# ARITHMETIC PROGRESSIONS IN
# THE SOLUTION SETS OF NORM FORM EQUATIONS

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**1. Introduction.** Let $K$ be an algebraic number field of degree $k$, and let $\alpha_1, \ldots, \alpha_n$ be linearly independent elements of $K$ over $\mathbf{Q}$. Denote by $D \in \mathbf{Z}$ the common denominator of $\alpha_1, \ldots, \alpha_n$, and put $\beta_i = D\alpha_i$, $i = 1, \ldots, n$. Note that $\beta_1, \ldots, \beta_n$ are algebraic integers of $K$. Let $m$ be a nonzero integer, and consider the norm form equation

$$(1.1) \qquad N_{K/\mathbf{Q}}(x_1\alpha_1 + \cdots + x_n\alpha_n) = m$$

in integers $x_1, \ldots, x_n$. Let $H$ denote the solution set of (1.1) and $|H|$ the size of $H$. Note that, if the $\mathbf{Z}$-module generated by $\alpha_1, \ldots, \alpha_n$ contains a submodule, which is a full module in a subfield of $\mathbf{Q}(\alpha_1, \ldots, \alpha_n)$ different from the imaginary quadratic fields and $\mathbf{Q}$, then this equation can have infinitely many solutions (see, e.g., Schmidt [**19**]). Various arithmetical properties of the elements of $H$ were studied in [**8, 11**]. In the present paper we are concerned with arithmetical progressions in $H$. Arranging the elements of $H$ in an $|H| \times n$ array $\mathcal{H}$, one may ask at least two natural questions about arithmetical progressions appearing in $H$. The "horizontal" one: do there infinitely many rows of $\mathcal{H}$ exist, which form arithmetic progressions; and the "vertical" one: do arbitrary long arithmetic progressions in some column of $\mathcal{H}$ exist? Note that the first question is meaningful only if $n > 2$.

The "horizontal" problem was treated by Bérczes and Pethő [**4**] by proving that if $\alpha_i = \alpha^{i-1}$, $i = 1, \ldots, n$, then in general $\mathcal{H}$ contains only finitely many effectively computable "horizontal" APs, and they were able to localize the possible exceptional cases. Later Bérczes and Pethő [**5**], Bérczes Pethő and Ziegler [**6**] and Bazsó [**2**]

computed all horizontal APs in the solution sets of norm form equations corresponding to the fields generated by the polynomials $x^n - a$, $2 \leq a \leq 100$, $x^3 - (a-1)x^2 - (a+2)x - 1$, $a \in \mathbf{Z}$ and $x^n + a$, $2 \leq a \leq 100$, respectively.

For quadratic norm form equations, which are called Pell equations if $K$ is a real quadratic field, only the "vertical" problem is interesting. In this direction Pethő and Ziegler [18] proved among others that the length of the "vertical" APs in $\mathcal{H}$ is bounded by a constant, which depends on the coefficients of the (quadratic) form and on $m$. On the other hand, they proved that every three term AP occurs in the second column of infinitely many $\mathcal{H}$. Dujella, Pethő and Tadić [7] were able to extend this result to four term APs.

The main goal of the present paper is to generalize the result of Pethő and Ziegler [18] to arbitrary norm form equations. In the sequel $AP$ *in H* always means a "vertical" arithmetical progression belonging to $\mathcal{H}$. A sequence in $H$, with the property that all the corresponding coordinate sequences form "vertical" APs, will be called an *algebraic* AP *in H*.

**2. Results.** Now we summarize our main results.

**Theorem 2.1.** *Let $(x_1^{(j)}, \ldots, x_n^{(j)})$, $j = 1, \ldots, t$, be a sequence of distinct elements in H such that $x_i^{(j)}$ is a nonzero arithmetic progression for some $i \in \{1, \ldots, n\}$. Then we have $t \leq c_1$, where $c_1 = c_1(k, m, D)$ is an explicitly computable constant.*

**Theorem 2.2.** *The set H contains at most $c_2$ arithmetic progressions of the form $\underline{x} + h\underline{d}$ $(h = -1, 0, 1)$. Here $c_2 = c_2(k, m, D)$ is an explicitly computable constant, $\underline{x} = (x_1, \ldots, x_n)$, $d$ is a nonzero integer, and $\underline{d}$ is the n-tuple with all entries equal to d.*

