ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 6, 2010

DIFFERENTIAL FORMS ON MODULI SPACES OF PARABOLIC BUNDLES

FRANCESCO BOTTACIN

ABSTRACT. Let X be a smooth projective variety, and let $\mathcal{PB}$ be a moduli space of stable parabolic bundles on X. For any flat family E_* of parabolic bundles on X parametrized by a smooth scheme Y, and for any integer m, with $1 \leq m \leq \dim X$, we construct a closed differential form $\Omega = \Omega_{E_*}$ on Y with values in $H^m(X, \mathcal{O}_X)$. By using the vector-valued differential form Ω we then prove that, for any $i \geq 0$, the choice of a (nonzero) element $\sigma \in H^i(X, \Omega_X^{i+m})$, determines, in a natural way, a closed differential m-form Ω_σ on the smooth locus of $\mathcal{PB}$.

Introduction. In this paper we want to provide another example of a general phenomenon, which can be roughly stated as follows: "geometric structures" on a variety X determine similar structures on various kinds of moduli spaces of sheaves on X.

Let X be a smooth n-dimensional projective variety, defined over an algebraically closed field k of characteristic 0. In [**4**] we proved that, if X admits nonzero differential forms of degree m, then the choice of any such m-form σ determines a closed differential m-form Ω_σ on the smooth locus of the moduli space $\mathcal{M}$ of stable sheaves on X.

In this paper we shall prove a similar result for moduli spaces of parabolic bundles. More precisely, let us denote by $\mathcal{PB}$ a moduli space of stable parabolic bundles on X. Then we show that, for any $m \leq n$ and any $i \leq n-m$, the choice of a (nonzero) element $\sigma \in H^i(X, \Omega_X^{i+m})$, determines a closed differential m-form Ω_σ on the smooth locus of the moduli space $\mathcal{PB}$. For instance, by taking $m = 2$, this result can be used to construct pre-symplectic structures on moduli spaces of parabolic bundles.

2010 AMS *Mathematics subject classification.* Primary 14J60, Secondary 14D20, 58A10.
Keywords and phrases. Closed forms, differential forms, moduli spaces of parabolic sheaves, parabolic bundles.
Received by the editors on February 7, 2008, and in revised form on June 3, 2008.
DOI:10.1216/RMJ-2010-40-6-1779

This paper is organized as follows. In Section 1 we recall the definitions of parabolic sheaves and parabolic bundles on a higher dimensional variety X, then in Section 2 we recall some useful results about cup-products and trace maps. We also introduce the symmetrized trace map and study its graded commutativity properties.

In Section 3 we construct, for any flat family E_* of parabolic bundles on X parametrized by a smooth scheme Y, and for any integer m with $1 \leq m \leq \dim X$, a closed differential m-form $\Omega = \Omega_{E_*}$ on Y with values in $H^m(X, \mathcal{O}_X)$.

Finally, in Section 4, we use the vector-valued differential form Ω to define, for any $\sigma \in H^i(X, \Omega_X^{i+m})$, a differential m-form Ω_σ on the smooth locus $\mathcal{PB}^{sm}$ of the moduli space $\mathcal{PB}$ of stable parabolic bundles on X. The closure of Ω_σ will then follow from the closure of Ω.

The paper ends with the discussion of some examples illustrating possible applications of this result.

1. Parabolic sheaves. In this section we shall briefly recall the definitions of parabolic sheaves and parabolic bundles on higher dimensional varieties. For more details we refer the reader to [**1, 6**].

Let X be a nonsingular projective variety of dimension n, defined over an algebraically closed field k of characteristic 0, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. We shall fix an effective Cartier divisor D on X.

Definition 1.1. A parabolic structure over D on a coherent, torsion-free $\mathcal{O}_X$-module E is the data of a filtration

$$F_* : E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D),$$

where $E(-D)$ denotes the image of $E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to E$, together with a sequence of real numbers $\alpha_* = (\alpha_1, \dots, \alpha_l)$, called weights, such that

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1.$$

A parabolic sheaf on X is a coherent, torsion-free $\mathcal{O}_X$-module E with a parabolic structure over D.

Remark 1.2. There is another definition of parabolic sheaves, which is closer to the original definition of parabolic bundles on curves (cf.

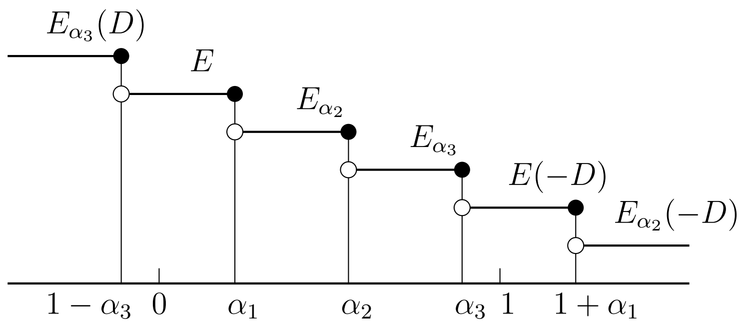

FIGURE 1. The $\mathbf{R}$-filtered sheaf E_* associated to a parabolic sheaf (E, F_*, α_*).

