ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 2, 2010

CHUNG'S LAW FOR HOMOGENEOUS BROWNIAN FUNCTIONALS

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Посвящается Ензо Орсингеру в честы его 60-летия.

ABSTRACT. Consider the first exit time $T_{a,b}$ from a finite interval $[-a,b]$ for a homogeneous fluctuating functional X of a linear Brownian motion. We show the existence of a finite positive constant $\mathcal{K}$ such that

$$\lim_{t\to\infty} t^{-1}\log \mathbf{P}[T_{ab}>t] = -\mathcal{K}.$$

Following Chung's original approach [**8**], we deduce a "liminf" law of the iterated logarithm for the two-sided supremum of X. This extends and gives a new point of view on a result of Khoshnevisan and Shi [**12**].

1. Introduction. Let $\{B_t,\ t\ge 0\}$ be a linear Brownian motion starting at 0 and $X=\{X_t,\ t\ge 0\}$ be the homogeneous fluctuating additive functional defined by

$$X_t = \int_0^t V(B_s)\,\mathrm{d}s, \quad t\ge 0,$$

where $V(x)=x^\alpha$ if $x\ge 0$ and $V(x)=-\lambda|x|^\alpha$ if $x\le 0$, for some fixed $\alpha,\lambda>0$. The process X appears in mathematical physics as the solution of a generalized Langevin equation involving a harmonic oscillator driven by a white noise, and we refer to [**14**] and the references therein for more details on this subject. Notice that X is $(1+\alpha/2)$ self-similar, but has no stationary increments. In the case $\alpha=\lambda=1$, it is the integrated Brownian motion:

$$X_t = \int_0^t B_s\,\mathrm{d}s, \quad t\ge 0,$$

2010 AMS *Mathematics subject classification.* Primary 60F99, 60G17, 60G18, 60J55, 60J65.
Keywords and phrases. First passage time, fluctuating additive functional, law of the iterated logarithm, small ball probabilities.
Received by the editors on April 29, 2007, and in revised form on October 17, 2007.

DOI:10.1216/RMJ-2010-40-2-561

and also a Gaussian process. However, in the other cases, it is not Gaussian any longer. For every $a, b > 0$ consider the bilateral exit time

$$T_{ab} = \inf\{t > 0,\ X_t \notin (-a, b)\}.$$

As a rule, studying the law of T_{ab} is a difficult issue because X alone is not Markov, so that no spectral theory is available. We refer however to [**14, 15**] for several distributional properties of the bivariate random variable $(T_{ab}, B_{T_{ab}})$ and for the solution to the two-sided exit problem, i.e., the computation of the probability $\mathbf{P}[X_{T_{ab}} = a]$. In [**14**], it was also shown that the variable T_{ab} has moments of any power, and an explicit upper bound was given on the latter, see Proposition 7.1 therein. Before this, the upper tails of T_{ab} in the case $\alpha = \lambda = 1$ had been precisely investigated in [**12**], with an elegant argument relying on Chung's law of the iterated logarithm. This result was then generalized in [**18**] to a broad class of Gaussian and sub-Gaussian processes, with a different method relying on wavelet decomposition. In this paper, we aim at extending the results of [**12**] to the above non-Gaussian functionals X, with a more elementary proof:

Theorem. *For every $a, b > 0$, there exists a finite positive constant $\mathcal{K}$ such that*

$$\lim_{t\to\infty} t^{-1} \log \mathbf{P}[T_{ab} > t] = -\mathcal{K}. \tag{1.1}$$

This exponential tail behavior is typical for exit-times from a finite interval for self-similar random processes. Actually, in most examples available, it appears that the upper tails of the variable T_{ab} are those of an exponential random variable. Some comments on this somewhat intriguing universal behavior are given in the last section of [**18**] in the case of a sub-Gaussian symmetric process exiting a symmetric interval. See, however, Example 3.3 in [**20**], where the tail behavior is shown to be subexponential. Notice also that the upper tails of the *unilateral* exit time $T_{a\infty}$ of X had been thoroughly studied in [**10, 11**] and exhibit an entirely different, polynomial, behavior which again in the framework of self-similar random processes is typical for exit-times from a semi-finite interval.

Taking $a = b = 1$, the estimate (1.1) entails by self-similarity that there exists a finite positive constant $\mathcal{K}'$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-2/(\alpha+2)} \log \mathbf{P}[||X||_\infty < \varepsilon] = -\mathcal{K}', \tag{1.2}$$

where $||.||_\infty$ stands for the supremum norm over $[0, 1]$. This other limit theorem is known as a small ball probability estimate, a subject which has given rise to intensive research over the last years, with interesting connections to different questions in analysis, probability and statistics. We refer to [**17, 23**] for recent accounts on this topic concerning both Gaussian and non Gaussian processes; see also [**16**, Chapter 7] for an abstract Wiener setting. Originally, this kind of estimate had been used by Chung [**8**] for random walks and Brownian motion, in connection with his celebrated law of the iterated logarithm. In [**12**], Khoshnevisan and Shi's original approach for integrated Brownian motion consisted in proving first Chung's LIL and then deducing the small deviation estimate (1.2). In this paper, we will follow the more standard approach viewing Chung's LIL as a consequence of (1.2). We introduce the notations

$$X_t^* = \sup\{|X_s|,\ s \le t\} \quad \text{and} \quad f(t) = (t/\log\log t)^{(\alpha+2)/2}$$

for every $t > \mathrm{e}$ and set $\mathcal{K}_1$ for the constant appearing in (1.1) when $a = b = 1$.