By Theorem 2.1 the length of any AP in $H$ is bounded. In the particular case $k = 2$, $H$ does not contain any algebraic AP (see Pethő and Ziegler [18]). However, it is not possible to give a bound for the number of APs in $H$ for $k \geq 3$. It is demonstrated by the following example. Let $P(x) = x(x - 1) \cdots (x - k + 1) + (-1)^k$ and denote by $\alpha$ one of its roots. It was proved in [14, Lemma 2.2], see also [1, 13, 17], that $P(X)$ is irreducible; thus $\alpha, \alpha - 1, \ldots, \alpha - (k - 1)$ are units

of norm 1 in the algebraic number field $\mathbf{Q}(\alpha)$; moreover, they form an AP of length $k$. If $\mu$ is an algebraic integer in $\mathbf{Q}(\alpha)$ of norm $m$, then $\mu\alpha, \mu(\alpha + 1), \ldots, \mu(\alpha + k - 1)$ also have norm $m$ and form an AP of length $k$.

The next theorem shows that in general if $H$ contains algebraic APs at all, then it contains infinitely many.

**Theorem 2.3.** *Suppose that $n = k \geq 3$. Let $t \geq 3$ be an integer. If $H$ contains a nonconstant $t$-term algebraic AP, then it contains infinitely many.*

Now we prove that the algebraic APs from the example before Theorem 2.3 are the longest ones. More precisely, we have the following theorem.

**Theorem 2.4.** *Let $K$ be an algebraic number field of degree $k$. Assume that $\alpha_1, \ldots, \alpha_t \in K$ have the same field norm and form a nontrivial AP. Then $t \leq k$.*

*Remark.* We note that Newman [16] (see also [17]) proved that the length of arithmetic progressions consisting of units of an algebraic number field of degree $k$ is at most $k$. Theorem 2.4 is a generalization of his result.

To formulate the next result, for a nonzero integer $a$, let $\omega(a)$ denote the number of prime divisors of $a$, and for a prime $p$ denote by $\mathrm{ord}_p(a)$ the highest exponent $u$ such that $p^u$ divides $a$.

**Theorem 2.5.** *Suppose that the Galois group of the normal closure of $K$ is doubly transitive. Then the number of those solutions $(x_1, \ldots, x_n)$ of equation (1.1), for which there exists another solution $(y_1, \ldots, y_n) \neq (x_1, \ldots, x_n)$, such that $\prod_{i=1}^{n}(x_i - y_i) = 0$, is bounded by*

$$\Psi\left(k, n, mD^k\right) \exp\left(k(12n)^{6n}\right),$$

*where*

$$\Psi(k, n, mD^k) := \binom{k}{n-1}^{\omega(mD^k)} \cdot \prod_{\substack{p \mid m \\ p \text{ prime}}} \binom{\mathrm{ord}_p(mD^k) + n - 1}{n-1}.$$

**Theorem 2.6.** *Let $S$ be a set of $s$ rational primes, and let $T$ be the set of integers without prime divisors outside $S$. Suppose that the Galois group of the normal closure of $K$ is doubly transitive. Then the number of those solutions $(x_1, \ldots, x_n)$ of equation (1.1), for which there exists another solution $(y_1, \ldots, y_n) \neq (x_1, \ldots, x_n)$, such that $x_i - y_i \in T$ for some $i \in \{1, \ldots, n\}$, is bounded by*

$$\Psi(k, n, mD^k) \cdot \exp\left((s+k)(12n)^{6n+3}\right),$$

*where $\Psi$ is the function defined in Theorem 2.5.*

*Remark.* With the help of Theorems 2.5 and 2.6 one can easily give a bound for the number of sequences $\mathbf{x}_j = (x_1^{(j)}, \ldots, x_n^{(j)}) \in H$ such that one of the coordinates of $\mathbf{x}_j$ forms an arithmetic progression whose difference is zero or is an $S$-unit, respectively.