[**1**]): a parabolic structure over D on a torsion-free sheaf E is given by a sequence of subsheaves of $E|_D$,

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{F}_D^2(E) \supset \cdots \supset \mathcal{F}_D^l(E) \supset \mathcal{F}_D^{l+1}(E) = 0,$$

together with a system of weights $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$.

Our definition is related to this one by setting

$$F_i(E) = \ker(E \to E|_D/\mathcal{F}_D^i(E)).$$

All definitions related to parabolic sheaves can be stated more efficiently in terms of $\mathbf{R}$-filtered sheaves (see [**8**] for the definition), whose introduction seems to be due originally to Simpson.

Given a parabolic sheaf (E, F_*, α_*), we define its associated $\mathbf{R}$-filtered sheaf $E_* = (E_x)$, for $0 \le x \le 1$, by setting $E_0 = E$ and $E_x = F_i(E)$ if $\alpha_{i-1} < x \le \alpha_i$, where we have set $\alpha_0 = 0$ and $\alpha_{l+1} = 1$. The definition of E_x can be extended to all $x \in \mathbf{R}$ by setting $E_{x+1} = E_x(-D)$. Figure 1 illustrates the $\mathbf{R}$-filtered sheaf corresponding to a parabolic sheaf (E, F_*, α_*) with weights $0 \le \alpha_1 < \alpha_2 < \alpha_3 < 1$.

From now on an $\mathbf{R}$-filtered sheaf $E_* = (E_x)_{x\in\mathbf{R}}$ associated to a parabolic sheaf (E, F_*, α_*) as above, will simply be called a parabolic sheaf.

If E_* is an $\mathbf{R}$-filtered sheaf, we shall always write E for the sheaf E_0.

Definition 1.3. A homomorphism of $\mathbf{R}$-filtered sheaves $\phi : E_* \to E'_*$ is a homomorphism of $\mathcal{O}_X$-modules $\phi : E \to E'$ such that $\phi(E_x) \subseteq E'_x$, for any $x \in \mathbf{R}$.

We shall denote by $\mathcal{H}om\,(E_*, E'_*)$ the sheaf of homomorphisms of $\mathbf{R}$-filtered sheaves from E_* to E'_*; it is a subsheaf of $\mathcal{H}om\,(E, E')$. We shall also write $\mathcal{E}nd\,(E_*)$ for $\mathcal{H}om\,(E_*, E_*)$.

With these definitions the notion of parabolic homomorphism of two parabolic sheaves becomes very simple:

Definition 1.4. If E_* and E'_* are two parabolic sheaves, a parabolic homomorphism $\phi : E_* \to E'_*$ is a homomorphism of $\mathbf{R}$-filtered sheaves.

Now let Y be a locally Noetherian scheme defined over k, and let us denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the canonical projections. We shall need the following definition:

Definition 1.5. A Y-flat family of parabolic sheaves on X is a triple (E, F_*, α_*), where E is a Y-flat family of coherent torsion-free sheaves, F_* is a filtration of E and α_* is a system of weights, such that all the sheaves $E/F_i(E)$ are flat over Y.

If (E, F_*, α_*) is a Y-flat family of parabolic sheaves, it follows from the definition that all the subsheaves $F_i(E)$ of E are flat over Y. This implies that we can associate to (E, F_*, α_*) an $\mathbf{R}$-filtered sheaf E_* as before; hence, we can denote a Y-flat family of parabolic sheaves (E, F_*, α_*) simply by its associated $\mathbf{R}$-filtered sheaf E_*.

In the sequel we shall be particularly interested in a special class of parabolic sheaves, namely locally free parabolic sheaves (also called parabolic bundles).

Definition 1.6. A parabolic sheaf E_* is said to be locally free, or a parabolic bundle, if, for any x, E_x is a locally free $\mathcal{O}_X$-module and, for any x, y, with $x \leq y < x + 1$, E_x/E_y is a locally free $\mathcal{O}_D$-module.

The obvious definition of a Y-flat family of locally free parabolic sheaves (or parabolic bundles) on X is left to the reader.

2. Preliminaries on trace maps.

2.1. Cup-product and trace maps. In this section we shall generalize, to the case of parabolic bundles, the description of trace maps given in [**4**] for ordinary sheaves.