Corollary (Chung's law of the iterated logarithm). *One has*

$$\liminf_{t \to +\infty} \frac{X_t^*}{f(t)} = \mathcal{K}_1^{(\alpha+2)/2} \quad a.s.$$

Notice that if we introduce the family of time-stretched functionals

$$X_t^n = \frac{X_{nt}}{(n/\log\log n)^{(\alpha+2)/2}}, \quad t \in [0, 1],$$

for every $n \ge 3$, then by a straightforward monotonicity argument our Chung's LIL is equivalent to

$$\liminf_{n \to +\infty} \|X^n\|_\infty = \mathcal{K}_1^{(\alpha+2)/2} \quad \text{a.s.}$$

From this fact and in the spirit of Wichura's functional LIL, it is an interesting question to determine the cluster set of the family of processes $\{X^n,\, n \geq 1\}$ for the weak topology. This was indeed recently investigated by Lin and Zhang [**19**] for m-fold integrated Brownian motion, yielding Chung's LIL for these processes as a corollary, see Theorem 1.1 and Corollary 1.1 therein. However, in our framework, the nonlinearity of the kernel $x \mapsto V(x)$ and the non Gaussianity of X makes the situation significantly more complicated in general, as will already appear in our proof. Setting now

$$\widetilde{X}_t^n = \frac{X_{nt}}{(n \log\log n)^{(\alpha+2)/2}}, \quad t \in [0,1],$$

for every $n \geq 3$, our result reads

$$\liminf_{n\to+\infty} (\log\log n)^{\alpha+2} \|\widetilde{X}^n\|_\infty = \mathcal{K}_1^{(\alpha+2)/2} \text{ almost surely.}$$

From this fact and in the spirit of Strassen's functional LIL, it is somewhat tantalizing to determine the set of functions f such that

$$\liminf_{n\to+\infty} (\log\log n)^{\alpha+2} \|\widetilde{X}^n - f\|_\infty \tag{1.3}$$

almost surely exists, as an explicit function of f and $\mathcal{K}_1$. In the case of Brownian motion, this (hard) problem had been initiated by Csáki [**9**] and De Acosta [**1**], hinging upon shifted Brownian small balls. Of course, before investigating (1.3) one should first determine the cluster set for the weak topology of the family of processes $\{\widetilde{X}^n,\, n \geq 1\}$. To the best of our knowledge, no results of this kind seem to exist even for integrated Brownian motion.

2. Proof of the theorem. Fix $a, b > 0$ once and for all, and introduce the notation $T = T_{ab}$ for concision. For every $x, y \in \mathbf{R}$, set $\mathbf{P}_{(x,y)}$ for the law of the strong Markov process $t \mapsto (B_t, X_t)$ starting at (x, y). We keep the notation $\mathbf{P} = \mathbf{P}_{(0,0)}$ for brevity. Considering the function

$$\varphi(t) = \sup\{\mathbf{P}_{(x,y)}[T > t],\ (x,y) \in \mathbf{R} \times (-a, b)\},$$

the simple Markov property yields for every $t, s \geq 0$

$$\begin{aligned}
\varphi(t+s) &= \sup\{\mathbf{P}_{(x,y)}[T > s, T > t+s],\ (x,y) \in \mathbf{R} \times (-a,b)\} \\
&= \sup\left\{ \int_{\mathbf{R}} \int_a^b \mathbf{P}_{(x,y)}[(B_s, X_s) \in \mathrm{d}u\,\mathrm{dv},\, T > s] \mathbf{P}_{(u,v)}[T > t], \right. \\
&\qquad\qquad\qquad \left. (x,y) \in \mathbf{R} \times (-a,b) \right\} \\
&\leq \varphi(t) \\
&\quad \times \sup\left\{ \int_{\mathbf{R}} \int_a^b \mathbf{P}_{(x,y)}[(B_s, X_s) \in \mathrm{d}u\,\mathrm{dv},\, T > s], \right. \\
&\qquad\qquad\qquad \left. (x,y) \in \mathbf{R} \times (-a,b) \right\} \\
&\leq \varphi(t)\varphi(s),
\end{aligned}$$

so that the function $\psi(t) = \log \varphi(t)$ is subadditive. Hence, there exists a $\mathcal{K} \in [0, +\infty]$ such that

$$\lim_{t \to +\infty} t^{-1}\psi(t) = \inf_{t>0}(t^{-1}\psi(t)) = -\mathcal{K}.$$

Besides from the second equality we see that $\kappa > 0$, since the function ψ is clearly not identically zero. This entails

$$\limsup_{t \to \infty} t^{-1} \log \mathbf{P}[T > t] = -\mathcal{K} < 0. \tag{2.1}$$

The remainder of the proof will be given in two steps. First, we will show the finiteness of $\mathcal{K}$, which is usually the difficult part in small deviation problems. In the case $\alpha = \lambda = 1$, it had been obtained in [**12**] through an original yet lengthy argument relying on random normalization and Chung's LIL. Here we will provide two proofs which are considerably simpler. The first one adapts the elementary arguments of Lemma 1 in [**5**] to the two-dimensional Markov process (B, X), while the second one is based on the time-substitution method which was used in [**10**] for unilateral passage times; let us stress that its main idea relying on the a.s. continuity of the Brownian paths was also implicitly used in [**12**, page 4258] to obtain Chung's LIL. The latter proof is slightly more involved than the former; nevertheless, it allows to bound the constant from above (see Remark 1 below).

Second, we will show that the above limit in (2.1) is actually a true limit, which appears to be more complicated. In the Gaussian case $\alpha = \lambda = 1$ and for a symmetric exit interval, it is an easy consequence of Anderson's inequality, as already noticed in [**12**]. However, no isoperimetric inequalities seem available when X is not Gaussian, and this argument breaks down, so that we had to use more barehand estimates, following roughly the outline of Lemma 1 in [**5**].