**3. Auxiliary results.** In this section we present some lemmas which will be needed in the proofs of our theorems. For this purpose we need to introduce some notation. Let $L$ be a number field of degree $l$, and denote by $U_L$ the unit group of $L$. The next statement is an immediate consequence of a result by Hajdu [12]. Note that a similar result was independently proved by Jarden and Narkiewicz [15].

**Lemma 3.1.** *Let $n$ be an integer, and let $A$ be a finite subset of $L^n$. There exists a constant $C_1 = C_1(l, n, |A|)$ such that the length of any nonconstant arithmetic progression in the set*

$$\left\{ \sum_{i=1}^n a_i y_i : (a_1, \ldots, a_n) \in A, \ (y_1, \ldots, y_n) \in U_L^n \right\}$$

*is at most $C_1$.*

For some other arithmetical properties of the set occurring in Lemma 3.1, see [11].

Let $K$ be a number field of degree $k$, $\alpha_1, \ldots, \alpha_n$ linearly independent algebraic integers in $K$, $m \in \mathbf{Z}$, and $\lambda \in K$. Consider now the equation

(3.2)        $N_{K/\mathbf{Q}}(\alpha_1 x_1 + \cdots + \alpha_n x_n + \lambda) = m$ in $x_1, \ldots, x_n \in \mathbf{Z}$.

The next lemma is a special case of Corollary 8 of [**3**].

**Lemma 3.2.** *Suppose that $\alpha_1, \ldots, \alpha_n$ and $\lambda$ are linearly independent over **Q**. Then the number of solutions of equation (3.2) does not exceed the bound*

$$\left(2^{17}k\right)^{((2/3)(n+1)(n+2)(2n+3)-4)(\omega(m)+1)}.$$

Let $F$ be an algebraically closed field of characteristic 0. Write $F^*$ for the multiplicative group of nonzero elements of $F$, and let $(F^*)^n$ be the direct product consisting of $n$-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_i \in F^*$ for $i = 1, \ldots, n$. For $x, y \in (F^*)^n$ write $x * y = (x_1 y_1, \ldots, x_n y_n)$. Let $\Gamma$ be a subgroup of $(F^*)^n$, and suppose that $(a_1, \ldots, a_n) \in (F^*)^n$. Consider the so-called generalized unit equation

$$(3.3) \qquad\qquad a_1 x_1 + \cdots + a_n x_n = 1$$

in $\mathbf{x} = (x_1, \ldots, x_n) \in \Gamma$. A solution $\mathbf{x}$ is called nondegenerate if no subsum of the lefthand side of (3.3) vanishes, that is, $\sum_{i \in I} a_i x_i \neq 0$ for any nonempty subset $I$ of $\{1, \ldots, n\}$. The next lemma is Theorem 1.1 of Evertse, Schlickewei and Schmidt [**10**].

**Lemma 3.3.** *Suppose that $\Gamma$ has finite rank $r$. Then the number of nondegenerate solutions $\mathbf{x} \in \Gamma$ of equation (3.3) is bounded by*

$$\exp\left((6n)^{3n}(r+1)\right).$$

Let $\mathcal{M}$ be the **Z**-module generated by the elements $\alpha_1, \ldots, \alpha_n$. Clearly, equation (1.1) can be transformed to the equation

$$(3.4) \qquad\qquad N_{K/\mathbf{Q}}(\delta) = m \text{ in } \delta \in \mathcal{M}.$$

**Lemma 3.4.** *The set of solutions of (3.4) is contained in some union $\delta_1 \Omega_K^* \cup \cdots \cup \delta_t \Omega_K^*$, where*

$$t \leq \Psi(k, n, m) = \binom{k}{n-1}^{\omega(m)} \cdot \prod_{\substack{p \mid m \\ p \text{ prime}}} \binom{\operatorname{ord}_p(m) + n - 1}{n - 1}$$

*and $\delta_1, \ldots, \delta_t$ are solutions of (3.4).*

*Proof.* This is a special case of Lemma 4 of [**9**]. □

## 4. Proofs.

*Proof of Theorem* 2.1. Recall that $H$ is the solution set of (1.1), $D$ is the common denominator of $\alpha_1, \dots, \alpha_n$, and $\beta_i = D\alpha_i$, $i = 1, \dots, n$.