Let E_* be a parabolic bundle on X, and let $\mathcal{E}nd\,(E_*)$ be the sheaf of parabolic endomorphisms of E_*. Since $\mathcal{E}nd\,(E_*)$ is a subsheaf of $\mathcal{E}nd\,(E)$, the usual trace map

$$\mathrm{tr}\ : \mathcal{E}nd\,(E) \longrightarrow \mathcal{O}_X$$

restricts to a trace map, denoted by the same symbol,

$$\mathrm{tr}\ : \mathcal{E}nd\,(E_*) \longrightarrow \mathcal{O}_X.$$

This map, in turn, induces natural maps (also called trace maps and denoted again by the same symbol)

$$\mathrm{tr}\ : H^i(X, \mathcal{E}nd\,(E_*)) \longrightarrow H^i(X, \mathcal{O}_X).$$

For any i and j there is a natural cup-product (or Yoneda composition) map

$$H^i(X, \mathcal{E}nd\,(E_*)) \times H^j(X, \mathcal{E}nd\,(E_*)) \stackrel{\circ}{\longrightarrow} H^{i+j}(X, \mathcal{E}nd\,(E_*))$$

and the composition of cup-product and trace is graded commutative in the following sense: if $\alpha \in H^i(X, \mathcal{E}nd\,(E_*))$ and $\beta \in H^j(X, \mathcal{E}nd\,(E_*))$, then

$$\mathrm{tr}\,(\alpha \circ \beta) = (-1)^{ij}\mathrm{tr}\,(\beta \circ \alpha) \tag{2.1}$$

as an element of $H^{i+j}(X, \mathcal{O}_X)$.

Analogous maps can also be defined in a relative situation. Let Y be a locally Noetherian scheme defined over k, and denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the canonical projections. Let E_* be a Y-flat family of parabolic bundles on X.

Then, for any i and j, there is a cup-product map

$$R^i q_* \mathcal{E}nd\,(E_*) \times R^j q_* \mathcal{E}nd\,(E_*) \stackrel{\circ}{\longrightarrow} R^{i+j} q_* \mathcal{E}nd\,(E_*)$$

and a trace map

$$\mathrm{tr}\,:R^iq_*\mathcal{E}nd\,(E_*)\longrightarrow R^iq_*\mathcal{O}_{X\times Y}\cong H^i(X,\mathcal{O}_X)\otimes_k\mathcal{O}_Y$$

satisfying (2.1) for any sections α and β of the sheaves $R^iq_*\mathcal{E}nd\,(E_*)$ and $R^jq_*\mathcal{E}nd\,(E_*)$, respectively.

2.2. The symmetrized trace map. Let E_* be a parabolic bundle on X. For any integer $m\geq 1$ let us consider the "symmetrized composition map"

$$\underbrace{\mathcal{E}nd\,(E_*)\times\cdots\times\mathcal{E}nd\,(E_*)}_{m}\xrightarrow{S}\mathcal{E}nd\,(E_*) \tag{2.2}$$

defined by setting

$$S(\phi_1,\phi_2,\ldots,\phi_m)=\frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\phi_{\sigma(1)}\circ\phi_{\sigma(2)}\circ\cdots\circ\phi_{\sigma(m)},$$

where the sum runs over the group $\mathfrak{S}_m$ of permutations of m elements.

We define the "symmetrized trace," denoted by Str, to be the composition of S with the usual trace map:

$$\mathrm{Str}\,(\phi_1,\phi_2,\ldots,\phi_m)=\frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\mathrm{tr}\,(\phi_{\sigma(1)}\circ\phi_{\sigma(2)}\circ\cdots\circ\phi_{\sigma(m)}). \tag{2.3}$$

The map

$$\mathrm{Str}:\mathcal{E}nd\,(E_*)\times\cdots\times\mathcal{E}nd\,(E_*)\longrightarrow\mathcal{O}_X \tag{2.4}$$

is totally symmetric and multi-linear.

For $m=2$, we have $\mathrm{Str}\,(\phi_1,\phi_2)=\mathrm{tr}\,(\phi_1\circ\phi_2)$, because the usual trace satisfies the following symmetry property:

$$\mathrm{tr}\,(\phi_1\circ\phi_2)=\mathrm{tr}\,(\phi_2\circ\phi_1).$$

The symmetrized trace map (2.4) induces a map, also denoted by Str,

$$\mathrm{Str}:H^{i_1}(X,\mathcal{E}nd\,(E_*))\times\cdots\times H^{i_m}(X,\mathcal{E}nd\,(E_*))\to H^{i_1+\cdots+i_m}(X,\mathcal{O}_X). \tag{2.5}$$

This map satisfies a kind of graded commutativity property similar to the one stated in (2.1).

Proposition 2.1. *Let* $\phi_h \in H^{i_h}(X, \mathcal{E}nd\,(E_*))$*, for* $h = 1, \dots, m$*. For any integer* p*, with* $1 \le p \le m-1$*, we have:*

$$\mathrm{Str}\,(\phi_1, \dots, \phi_p, \phi_{p+1}, \dots, \phi_m) = (-1)^{i_p i_{p+1}} \mathrm{Str}\,(\phi_1, \dots, \phi_{p+1}, \phi_p, \dots, \phi_m),$$

i.e., whenever we mutually exchange two adjacent elements ϕ_p *and* ϕ_{p+1}*, the value of* Str *acquires the factor* $(-1)^{\deg(\phi_p)\deg(\phi_{p+1})} = (-1)^{i_p i_{p+1}}$.