First proof of the finiteness of the constant. Fixing $A < 0 < B$ and $a < c < 0 < d < b$, introduce the functions $\widetilde{\varphi}(t) = \inf\{\mathbf{P}_{(x,y)}[T > t],\ (x,y) \in [A,B] \times [c,d]\}$ and

$$\Phi(t) = \inf\{\mathbf{P}_{(x,y)}[(B_t, X_t) \in [A,B] \times [c,d],\, T > t], \\ (x,y) \in [A,B] \times [c,d]\}, \quad t \geq 0.$$

For every $(x,y) \in [A,B] \times [c,d]$ and $t, s \geq 0$ the simple Markov property entails

$$\begin{aligned}
&\mathbf{P}_{(x,y)}[T > t+s] \\
&\qquad \geq \mathbf{P}_{(x,y)}[(B_s, X_s) \in [A,B] \times [c,d],\ T > t+s] \\
&\qquad = \int_A^B \int_c^d \mathbf{P}_{(x,y)}[(B_s, X_s) \in \mathrm{d}u\,\mathrm{dv},\, T > s] \times \mathbf{P}_{(u,v)}[T > t] \\
&\qquad \geq \mathbf{P}_{(x,y)}[(B_s, X_s) \in [A,B] \times [c,d],\, T > s] \times \widetilde{\varphi}(t) \\
&\qquad \geq \Phi(s)\widetilde{\varphi}(t),
\end{aligned}$$

so that $\widetilde{\varphi}(t+s) \geq \widetilde{\varphi}(s)\Phi(t)$ for every $t, s \geq 0$. In particular,

$$\varphi(n) \geq \widetilde{\varphi}(n) \geq \Phi(1)\widetilde{\varphi}(n-1) \geq \cdots \geq \Phi(1)^n \widetilde{\varphi}(0) = \Phi(1)^n$$

for every $n \in \mathbf{N}$, which entails $t^{-1}\psi(t) \geq \log \Phi(1)$ for every $t > 0$, since the function $t \mapsto t^{-1}\psi(t)$ is decreasing. We finally get

$$\mathcal{K} \leq -\log \Phi(1).$$

Now the function $(x,y,t) \mapsto \mathbf{P}_{(x,y)}[(B_t, X_t) \in [A,B] \times [c,d],\, T > t]$ is continuous on the compact $[A,B] \times [c,d] \times [0,2]$, since it satisfies the heat equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(y)\frac{\partial}{\partial y} = \frac{\partial}{\partial t}$$

on $\mathbf{R} \times (-a, b) \times \mathbf{R}^+$. In particular, the function

$$(x, y) \longmapsto \mathbf{P}_{(x,y)}[(B_1, X_1) \in [A, B] \times [c, d],\ T > 1]$$

is continuous on the compact $[A, B] \times [c, d]$ and, since it is obviously everywhere positive, one has $\Phi(1) > 0$, which completes the proof. □

Second proof of the finiteness of the constant. Let $L = \{L(t, x),\ t \geq 0,\ x \in \mathbf{R}\}$ be the local-time process associated with B and

$$\tau_t = \inf\{u \geq 0,\ L(0, u) > t\}, \quad t \geq 0$$

be the inverse local time of B at zero. It follows easily from the Markov property and a scaling argument that the process $t \mapsto (\tau_t, X_{\tau_t})$ is a two-dimensional Lévy process such that $t \mapsto \tau_t$ is a $(1/2)-$stable subordinator and $Y : t \mapsto Y_t = X_{\tau_t}$ a $1/(\alpha + 2)$-stable process. Introducing

$$\Theta = \inf\{t > 0,\ X_{\tau_t} \notin (-a, b)\},$$

the a.s. continuity of Brownian trajectories yields the key-inequality

$$T \geq \tau_{\Theta-} \quad \text{a.s.} \tag{2.2}$$

As in the proof of Theorem B in [**22**], we now decompose, for every $c > 0$,

$$\begin{aligned} \mathbf{P}[\Theta > t] &\leq \mathbf{P}[\tau_t < ct] + \mathbf{P}[\Theta > t,\ \tau_t \geq ct] \\ &\leq \mathbf{P}[\tau_1 < ct^{-1}] + \mathbf{P}[\tau_{\Theta-} \geq ct] \\ &\leq \mathbf{P}[\tau_1 < ct^{-1}] + \mathbf{P}[T \geq ct], \end{aligned}$$

where we used the 2-self-similarity and the a.s. increasingness of τ in the second line, and (2.2) in the third. By [**4**, Proposition VIII.3] and a scaling argument, there exists $\mathcal{K}_0$ finite such that

$$\lim_{t\to\infty} t^{-1} \log \mathbf{P}[\Theta > t] = -\mathcal{K}_0.$$

By Theorem 5.12.9 in [**7**] there exists $\mathcal{K}_c \to +\infty$ as $c \to 0$ such that

$$\lim_{t\to\infty} t^{-1} \log \mathbf{P}[\tau_1 < ct^{-1}] = -\mathcal{K}_c.$$

Taking c small enough and putting everything together yields

$$\liminf_{t\to\infty} t^{-1} \log \mathbf{P}[T > t] \geq -\mathcal{K}_0/c > -\infty,$$

which entails $\mathcal{K} < +\infty$ as desired. $\square$

Remark 1. The positivity parameter $\mathbf{P}[Y_1 > 0]$ of the non completely asymmetric Lévy $1/(\alpha+2)$-stable process Y had been computed in [**11**]; see Remark 4 therein. This makes it possible to bound from above the constant $\mathcal{K}_0$ explicitly: when $\lambda = 1$, i.e., Y is symmetric; this can be done in subordinating Y to some Brownian motion, see [**3**, Theorem 4] or [**21**, Proposition 8], whereas when $\lambda \neq 1$, the same method works in subordinating Y to some completely asymmetric stable process with infinite variation, see [**4**, Exercise VIII.1], and using the explicit calculations of [**5**] in the completely asymmetric case. On the other hand, the scaling parameter of the stable subordinator τ is explicit, so that the constants $\mathcal{K}_c$ are also explicit, again by [**7**, Theorem 5.12.9]. To put it in a nutshell, our second proof allows to exhibit an explicit upper bound on $\mathcal{K}$, which we will however not include here for the sake of brevity. Notice that in the case of integrated Brownian motion in a symmetric interval, a lower bound had been given in [**12**, Remark 1.4]. Recall also that in the non completely asymmetric framework, the exact computation of $\mathcal{K}_0$ is a long-standing and challenging problem, see [**2, 3, 5**] and the references therein.

