b\alpha a\alpha b)+bk(\alpha)a\alpha a\alpha b = 0. \tag{20}$$

(ix) Finally, using (19) in (20), we get

$$(b\alpha b\alpha a-2b\alpha a\alpha b+a\alpha b\alpha b)\alpha d(a)+ak(\alpha)b\alpha b\alpha a+bk(\alpha)(b\alpha a\alpha a-2a\alpha b\alpha a) = 0.$$

The proof of the lemma is thus completed. □

Theorem 2.1. *Let M be a 2-torsion free completely prime Γ-ring, and let d be a nonzero Jordan left k-derivation of M such that $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for every $a, b, c \in M$ and $\alpha \in \Gamma$. Then M is commutative, and accordingly, d is a left k-derivation of M.*

Proof. From Lemma 2.2 (vii), for every $a, b \in M$ and $\alpha \in \Gamma$, we have

$$(a\alpha b - b\alpha a)\alpha(d(a\alpha b - b\alpha a) + ak(\alpha)b - bk(\alpha)a) = 0.$$

Since M is completely prime, we get

$$a\alpha b - b\alpha a = 0 \text{ or } d(a\alpha b - b\alpha a) + ak(\alpha)b - bk(\alpha)a = 0.$$

If $a\alpha b - b\alpha a = 0$, i.e., $a\alpha b = b\alpha a$ for every $a, b \in M$ and $\alpha \in \Gamma$, then M is commutative (by definition).

And, if $d(a\alpha b - b\alpha a) + ak(\alpha)b - bk(\alpha)a = 0$, then we have

$$d(a\alpha b - b\alpha a) = bk(\alpha)a - ak(\alpha)b;$$

which gives

$$d(a\alpha b) = d(b\alpha a) - ak(\alpha)b + bk(\alpha)a.$$

Replacing $a\alpha b$ for b in the last equation, we obtain

$$d(a\alpha a\alpha b) = d(a\alpha b\alpha a) - ak(\alpha)a\alpha b + a\alpha bk(\alpha)a.$$

Hence, by Lemma 2.2 (i),(v) and applying the hypothesis $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for every $a, b, c \in M$ and $\alpha \in \Gamma$, this yields

$$2(a\alpha b - b\alpha a)\alpha d(a) + 2(a\alpha b - b\alpha a)k(\alpha)a = 0.$$

But, since M is 2-torsion free, we obtain

$$(a\alpha b - b\alpha a)\alpha d(a) + (a\alpha b - b\alpha a)k(\alpha)a = 0.$$

Putting $a\alpha b$ for b here again, we have

$$a\alpha(a\alpha b - b\alpha a)\alpha d(a) + a\alpha(a\alpha b - b\alpha a)k(\alpha)a = 0.$$

Hence, by Lemma 2.2 (iii) using the hypothesis $a\alpha bk(\alpha)c = ak(\alpha)b\alpha c$ for every $a, b, c \in M$ and $\alpha \in \Gamma$, we get

$$(a\alpha b - b\alpha a)\alpha(a\alpha d(a) + ak(\alpha)a) = 0 \text{ for all } a, b \in M \text{ and } \alpha \in \Gamma.$$

Since M is completely prime, it follows that $a\alpha b - b\alpha a = 0$ or $a\alpha d(a) + ak(\alpha)a = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$. If $a\alpha d(a) + ak(\alpha)a = 0$ for every $a \in M$ and $\alpha \in \Gamma$, then $(2a\alpha d(a) + ak(\alpha)a) - a\alpha d(a) = 0$, and therefore, it gives $d(a\alpha a) = a\alpha d(a)$, which is a contradiction to the definition of d (since we assumed that $d \neq 0$). Hence, we conclude that $a\alpha b - b\alpha a = 0$ for every $a, b \in M$ and $\alpha \in \Gamma$, and consequently, M is commutative.

Accordingly, since M is commutative, from Lemma 2.1, we obtain $2d(a\alpha b) = 2a\alpha d(b) + 2ak(\alpha)b + 2b\alpha d(a)$. But, since M is 2-torsion free, we get $d(a\alpha b) = a\alpha d(b) + ak(\alpha)b + b\alpha d(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$, which indicates that d is a left k-derivation of M. This completes the proof of the theorem. □

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