Proof. The proof is analogous to the proof of Proposition 1.1 in [**4**]. □

In the sequel we shall be interested in a special case of (2.5). By taking all i_h equal to 1, we get the map

$$\mathrm{Str} : \underbrace{H^1(X, \mathcal{E}nd\,(E_*)) \times \cdots \times H^1(X, \mathcal{E}nd\,(E_*))}_{m} \longrightarrow H^m(X, \mathcal{O}_X),$$

satisfying

$$\mathrm{Str}\,(\phi_1, \dots, \phi_p, \phi_{p+1}, \dots, \phi_m) = -\mathrm{Str}\,(\phi_1, \dots, \phi_{p+1}, \phi_p, \dots, \phi_m),$$

for every $p \in [1, m-1]$.

Since every permutation σ of $\{1, 2, \dots, m\}$ may be expressed as the product of a certain number s of transpositions of adjacent elements $\pi_{p,p+1}$, and since the sign of σ is given by $\mathrm{sgn}\,(\sigma) = (-1)^s$, we deduce the following result:

Corollary 2.2. *For any* $m \ge 1$*, the map*

$$\mathrm{Str} : \underbrace{H^1(X, \mathcal{E}nd\,(E_*)) \times \cdots \times H^1(X, \mathcal{E}nd\,(E_*))}_{m} \longrightarrow H^m(X, \mathcal{O}_X)$$

is alternating, i.e., for any permutation σ *of* $\{1, 2, \dots, m\}$*, we have*:

$$\mathrm{Str}\,(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots, \alpha_{\sigma(m)}) = \mathrm{sgn}\,(\sigma)\mathrm{Str}\,(\alpha_1, \alpha_2, \dots, \alpha_m).$$

Now let Y be a locally Noetherian scheme defined over k, and let E_* be a Y-flat family of parabolic bundles on X. The preceding constructions can be generalized to this relative situation (exactly as in the case of the usual trace map, described in subsection 2.1). We leave the details to the reader and just state the relative version of Corollary 2.2:

Corollary 2.3. *Let E_* be a Y-flat family of parabolic bundles on X. For any $m \geq 1$, the map*

(2.6)
$$\mathrm{Str} : \underbrace{R^1 q_* \mathcal{E}nd\,(E_*) \times \cdots \times R^1 q_* \mathcal{E}nd\,(E_*)}_{m} \longrightarrow R^m q_* (\mathcal{O}_{X \times Y}) \cong H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_Y$$

is alternating.

Remark 2.4. In this section we have assumed, for simplicity of exposition, that E_* is a parabolic bundle (or a family of parabolic bundles) on X. If, more generally, E_* is a parabolic sheaf on X (i.e., if it is not necessarily locally free), it is necessary to replace the cohomology groups $H^i(X, \mathcal{E}nd\,(E_*))$ with the "parabolic Ext" groups $\mathrm{Ext}^i(E_*, E_*)$ (see [**8**] for the definition). The construction of trace maps, cup-products and symmetrized trace maps can be naturally extended to this more general setting.

3. Vector-valued differential forms. Let Y be a smooth scheme of finite type over k and E_* a Y-flat family of parabolic bundles on X. In this section we shall define, for any such E_* and any integer m, with $1 \leq m \leq n = \dim X$, a vector-valued m-form on Y (more precisely, a differential form of degree m on Y with values in $H^m(X, \mathcal{O}_X)$). Then we shall prove that these differential forms are closed.

Let us begin by recalling that, for any parabolic bundle F_* on X, the set of isomorphism classes of infinitesimal deformations of F_* is canonically identified with $H^1(X, \mathcal{E}nd\,(F_*))$. It follows that, for any family E_* of parabolic bundles on X parametrized by Y, there is a map, called the Kodaira-Spencer map,

$$\rho : TY \longrightarrow R^1 q_* \mathcal{E}nd\,(E_*), \tag{3.1}$$

that sends a tangent vector $v \in T_yY$ to the class $\rho(v) \in H^1(X, \mathcal{E}nd\,(E_{*y}))$ corresponding to the infinitesimal deformation of the parabolic bundle E_{*y} along the direction of v.

Now, for any m as above, we define an $H^m(X, \mathcal{O}_X)$-valued differential m-form $\Omega = \Omega_{E_*}$ on Y by setting

$$\Omega : \underbrace{TY \times \cdots \times TY}_{m} \longrightarrow R^1q_*\mathcal{E}nd\,(E_*) \times \cdots \times R^1q_*\mathcal{E}nd\,(E_*) \longrightarrow H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_Y,$$

where the first map is induced by the Kodaira-Spencer map (3.1), and the second one is the symmetrized trace map (2.6). In other words, we set

$$\Omega(v_1, \ldots, v_m) = \mathrm{Str}\,(\rho(v_1), \ldots, \rho(v_m)), \tag{3.2}$$

for any sections $v_1, \ldots, v_m$ of the tangent bundle TY. It follows from Corollary 2.3 that Ω is a vector-valued differential form of degree m.