Proof of the existence of the constant. Suppose first that $\alpha = \lambda = 1$ and $a = b$. Then, by self-similarity and by linearity of the integral one has, for every $x, y \in \mathbf{R}$ and $t > 0$,

$$\begin{aligned}\mathbf{P}_{(x,y)}[T > t] &= \mathbf{P}_{(xt^{-1/2}, yt^{-3/2})}[\|X\|_\infty < at^{-3/2}] \\ &= \mathbf{P}[\|X + f^{x,y,t}\|_\infty < at^{-3/2}]\end{aligned}$$

where $\|\cdot\|_\infty$ stands for the supremum norm over $[0, 1]$ and $f^{x,y,t} : u \mapsto yt^{-3/2} + uxt^{-1/2}$. Hence, Anderson's inequality, see e.g. (7.5) in [**16**], entails

$$\begin{aligned}\mathbf{P}_{(x,y)}[T > t] &= \mathbf{P}[\|X + f^{x,y,t}\|_\infty < at^{-3/2}] \leq \mathbf{P}[\|X\|_\infty < at^{-3/2}] \\ &= \mathbf{P}[T > t],\end{aligned}$$

so that $\varphi(t) = \mathbf{P}[T > t]$ for every $t > 0$, and (2.1) is a true limit. Unfortunately, this simple Gaussian argument cannot be used in general, and we will have to use a lengthier yet elementary method, which will be divided into three lemmas. For every $\varepsilon > 0$, we introduce

$$T_\varepsilon = \inf\{t > 0,\ X_t \notin (-a+\varepsilon, b-\varepsilon)\}.$$

Lemma 2. *There exist c_1, c_2, $K > 0$ such that for every ε small enough and every t large enough, there exist $x_t^\varepsilon \in (-K, K)$ and $y_t^\varepsilon \in (-a+\varepsilon, b-\varepsilon)$ such that*

$$\mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}[T_\varepsilon > t] \geq c_2 e^{-\mathcal{K}(1+c_1\varepsilon)t}. \tag{2.3}$$

Proof. For every $t > 0$, we can choose $(x_t^\varepsilon, y_t^\varepsilon) \in \mathbf{R} \times (-a+\varepsilon, b-\varepsilon)$ such that

$$\begin{aligned} \mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}&[T_\varepsilon > t+1] \\ &\geq \frac{1}{2} \sup\{\mathbf{P}_{(x,y)}[T_\varepsilon > t+1], (x,y) \in \mathbf{R} \times (-a+\varepsilon, b-\varepsilon)\}. \end{aligned} \tag{2.4}$$

Besides, by scaling and translation, we have for every $(x,y) \in \mathbf{R} \times (-a,b)$,

$$\mathbf{P}_{(x,y)}[T_\varepsilon > t+1] = \mathbf{P}_{(x_\varepsilon, y_\varepsilon)}[T_{ab} > t_\varepsilon]$$

with the notations $x_\varepsilon = x/(1-2\varepsilon/(a+b))^{1/(\alpha+2)}$, $y_\varepsilon = (y-(b-a)/2)/(1-2\varepsilon/(a+b)) + (b-a)/2$, and $t_\varepsilon = (t+1)/(1-2\varepsilon/(a+b))^{2/(\alpha+2)}$. Hence, choosing some constant $c_1 > 0$ such that $1 + c_1\varepsilon > (1-2\varepsilon/(a+b))^{-2/(\alpha+2)}$ for every ε small enough and by the definition of $\mathcal{K}$, we get

$$\begin{aligned} \sup\{\mathbf{P}_{(x,y)}[T_\varepsilon > t+1],\ (x,y) &\in \mathbf{R} \times (-a+\varepsilon, b-\varepsilon)\} \\ &= \sup\{\mathbf{P}_{(x,y)}[T_{ab} > t_\varepsilon],\ (x,y) \in \mathbf{R} \times (-a,b)\} \\ &\geq \sup\{\mathbf{P}_{(x,y)}[T_{ab} > (1+c_1\varepsilon)(t+1)], \\ &\qquad (x,y) \in \mathbf{R} \times (-a,b)\} \\ &\geq e^{-\mathcal{K}(1+c_1\varepsilon)(t+1)} \end{aligned}$$

for t large enough, so that by (2.4),

$$\mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}[T_\varepsilon > t+1] \geq c_2 e^{-\mathcal{K}(1+c_1\varepsilon)t} \tag{2.5}$$

for t large enough with $c_2 = e^{-\mathcal{K}(1+c_1)}/2$. Set now $K = 2(1 \vee \lambda^{-1/\alpha})(a+b)^{1/\alpha}$, fix $\varepsilon > 0$ and t large enough. If $|x_t^\varepsilon| < K$, then by (2.5)

$$\mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}[T_\varepsilon > t] \geq \mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}[T_\varepsilon > t+1] \geq c_2 e^{-\mathcal{K}(1+c_1\varepsilon)t}$$

and (2.3) holds since necessarily $y_t^\varepsilon \in (-a+\varepsilon, b-\varepsilon)$. If $x_t^\varepsilon \geq K$, then introducing the stopping time

$$S = \inf\{s > 0,\ B_s = K/2\},$$

the definition of K and the strong Markov property at S entail

$$\mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}[T_\varepsilon > t+1] = \mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}[S \leq 1, T_\varepsilon > t+1].$$