Suppose first that we have a nonconstant sequence $(x_1^{(j)}, \dots, x_n^{(j)})$, $j = 1, \dots, t$, in $H$ such that $x_i^{(j)}$ is constant (but nonzero) for some $i \in \{1, \dots, n\}$. Let $\lambda := x_i^{(j)} \cdot \beta_i \neq 0$. Then equation (1.1) is of the shape (3.2) and by Lemma 3.2 we see that the number of such solutions of (1.1), i.e., $t$, is bounded by

$$\left(2^{17}k\right)^{((2/3)n(n+1)(2n+1)-4)(\omega(mD^k)+1)} \leq c_1(k, m, D).$$

Assume next that $(x_1^{(j)}, \dots, x_n^{(j)}) \in H$ for $j = 1, \dots, t$ such that $x_i^{(j)}$ forms a nonconstant arithmetic progression for some $i \in \{1, \dots, n\}$. Writing $\sigma_1, \dots, \sigma_k$ for the isomorphisms of $K$ into $\mathbf{C}$, for $u = 1, \dots, k$, we have

$$x_1\sigma_u(\beta_1) + \dots + x_n\sigma_u(\beta_n) = \sigma_u(\varepsilon)\sigma_u(\mu)$$

where $\mu$ is an element of norm $mD^k$ and $\varepsilon$ is a unit in the $\mathbf{Z}$-module $\mathbf{Z}[\beta_1, \dots, \beta_n]$. By Lemma 3.4 $\mu$ can be chosen from a set having at most $\Psi(k, n, mD^k)$ elements. Consider a fixed value of $\mu$. Choose the order of the isomorphisms $\sigma_1, \dots, \sigma_k$ such that the matrix

$$(4.5) \qquad B = \begin{pmatrix} \sigma_1(\beta_1) & \dots & \sigma_1(\beta_n) \\ \vdots & \ddots & \vdots \\ \sigma_n(\beta_1) & \dots & \sigma_n(\beta_n) \end{pmatrix}$$

has nonzero determinant. Hence, we have

$$(4.6) \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B^{-1} \begin{pmatrix} \sigma_1(\varepsilon)\sigma_1(\mu) \\ \vdots \\ \sigma_n(\varepsilon)\sigma_n(\mu) \end{pmatrix}.$$

Writing

$$(4.7) \qquad B^{-1} = \begin{pmatrix} \gamma_{11} & \dots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \dots & \gamma_{nn} \end{pmatrix}$$

we get

$$x_i = a_{i1}y_1 + \cdots + a_{in}y_n$$

for all $i = 1, \ldots, n$, where $a_{ih} = \gamma_{ih}\sigma_h(\mu)$ and $y_h = \sigma_h(\varepsilon)$ for $h = 1, \ldots, n$. Noting that the $y_h$, $h = 1, \ldots, n$, are units in the splitting field $L$ of $K$, and $\deg(L) \leq k!$, using $n \leq k$ the theorem follows from Lemma 3.1.    □

*Proof of Theorem* 2.2. Obviously, in view of Theorem 2.1, it is sufficient to give an upper bound for the number of three-term progressions in $H$. For this purpose, assume that $(x_1, \ldots, x_n)$ is the middle term of a three-term arithmetic progression in $H$, with common difference $\underline{d}$. Denote by $U_K$ the unit group of the ring of algebraic integers of the field $K$. Put

$$\mu_{\pm 1} = (x_1 \pm d)\beta_1 + \cdots + (x_n \pm d)\beta_n \text{ and } \mu_0 = x_1\beta_1 + \cdots + x_n\beta_n.$$

Note that $N_{K/\mathbf{Q}}(\mu_{-1}) = N_{K/\mathbf{Q}}(\mu_0) = N_{K/\mathbf{Q}}(\mu_1) = mD^k$, and further that $\mu_h = \varepsilon_h\mu_h^*$, $h = -1, 0, 1$, where $\varepsilon_{-1}, \varepsilon_0, \varepsilon_1 \in U_K$ and $\mu_{-1}^*, \mu_0^*, \mu_1^*$ belong to a finite set whose cardinality is bounded in terms of $k, m, D$. Thus, we have

$$\mu_{-1}^*\varepsilon_{-1} - 2\mu_0^*\varepsilon_0 + \mu_1^*\varepsilon_1 = 0.$$