Remark 3.1. Let E_* be a Y-flat family of parabolic bundles on X and L be a line bundle on Y. We can define another Y-flat family of parabolic bundles E'_* on X by setting $E'_* = E_* \otimes q^*(L)$. These two families of parabolic bundles may be considered as equivalent because, for every closed point $y \in Y$, the parabolic bundles E_{*y} and E'_{*y} on X are isomorphic. Under these hypotheses, the differential m-forms Ω_{E_*} and $\Omega_{E'_*}$ are equal.

Remark 3.2. Let us observe that in the definition of Ω_{E_*} we do not use directly the parabolic bundle E_*, but rather the sheaf $R^1q_*\mathcal{E}nd\,(E_*)$. This is very important because in most interesting applications, when we take as Y a suitable moduli space of stable parabolic bundles on X, a global universal family of parabolic bundles E_* does not exist, but the sheaf $R^1q_*\mathcal{E}nd\,(E_*)$ on Y is, nevertheless, well defined (cf. Remark 4.3). It follows that our definition of the differential form Ω_{E_*} remains valid also in these more general situations.

Then we have:

Theorem 3.3. *For any Y-flat family E_* of parabolic bundles on X, and any integer m, with $1 \leq m \leq \dim X$, the construction described above defines a closed m-form on Y with values in $H^m(X, \mathcal{O}_X)$.*

The proof of this result can be obtained by a suitable modification of the proof of the analogous result for a Y-flat family of ordinary vector bundles (see [**4**, Theorem 2.8]).

4. Differential forms on moduli spaces. In this section we shall apply the results of Section 3 to the construction of closed differential forms on moduli spaces of parabolic bundles.

Let X be a nonsingular projective variety defined over an algebraically closed field k of characteristic 0, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X.

In order to construct moduli spaces of parabolic sheaves we need, as usual, a suitable notion of stability. This was introduced in [**6**], where moduli spaces of semi-stable parabolic sheaves were constructed in great generality. We only state here the results we shall need in the sequel.

Proposition 4.1. *Let us fix a sequence of rational numbers $\alpha_* = (\alpha_1, \dots, \alpha_l)$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$, and polynomials $H, H_1, \dots, H_l$. Then there exists a quasi-projective moduli space $\mathcal{PS}$ parametrizing isomorphism classes of stable parabolic sheaves E_* having α_* as system of weights and such that the Hilbert polynomial of E is H and the Hilbert polynomial of $E/F_{i+1}(E)$ is H_i, for $i = 1, \dots, l$.*

Let us restrict our attention to the open subset $\mathcal{PB}$ of the moduli space $\mathcal{PS}$ which parametrizes isomorphism classes of parabolic bundles.

Infinitesimal deformation theory for parabolic sheaves (cf. [**8**]) yields the following result:

Proposition 4.2. *The tangent space $T_{E_*}\mathcal{PB}$ to the moduli space $\mathcal{PB}$ at a point E_* is canonically identified with the cohomology group $H^1(X, \mathcal{E}nd\,(E_*))$ and the obstruction to the smoothness of $\mathcal{PB}$ at the point E_* lies in $H^2(X, \mathcal{E}nd\,(E_*))$.*

If X is a nonsingular curve, the moduli space $\mathcal{PB}$ is smooth because the obstruction group $H^2(X, \mathcal{E}nd\,(E_*))$ trivially vanishes. The same thing happens if X is a smooth projective surface with an effective divisor D such that $H^0(X, \omega_X^{-1}(-D)) \neq 0$: in this case the vanishing of the obstruction group is a consequence of the stability of the parabolic bundle (cf. [**3**, Proposition 3.2]). On the other hand, when $\dim X \geq 3$, the moduli spaces of stable bundles or stable parabolic bundles on X are usually singular. We shall denote by $\mathcal{PB}^{sm}$ the smooth locus of the moduli space $\mathcal{PB}$.

Remark 4.3. On the moduli space $\mathcal{PB}$ there does not exist, in general, a universal family of parabolic bundles $\mathcal{E}_*$, not even locally in the Zariski topology. In any case, a universal family $\mathcal{E}_*$ on $\mathcal{PB}$ exists locally in the étale topology (or in the complex analytic topology, if $k = \mathbf{C}$). As noted in Remark 3.1, these local universal families are not uniquely determined, in fact they are defined only up to tensoring with the pull-back of a line bundle on $\mathcal{PB}$. In general, these ambiguities prevent the local universal families from glueing together to a globally defined one. On the other hand, when we consider the sheaf of parabolic endomorphisms $\mathcal{E}nd\,(\mathcal{E}_*)$, these ambiguities disappear, and these locally defined sheaves glue together to a globally defined one on $\mathcal{PB}$. For this reason, we shall abuse the notation and write $\mathcal{E}nd\,(\mathcal{E}_*)$ even if the universal family $\mathcal{E}_*$ does not exist on $\mathcal{PB}$.