Indeed, if $S > 1$, then $B_s \geq K/2$ for every $s \leq 1$, so that $X_1 > -a + \varepsilon + (K/2)^\alpha > b - \varepsilon$ and $T_\varepsilon < 1$. Hence,

$$\begin{aligned}
&\mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}[T_\varepsilon > t+1] \\
&\qquad \leq \mathbf{E}_{(x_t^\varepsilon, y_t^\varepsilon)}[\mathbf{1}_{\{S \leq 1, X_S \in (-a+\varepsilon, b-\varepsilon)\}} \mathbf{P}_{(K/2, X_S)}[T_\varepsilon > t]] \\
&\qquad \leq \mathbf{P}_{(x_t^\varepsilon, y_t^\varepsilon)}[S \leq 1] \sup\{\mathbf{P}_{(K/2,y)}[T_\varepsilon > t],\ y \in (-a+\varepsilon, b-\varepsilon)\} \\
&\qquad \leq \sup\{\mathbf{P}_{(K/2,y)}[T_\varepsilon > t],\ y \in (-a+\varepsilon, b-\varepsilon)\}.
\end{aligned}$$

In particular, setting $c_2' = e^{-\mathcal{K}(1+c_1)}/4$ and $\tilde{x}_t^\varepsilon = K/2$, we see by (2.5) that there exists $\tilde{y}_t^\varepsilon \in (-a+\varepsilon, b-\varepsilon)$ such that

$$\mathbf{P}_{(\tilde{x}_t^\varepsilon, \tilde{y}_t^\varepsilon)}[T_\varepsilon > t] \geq c_2' e^{-\mathcal{K}(1+c_1\varepsilon)t}.$$

The case $x_t^\varepsilon \leq -K$ can be handled similarly, and the proof of Lemma 2 is complete. $\square$

We now need to show that the estimate (2.3) remains true in a suitable neighborhood of $(x_t^\varepsilon, y_t^\varepsilon)$. Fixing $\varepsilon > 0$ and $(x_t^\varepsilon, y_t^\varepsilon) \in (-K, K) \times (-a+\varepsilon, b-\varepsilon)$ as above for t large enough, introduce

$$\mathcal{V}_t^\varepsilon = \begin{cases} [x_t^\varepsilon, x_t^\varepsilon + 1] \times [y_t^\varepsilon - \varepsilon/2, y_t^\varepsilon + \varepsilon/2] & \text{if } x_t^\varepsilon \geq 0, \\ [x_t^\varepsilon - 1, x_t^\varepsilon] \times [y_t^\varepsilon - \varepsilon/2, y_t^\varepsilon + \varepsilon/2] & \text{if } x_t^\varepsilon < 0. \end{cases}$$

The key feature of this neighborhood is that its volume does not depend on t and, for this reason, the proof of the following lemma is a bit technical:

Lemma 3. *There exists a $c_3 > 0$ such that for every $\varepsilon > 0$,*

$$\inf\{\mathbf{P}_{(x,y)}[T > t],\ (x,y) \in \mathcal{V}\} > c_3 e^{-\mathcal{K}(1+c_1\varepsilon)t}, \qquad t \to +\infty.$$

Proof. First, by translation invariance, one has

$$\begin{aligned}(2.6)\quad \inf\{\mathbf{P}_{(x_t^\varepsilon,y)}[T > t],\ y \in [y_t^\varepsilon - \varepsilon, y_t^\varepsilon + \varepsilon]\}& \\ \geq \mathbf{P}_{(x_t^\varepsilon,y_t^\varepsilon)}[T_\varepsilon > t] \geq c_2 e^{-\mathcal{K}(1+c_1\varepsilon)t}&\end{aligned}$$

as $t \to +\infty$, where c_2 is the constant in (2.3). Suppose now $x_t^\varepsilon \geq 0$ and introduce the stopping time

$$\sigma_t^\varepsilon = \inf\{s > 0,\ B_s = x_t^\varepsilon\}.$$

For every $(x,y) \in \mathcal{V}_t^\varepsilon$, one gets from the Markov property

$$\begin{aligned}\mathbf{P}_{(x,y)}[T > t] \geq \mathbf{P}_{(x,y)}[T > t > \sigma_t^\varepsilon]& \\ = \int_0^t \int_a^b \mathbf{P}_{(x,y)}[\sigma_t^\varepsilon \in ds, X_{\sigma_t^\varepsilon} \in dv]& \\ \times \mathbf{P}_{(x_t^\varepsilon,v)}[T > t - s]& \\ \geq \int_0^t \int_a^b \mathbf{P}_{(x,y)}[\sigma_t^\varepsilon \in ds, X_{\sigma_t^\varepsilon} \in dv]\mathbf{P}_{(x_t^\varepsilon,v)}[T > t]& \\ \geq \int_0^t \int_{y_t^\varepsilon-\varepsilon}^{y_t^\varepsilon+\varepsilon} \mathbf{P}_{(x,y)}[\sigma_t^\varepsilon \in ds, X_{\sigma_t^\varepsilon} \in dv]& \\ \times \inf\{\mathbf{P}_{(x_t^\varepsilon,z)}[T > t],\ |z - y_t^\varepsilon| \leq \varepsilon\}& \\ \geq c_2 \mathbf{P}_{(x,y)}[\sigma_t^\varepsilon \leq t,\ |X_{\sigma_t^\varepsilon} - y_t^\varepsilon| \leq \varepsilon] e^{-\mathcal{K}(1+c_1\varepsilon)t},&\end{aligned}$$

where we used (2.6) in the last step. Hence, since $[-\varepsilon/2, \varepsilon/2] \subset [y_t^\varepsilon - y - \varepsilon, y_t^\varepsilon - y + \varepsilon]$, it suffices to bound

$$\mathbf{P}_{(x,y)}[\sigma_t^\varepsilon \leq t,\ |X_{\sigma_t^\varepsilon} - y_t^\varepsilon| \leq \varepsilon] \geq \mathbf{P}_{(x,0)}[\sigma_t^\varepsilon \leq t,\ |X_{\sigma_t^\varepsilon}| \leq \varepsilon/2]$$

from below. Now, since $\alpha \geq 0$, there exists an $M > 0$ such that

$$(2.7)\qquad |u + v|^\alpha \leq M(|u|^\alpha + |v|^\alpha)$$

for every $u, v \in \mathbf{R}$, so that $\mathbf{P}_{(x,0)}$ a.s.