With these notations, we can state the global version of Proposition 4.2:

Proposition 4.4. *Let $\mathcal{E}_*$ be a (locally defined) universal family of parabolic bundles on $\mathcal{PB}^{sm}$. Then there is a natural isomorphism (given by the Kodaira-Spencer map)*

$$T\mathcal{PB}^{sm} \cong R^1 q_* \mathcal{E}nd\,(\mathcal{E}_*).$$

We can now apply the results of the preceding section to construct natural differential forms on the moduli space $\mathcal{PB}^{sm}$.

More precisely, by setting $Y = \mathcal{PB}^{sm}$ and denoting by $\mathcal{E}_*$ a locally defined universal family on Y, we have, for any m with $1 \leq m \leq n = \dim X$, a vector-valued m-form

$$\Omega : T\mathcal{PB}^{sm} \times \cdots \times T\mathcal{PB}^{sm} \longrightarrow H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_{\mathcal{PB}^{sm}}. \tag{4.1}$$

Let us now assume that, for some $i \leq \dim X$, there exists a nonzero element $\sigma \in H^i(X, \Omega_X^{i+m})$. The multiplication by σ defines a map

$$H^m(X, \mathcal{O}_X) \xrightarrow{\sigma} H^{i+m}(X, \Omega_X^{i+m}). \tag{4.2}$$

Finally, if we denote by $\eta_X \in H^1(X, \Omega_X^1)$ the cohomology class of the polarization $\mathcal{O}_X(1)$, we have a map

$$H^{i+m}(X, \Omega_X^{i+m}) \xrightarrow{\eta_X^{n-i-m}} H^n(X, \Omega_X^n) \cong k. \tag{4.3}$$

By composing the vector-valued differential form Ω with the maps (4.2) and (4.3), we obtain an ordinary (scalar-valued) m-form, which we denote by Ω_σ:

$$\Omega_\sigma : T\mathcal{PB}^{sm} \times \cdots \times T\mathcal{PB}^{sm} \longrightarrow \mathcal{O}_{\mathcal{PB}^{sm}}.$$

Since the vector-valued form Ω is closed, it follows that Ω_σ is a closed differential m-form.

We can summarize these results as follows:

Theorem 4.5. *Let X be a smooth n-dimensional projective variety. For any $m \leq n$, $i \leq n - m$, and any $\sigma \in H^i(X, \Omega_X^{i+m})$, there is a naturally defined closed differential m-form Ω_σ on the smooth locus $\mathcal{PB}^{sm}$ of the moduli space of stable parabolic bundles on X.*

Remark 4.6. The construction of the differential m-form Ω (hence also of Ω_σ) can be extended to the smooth locus $\mathcal{PS}^{sm}$ of the moduli space $\mathcal{PS}$ of stable parabolic sheaves. Then, whenever $\mathcal{PB}^{sm}$ is a dense open subscheme of $\mathcal{PS}^{sm}$, the closure of Ω on $\mathcal{PB}^{sm}$ obviously implies the closure of Ω on $\mathcal{PS}^{sm}$. We conjecture that this is always true, i.e., that the m-form Ω on $\mathcal{PS}^{sm}$ is always closed.

Remark 4.7. Let us analyze in more detail the construction of 1-forms on the moduli space $\mathcal{PB}$. Let

$$\det : \mathcal{PB} \longrightarrow \operatorname{Pic}(X) \tag{4.4}$$

be the map sending a parabolic bundle E_* to the determinant line bundle of the underlying vector bundle E. Then, for any $E_* \in \mathcal{PB}$, the trace map

$$\mathrm{tr} : H^1(X, \mathcal{E}nd\,(E_*)) \longrightarrow H^1(X, \mathcal{O}_X)$$

is naturally identified with the tangent map to (4.4) at the point E_*.

We remark that the choice of an element $\sigma \in H^i(X, \Omega_X^{i+1})$ determines a (translation-invariant) closed 1-form ω_σ on $\mathrm{Pic}\,(X)$, defined as follows: for any $L \in \mathrm{Pic}\,(X)$, the map

$$\omega_\sigma(L) : T_L \mathrm{Pic}\,(X) \longrightarrow k$$

is given by the composition of the following maps:

$$H^1(X, \mathcal{O}_X) \xrightarrow{\sigma} H^{i+1}(X, \Omega_X^{i+1}) \xrightarrow{\eta_X^{n-i-1}} H^n(X, \Omega_X^n) \cong k.$$

It follows that the 1-form Ω_σ on $\mathcal{PB}$ constructed from $\sigma \in H^i(X, \Omega_X^{i+1})$ is the pull-back, via the determinant map, of the 1-form ω_σ on $\mathrm{Pic}\,(X)$; $\Omega_\sigma = (\det)^* \omega_\sigma$.

Let us now describe some examples of applications of Theorem 4.5.

Let X be a smooth variety with an effective divisor D, and let us consider parabolic vector bundles on X with trivial parabolic structure over D, i.e., vector bundles E with parabolic structure given by

$$E = F_1(E) \supset F_2(E) = E(-D),$$

and weight $\alpha_1 = 0$. In this case the moduli space $\mathcal{PB}$ coincides with the moduli space $\mathcal{M}$ of stable vector bundles on X, and the construction of closed differential forms on $\mathcal{PB}$ specializes to the construction of closed forms on $\mathcal{M}$, described in [**4**].

As another application let us recall that in [**5**] Maruyama proved that a certain moduli space of parabolic bundles on the projective plane, with parabolic structure over a line, can be identified with a moduli space of marked $\mathrm{SU}\,(r)$-instantons on the sphere.

Generalizing this result, Biquard proved in [**2**] that, for a smooth projective surface X and an effective divisor D, a moduli space of irreducible anti self-dual connections on the restrictions to $X \setminus D$ of an $\mathrm{SU}\,(2)$-bundle on X can be identified with a moduli space of stable parabolic $\mathrm{SL}\,(2)$-bundles on X. In this case our results can be applied to construct natural differential forms on these moduli spaces of anti self-dual connections.

Let us now describe in more detail our construction applied to the case of a smooth projective surface X with a holomorphic 2-form $\sigma \in H^0(X, \omega_X)$.

Let D be an effective divisor on X, and let $\mathcal{PB}$ be a moduli space of parabolic bundles on X with parabolic structure over D. The choice of σ determines a closed holomorphic 2-form Ω_σ on $\mathcal{PB}^{sm}$. This 2-form will usually be degenerate, i.e., it will define a so-called *pre-symplectic structure* on $\mathcal{PB}^{sm}$. Giving the 2-form Ω_σ is equivalent to giving the corresponding "Hamiltonian" morphism

$$H_\sigma(E_*) : T_{E_*}\mathcal{PB} \longrightarrow T^*_{E_*}\mathcal{PB},$$

for any $E_* \in \mathcal{PB}^{sm}$.

Since $T_{E_*}\mathcal{PB} \cong H^1(X, \mathcal{H}om\,(E_*, E_*))$, by the version of Serre duality for parabolic bundles ([**8**, Proposition 3.7]), we have $T^*_{E_*}\mathcal{PB} \cong H^1(X, \mathcal{H}om\,(E_*, \hat{E}_* \otimes \omega_X(D)))$, where $\hat{E}_*$ is the $\mathbf{R}$-filtered sheaf defined by setting, for any $x \in [0,1]$,

$$\widehat{E}_x = \begin{cases} E_x & \text{if } x \neq \alpha_i, \\ E_{\alpha_{i+1}} & \text{if } x = \alpha_i, \end{cases}$$

and by extending the definition to all $x \in \mathbf{R}$ by setting $\widehat{E}_{x+1} = \widehat{E}_x(-D)$.

We remark that there is a natural injective morphism of $\mathbf{R}$-filtered sheaves

$$j : E_* \hookrightarrow \widehat{E}_* \otimes \mathcal{O}_X(D).$$

For $x \neq \alpha_i$ this is the map

$$E_x \xrightarrow{s} E_x \otimes \mathcal{O}_X(D)$$

induced by the multiplication by s, where s is a section defining the effective divisor D, while for $x = \alpha_i$ the map j is given by the natural inclusion

$$E_{\alpha_i} \hookrightarrow E_{\alpha_{i+1}} \otimes \mathcal{O}_X(D) = E_{\alpha_{i+1}-1}$$

defined by the parabolic structure over D.

Let us now consider the composition of the following maps:

$$E_* \xrightarrow{\sigma} E_* \otimes \omega_X \xrightarrow{j} \widehat{E}_* \otimes \omega_X(D).$$

By applying the functor $\mathcal{H}om\,(E_*, \cdot)$ we get an induced map on the first cohomology groups

$$H^1(X, \mathcal{H}om\,(E_*, E_*)) \xrightarrow{j\circ\sigma} H^1(X, \mathcal{H}om\,(E_*, \widehat{E}_* \otimes \omega_X(D))).$$

Finally, by recalling the construction of the 2-form Ω_σ on $\mathcal{PB}$, it is now easy to prove that this map is precisely the Hamiltonian map $H_\sigma(E_*)$.

Remark 4.8. An explicit description of the Hamiltonian morphism corresponding to a pre-symplectic structure on $\mathcal{PB}$ can also be given when X is a variety of dimension ≥ 3. We shall describe it very briefly, skipping all details.