$$|X_{\sigma_t^\varepsilon}| \leq M\sigma_t^\varepsilon(x^\alpha + (B^*_{\sigma_t^\varepsilon})^\alpha),$$

with the notation $B^*_t = \max\{|\beta_s|, s \leq t\}$ for every $t \geq 0$, where $\{\beta_s, s \geq 0\}$ is a Brownian motion starting at zero. With the notations $\delta_t^\varepsilon = x - x_t^\varepsilon$, $\rho_t^\varepsilon = \inf\{s > 0, \beta_s = -\delta_t^\varepsilon\}$ and $\theta_z = \inf\{s > 0, \beta_s = z\}$ for every $z \in \mathbf{R}$, this entails

$$\begin{aligned}
\mathbf{P}_{(x,0)}[\sigma_t^\varepsilon \leq t,\ |X_{\sigma_t^\varepsilon}| \leq \varepsilon/2] & \\
&\geq \mathbf{P}[\rho_t^\varepsilon \leq t,\ \rho_t^\varepsilon((B^*_{\rho_t^\varepsilon})^\alpha + x^\alpha) \leq \varepsilon/2M] \\
&\geq \mathbf{P}[\rho_t^\varepsilon \leq t,\ \rho_t(B^*_{\rho_t^\varepsilon})^\alpha \leq \varepsilon/4M,\ \rho_t^\varepsilon x^\alpha \leq \varepsilon/4M] \\
&\geq \mathbf{P}[\rho_t^\varepsilon \leq t \wedge (\varepsilon/4Mx^\alpha),\ B^*_{\rho_t^\varepsilon} \leq x] \\
&\geq \mathbf{P}[\rho_t^\varepsilon \leq t \wedge (\varepsilon/4Mx^\alpha) \wedge \theta_x],
\end{aligned}$$

where in the fourth line we used the obvious fact that $\rho_t^\varepsilon \leq \theta_{-x}$ a.s. By scaling, and since $0 \leq \delta_t^\varepsilon \leq x$, we know that

$$(\rho_t^\varepsilon, \theta_x) \stackrel{d}{=} (\delta_t^\varepsilon)^2(\theta_{-1}, \theta_{x/\delta_t^\varepsilon}) \quad \text{and} \quad \theta_{x/\delta_t^\varepsilon} \geq \theta_1 \text{ a.s.}$$

By Lemma 2 we know that $x \leq K + 1$ and, since $\delta_t^\varepsilon \in [0, 1]$, we finally get

$$\begin{aligned}
\mathbf{P}_{(x,0)}[\sigma_t^\varepsilon \leq t,\ |X_{\sigma_t^\varepsilon}| \leq \varepsilon/2] &\geq \mathbf{P}\left[\theta_{-1} \leq \frac{t \wedge (\varepsilon/4Mx^\alpha)}{(\delta_t^\varepsilon)^2} \wedge \theta_{x/\delta_t^\varepsilon}\right] \\
&\geq \mathbf{P}[\theta_{-1} \leq (\varepsilon/4M|K+1|^\alpha) \wedge \theta_1],
\end{aligned}$$

which finishes the proof of Lemma 3 because the righthand side does not depend on t. □

Our last lemma is intuitively obvious, but we will give a proof for the sake of completeness.

Lemma 4. *For every $\varepsilon > 0$, there is a constant c_ε such that*

$$\mathbf{P}[(B_1, X_1) \in \mathcal{V}_t^\varepsilon,\ T > 1] > c_\varepsilon$$

for every t large enough.

Proof. Fix $\varepsilon > 0$ and define K as in Lemma 2. For every $(x, y) \in (-K, K) \times (-a + \varepsilon, b - \varepsilon)$, there exists a piecewise linear function $f^{x,y} : [0, 1] \to \mathbf{R}$ starting at zero such that $f_1^{x,y} = x + 1/2$ if $x \geq 0$ and $f_1^{x,y} = x - 1/2$ if $x < 0$, $g_1^{x,y} = y$ and $\tau^{x,y} > 1$, with the notations

$$g_t^{x,y} = \int_0^t V(f_s^{x,y})\,\mathrm{d}s,\ t \geq 0, \quad \text{and} \quad \tau^{x,y} = \inf\{t > 0,\ g_t^{x,y} \notin (-a, b)\}.$$

Besides, since from (2.7) we know that a.s. $\|X - g^{x,y}\|_\infty \leq M\|B - f^{x,y}\|_\infty^\alpha$ for every (x, y), by the definition of $\mathcal{V}_t^\varepsilon$ we have for every $t > 0$,

$$\{\|B - f^{x_t^\varepsilon, y_t^\varepsilon}\|_\infty < (\varepsilon/2M)^{1/\alpha}\} \subset \{(B_1, X_1) \in \mathcal{V}_t^\varepsilon,\ T > 1\}.$$

On the one hand, by compacity, we can clearly choose the functions $f^{x,y}$ such that

$$M := \sup\left\{\int_0^1 \left(\frac{\mathrm{d}f_s^{x,y}}{\mathrm{d}s}\right)^2 \mathrm{d}s,\ (x, y) \in (-K, K)\times(-a+\varepsilon, b-\varepsilon)\right\} < +\infty.$$