So, let X be a smooth n-dimensional variety, D an effective divisor on X and σ a holomorphic 2-form on X, as above. In this case, Serre duality implies that $T^*_{E_*}\mathcal{PB} \cong H^{n-1}(X, \mathcal{H}om\,(E_*, \widehat{E}_* \otimes \omega_X(D)))$, hence the Hamiltonian morphism corresponding to the 2-form Ω_σ on $\mathcal{PB}^{sm}$ will be a map

(4.5)

$$H_\sigma(E_*) : H^1(X, \mathcal{H}om\,(E_*, E_*)) \longrightarrow H^{n-1}(X, \mathcal{H}om\,(E_*, \widehat{E}_* \otimes \omega_X(D))),$$

for any $E_* \in \mathcal{PB}^{sm}$. To describe this map let us consider the "parabolic Atiyah class" $a(E_*) \in H^1(X, \mathcal{H}om\,(E_*, E_* \otimes \Omega^1_X))$ associated to the parabolic bundle E_* (this is the parabolic analogue of the usual Atiyah class $a(E) \in H^1(X, \mathcal{H}om\,(E, E \otimes \Omega^1_X))$ of a vector bundle E). By repeatedly composing the parabolic Atiyah class $a(E_*)$ with itself, we obtain classes in $H^i(X, \mathcal{H}om\,(E_*, E_* \otimes (\Omega^1_X)^{\otimes i}))$, for any $i \geq 1$. Then, by composing with the map induced by the natural map $(\Omega^1_X)^{\otimes i} \to \Omega^i_X$, we get classes $a(E_*)^i \in H^i(X, \mathcal{H}om\,(E_*, E_* \otimes \Omega^i_X))$. We can now describe the Hamiltonian morphism (4.5) (possibly up to a nonzero constant factor) as the composition of the following maps: first we consider the map

$$H^1(X, \mathcal{H}om\,(E_*, E_*)) \longrightarrow H^{n-1}(X, \mathcal{H}om\,(E_*, E_* \otimes \Omega^{n-2}_X))$$

induced by the composition with the cohomology class $a(E_*)^{n-2}$; then we compose with the map

$$H^{n-1}(X, \mathcal{H}om\,(E_*, E_* \otimes \Omega_X^{n-2})) \xrightarrow{\sigma} H^{n-1}(X, \mathcal{H}om\,(E_*, E_* \otimes \omega_X))$$

determined by the wedge product with $\sigma \in H^0(X, \Omega_X^2)$, and finally we compose with the map

$$H^{n-1}(X, \mathcal{H}om\,(E_*, E_* \otimes \omega_X)) \xrightarrow{j} H^{n-1}(X, \mathcal{H}om\,(E_*, \hat{E}_* \otimes \omega_X(D)))$$

induced by the natural inclusion $j : E_* \hookrightarrow \widehat{E}_* \otimes \mathcal{O}_X(D)$.

As another example let us take $m = n = \dim X$ and assume that there exists a nonzero section σ of the canonical line bundle $K_X = \Omega_X^n$. In this case the map (4.3) is the identity; hence, the n-form Ω_σ is given by the composition of Ω in (4.1) with the map

$$H^n(X, \mathcal{O}_X) \xrightarrow{\sigma} H^n(X, K_X) \cong k.$$

A particularly interesting situation arises when X has an effective canonical divisor and we take $D \in |K_X|$. In this case there is a canonical choice (up to a scalar multiple) of a section $\sigma \in H^0(X, K_X)$, namely, the section defining the effective divisor D. It follows that, on the moduli space $\mathcal{PB}^{sm}$ of stable parabolic bundles on X with parabolic structure over D, there is a canonical closed differential n-form Ω_σ.

Another special case where there is a canonical choice of the section σ is when the canonical line bundle of X is trivial. In this case the natural choice for σ is $\sigma = 1 \in H^0(X, K_X) \cong k$. It follows that, in this case too, there is a canonical n-form on the moduli space $\mathcal{PB}^{sm}$.

REFERENCES

1. U. Bhosle, *Parabolic sheaves on higher dimensional varieties*, Math. Ann. **293** (1992), 177–192.

2. O. Biquard, *Sur les fibrés paraboliques sur une surface complexe*, J. London Math. Soc. **53** (1996), 302–316.

3. F. Bottacin, *Poisson structures on moduli spaces of parabolic bundles on surfaces*, Manuscr. Math. **103** (2000), 31–46.

4. ———, *Closed differential forms on moduli spaces of sheaves*, Rend. Mat. Univ. Roma. **28** (2008), 139–162.

5. M. Maruyama, *Instantons and parabolic sheaves*, in *Geometry and analysis*, Tata Inst. Fund. Res., Bombay, 1995.

6. M. Maruyama and K. Yokogawa, *Moduli of parabolic stable sheaves*, Math. Ann. **293** (1992), 77–99.

7. K. Yokogawa, *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves*, J. Math. Kyoto Univ. **33** (1993), 451–504.

8. ———, *Infinitesimal deformations of parabolic Higgs sheaves*, International J. Math. **6** (1995), 125–148.

Dip. di Matematica Pura e Appl., Università degli Studi di Padova, Via Trieste 63, 35121, Padova, Italy
Email address: bottacin@math.unipd.it