On the other hand, the Onsager-Machlup formula, see e.g. [**16**, Theorem 7.8], entails

$$\begin{aligned}\mathbf{P}[\|B - f^{x_t^\varepsilon, y_t^\varepsilon}\|_\infty &< (\varepsilon/2M)^{1/\alpha}] \\ &\geq c'_\varepsilon \exp\left[-\frac{1}{2}\int_0^1 \left(\frac{\mathrm{d}f_s^{x_t^\varepsilon, y_t^\varepsilon}}{\mathrm{d}s}\right)^2 \mathrm{d}s\right] \geq c'_\varepsilon e^{-M/2}\end{aligned}$$

where $c'_\varepsilon = \mathbf{P}[\|B\|_\infty < (\varepsilon/2M)^{1/\alpha}]$. Putting everything together and setting $c_\varepsilon = c'_\varepsilon e^{-M/2}$ completes the proof of Lemma 4. □

We can now conclude the proof of existence of the constant. Fix $\varepsilon > 0$, take $t > 0$ large enough and suppose first that $x_t^\varepsilon \geq 0$. By the Markov property at time 1,

$$\begin{aligned}\mathbf{P}[T > t] &\geq \mathbf{P}[(B_1, X_1) \in \mathcal{V}_t^\varepsilon,\ T > t] \\ &\geq \mathbf{P}[(B_1, X_1) \in \mathcal{V}_t^\varepsilon,\ T > 1] \\ &\quad \times \inf\{\mathbf{P}_{(x,y)}[T > t - 1], (x, y) \in \mathcal{V}_t^\varepsilon\} \\ &\geq c_\varepsilon \inf\{\mathbf{P}_{(x,y)}[T > t], (x, y) \in \mathcal{V}_t^\varepsilon\} \\ &\geq c_\varepsilon c_3 e^{-\mathcal{K}(1+c_1\varepsilon)t},\end{aligned}$$

where we used Lemma 4 in the third line and Lemma 3 in the fourth. The case $x_t^\varepsilon < 0$ being handled analogously, we finally obtain, for every $\varepsilon > 0$,

$$\liminf_{t\to+\infty} \frac{1}{t}\log \mathbf{P}[T > t] \geq -\mathcal{K}(1 + c_1\varepsilon),$$

which completes the proof in letting ε tend to 0. □

Remark 5. (a) By the self-similarity of B, one can actually extend the definition of the functionals X to every $\alpha > -1$ with an absolute convergence of the integral. In the symmetric case $\lambda = 1$, it is even possible to extend this definition to every $\alpha \in (-3/2, 1]$, viewing X as a Cauchy principal value process:

$$\begin{aligned} X_t &= \lim_{\varepsilon\to 0}\int_0^t \mathbf{1}_{\{|B_s|>\varepsilon\}}|B_s|^\alpha \operatorname{sgn}(B_s)\,\mathrm{d}s \\ &= \lim_{\varepsilon\to 0}\int_{\mathbf{R}} \mathbf{1}_{\{|x|>\varepsilon\}}|x|^\alpha \operatorname{sgn}(x)(L(t,x) - L(0,x))\mathrm{d}x, \end{aligned}$$

where in the second equality we used the occupation formula and where the second limit exists almost surely since the map $x \mapsto L(t,x)$ is almost surely η-Hölder for every $\eta < 1/2$. For $\alpha = -1$, the process X is then, up to a multiplicative constant, the Hilbert transform of L, while for $\alpha < -1$, it can be viewed as a fractional derivative of L, and we refer to the seminal paper [**6**] and [**4**, Chapter 5] for much more on this topic.

Above, the subadditivity argument and the finiteness of the constant do not rely on the specific value of α, so that one gets with the same notations

$$-\infty < \liminf_{t\to\infty} t^{-1}\log \mathbf{P}[T_{ab} > t] \leq \limsup_{t\to\infty} t^{-1}\log \mathbf{P}[T_{ab} > t] < 0,$$

which is a weaker version of our main result. However, the *positivity* assumption on α is crucial for Lemma 2 which is the key-step in our proof of the existence of the constant. We believe that the limit in (2.1) is also a true limit when α is negative, but the proof requires probably less bare-hand arguments than ours.

(b) In the case $\alpha = \lambda = 1$, the process (B, X) is a Gaussian diffusion and in this case it is known that the function $f_t : (x,y) \mapsto \mathbf{P}_{(x,y)}[T > t]$ is log-concave for every $t > 0$, see e.g. [**13**, Proposition 1.3]. Hence, in the case of a symmetric interval, its maximum is attained in $(0,0)$, and this gives another proof of the existence of the constant. Despite Theorem 1.2 in [**13**], our intuition is that the function f_t remains log-concave in general, but we were unable to prove this. If this were true, the existence of the constant would follow immediately in the case $\lambda = 1$ and for a symmetric interval. Let us stress that the function f_t already exhibits some concavity properties in the framework of non Gaussian symmetric stable processes [**2**].

3. Proof of the corollary. We will follow the outline of [**12**, subsections 2.4 and 2.5], which are themselves a variation on Chung's original argument. First, arguing with (1.2) and the first Borel-Cantelli lemma exactly as in subsection 2.4 of [**12**], one can show that

$$\liminf_{t\to+\infty} \frac{X_t^*}{f(t)} \geq \mathcal{K}_1^{(\alpha+2)/2} \quad \text{a.s.,} \tag{3.1}$$

and we leave verification to the reader (beware the minor correction $R \to \log R$ on the last line page 4258). Moreover, the arguments of subsection 2.3 in [**12**] applied to our Lévy $(1+\alpha/2)$-stable process $Y : t \mapsto X_{\tau_t}$ entail without major modification

$$\liminf_{t\to+\infty} \frac{X_t^*}{f(t)} < \infty \quad \text{a.s.} \tag{3.2}$$

By the 0-1 law, we know that the liminf on the lefthand side is a.s. deterministic, so that Chung's law holds by (3.1) and (3.2), with an unknown finite positive constant. Notice in passing that (3.1) and (3.2) give also a third proof of the finiteness of $\mathcal{K}$ in the symmetric case $a = b$, which is actually Khoshnevisan and Shi's in the case of integrated Brownian motion.

However, to prove that

$$\liminf_{t\to+\infty} \frac{X_t^*}{f(t)} \leq \mathcal{K}_1^{(\alpha+2)/2} \quad \text{a.s.,} \tag{3.3}$$

we will have to modify slightly the arguments of subsection 2.5 in [**12**], since the kernel $x \mapsto V(x)$ is not linear in general. Fixing a small $\varepsilon > 0$, introduce the numbers $t_n = n^{4n}$, $s_n = n^{4n+3}$ and

$y_n = (1+2\varepsilon)\mathcal{K}_1^{(\alpha+2)/2} f(t_n)$ for every $n \geq 1$. Define the sequence of stopping times

$$S_0 = 0 \quad \text{and} \quad S_n = \inf\{t > t_n + S_{n-1},\ B_t = 0\}, \quad n \geq 1.$$

Finally, consider the events

$$\begin{aligned} E_n &= \left\{ \sup_{S_n \leq t \leq t_{n+1} + S_n} \| \int_{S_n}^t V(B_s)\mathrm{d}s < y_{n+1} \right\} \text{ and} \\ F_n &= \{S_n < s_n + S_{n-1}\} \end{aligned}$$

for every $n \geq 1$. On the one hand, setting $r_n = s_n - t_n$, $\mathbf{P}_x$ for the law of B starting at x, and resuming the notations of Lemma 3, the strong Markov property, the symmetry of Brownian motion and a scaling argument yield

$$\begin{aligned} \mathbf{P}[F_n^c] &= \int_{\mathbf{R}} \mathbf{P}[B_{S_{n-1}+t_n} \in \mathrm{d}x]\mathbf{P}_x[\theta_0 > r_n] \\ &= \int_{\mathbf{R}} \mathbf{P}[B_{t_n} \in \mathrm{d}x]\mathbf{P}[B_t < |x|,\ \forall t \leq r_n] \\ &= \int_{\mathbf{R}} \mathbf{P}[B_1 \in \mathrm{d}u]\mathbf{P}[B_t < |u|\sqrt{t_n r_n^{-1}},\ \forall t \leq 1] \\ &\sim c\sqrt{t_n r_n^{-1}} \sim c n^{-3/2}, \quad n \to \infty \end{aligned}$$

for some positive finite constant c, so that

$$\sum_{n \geq 1} \mathbf{P}[F_n^c] < +\infty.$$

By the Borel-Cantelli lemma, for almost every ω there exists $n_0(\omega)$ such that

$$S_n(\omega) < S_{n_0(\omega)}(\omega) + s_{n_0(\omega)+1} + \cdots + s_n$$

for every $n > n_0(\omega)$. Hence, by the definition of s_n, there exists $n_1(\omega) > n_0(\omega)$ such that

$$S_n(\omega) < 2s_n \tag{3.4}$$

for every $n \geq n_1(\omega)$. On the other hand, since

$$E_n = \left\{ \sup_{0 \leq t \leq t_{n+1}} \left| \int_0^t V(B_{S_n+s} - B_{S_n}) \mathrm{d}s \right| < y_{n+1} \right\},$$

it follows readily from the strong Markov property and the definition of S_n that the events E_n are mutually independent. Besides, using (1.2) and reasoning exactly as in [**12**, page 4259] entails

$$\sum_{n \geq 1} \mathbf{P}[E_n] = +\infty.$$

By the second Borel-Cantelli lemma, an infinity of events E_n occur almost surely and by (3.4), we know that almost surely eventually $[2s_n, t_{n+1}] \subset [S_n, t_{n+1} + S_n]$. This entails

$$\sup_{2s_n \leq t \leq t_{n+1}} \left| \int_{S_n}^t V(B_s) \mathrm{d}s \right| < (1 + 2\varepsilon) \mathcal{K}_1^{(\alpha+2)/2} f(t_{n+1}) \text{ i.o.}$$

By Khintchine's LIL for Brownian motion,

$$\liminf_{n \to +\infty} \frac{1}{f(t_{n+1})} \left\| \int_{S_n}^{2s_n} V(B_s)\, \mathrm{d}s \right\| \leq \liminf_{n \to +\infty} \frac{2(1 \vee \lambda) s_n}{f(t_{n+1})} (B^*_{s_n})^\alpha = 0 \quad \text{a.s.}$$

Putting everything together and letting $\varepsilon \to 0$ yields

$$\liminf_{n \to +\infty} \frac{1}{f(t_n)} \sup_{2s_{n-1} \leq t \leq t_n} \left| \int_{2s_{n-1}}^t V(B_s) \mathrm{d}s \right| \leq \mathcal{K}_1^{(\alpha+2)/2} \quad \text{a.s.} \tag{3.5}$$

Finally, we know from (3.2) that

$$\frac{X^*_{2s_{n-1}}}{f(t_n)} \longrightarrow 0 \quad \text{a.s.}$$

which together with (3.5), the usual monotonicity argument, and the fact that a.s.

$$X^*_{t_n} \le X^*_{2s_{n-1}} + \sup_{2s_{n-1} \le t \le t_n} \left| \int_{2s_{n-1}}^{t} V(B_s)\mathrm{d}s \right|,$$

yields (3.3) as desired. □

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