Hence, Lemma 3.3 implies that

$$(\varepsilon_{-1}, \varepsilon_0, \varepsilon_1) = \varepsilon(\varepsilon_{-1}^*, \varepsilon_0^*, \varepsilon_1^*)$$

with some $\varepsilon \in U_K$, where $(\varepsilon_{-1}^*, \varepsilon_0^*, \varepsilon_1^*)$ belongs to a finite subset of $U_K^3$, of cardinality bounded by some constant depending only on $k, m, D$. Thus, we conclude that

$$\mu_h = \varepsilon\lambda_h, \quad h = -1, 0, 1$$

holds, where $\varepsilon \in U_K$ and $\lambda_{-1}, \lambda_0, \lambda_1$ belong to a finite set of cardinality depending only on $k, m, D$ again. Observe that $d = \varepsilon(\lambda_1 - \lambda_0)$ holds, and further that this $d$ can be rational for at most one choice of $\varepsilon \in U_K$ (up to a factor $-1$), for any fixed $(\lambda_{-1}, \lambda_0, \lambda_1)$. Hence, the theorem follows.    □

*Proof of Theorem* 2.3. Suppose that $(x_1^{(j)}, \ldots, x_n^{(j)})$, $j = 1, \ldots, t$, is a nonconstant algebraic AP in $H$. Let $\varepsilon$ be an arbitrary unit in $\mathbf{Z}[\beta_1, \ldots, \beta_n]$ of norm 1, and define $(y_1^{(j)}, \ldots, y_n^{(j)})$ by

$$y_1^{(j)}\beta_1 + \cdots + y_n^{(j)}\beta_n = \varepsilon(x_1^{(j)}\beta_1 + \cdots + x_n^{(j)}\beta_n) \text{ for } j = 1, \ldots, t.$$

Obviously, then $(y_1^{(j)}, \ldots, y_n^{(j)})$, $j = 1, \ldots, t$, is a nonconstant algebraic AP in $H$. As there are infinitely many units in $\mathbf{Z}[\beta_1, \ldots, \beta_n]$ of norm 1, the theorem follows.    □

*Proof of Theorem* 2.4. Denote by $m$ the common norm of $\alpha_1, \ldots, \alpha_t$. As these numbers form an AP, we have $\alpha_i = \alpha_1 + (i-1)(\alpha_2 - \alpha_1), i = 1, \ldots, t$. This implies $\alpha_i/\beta = (\alpha_1/\beta) + i - 1$ with $\beta = \alpha_2 - \alpha_1$. Put $M$ for the norm of $\beta$ and $P(x) = x^u + p_{u-1}x^{u-1} + \cdots + p_0, p_j \in \mathbf{Q}$ for the minimal polynomial of $\alpha_1/\beta$. It is well known that the defining polynomial of $\alpha_1/\beta$ is a power of its minimal polynomial, i.e., $u|k$ and $p_0^{k/u} = (-1)^k m/M$. If $k = u$, then we even have $p_0 = (-1)^k m/M$; otherwise, because both $p_0$ and $m/M$ are rational numbers, there are at most two possibilities for $p_0$, which differ from each other only in their sign.

Consider the polynomials $P_i(x) = P(x - (i-1))$, $i = 1, \ldots, t$. They are with $P(x)$ irreducible and we have

$$P_i\left(\frac{\alpha_i}{\beta}\right) = P\left(\frac{\alpha_i}{\beta} - (i-1)\right)P\left(\frac{\alpha_1}{\beta}\right) = 0,$$

i.e., $\alpha_i/\beta$ is a root of $P_i(x)$, which together with the irreducibility of $P_i(x)$ implies that it is the minimal polynomial of $\alpha_i/\beta$. Thus, its constant term is equal to $p_0$ if $k = u$ and may differ from $p_0$ only in its sign, otherwise. Hence, $P(-i+1)$, $i = 1, \ldots, t$ is constant if $k = u$ or can assume only at most two different values. If $k = u$ this implies $P(x) = x(x-1)\cdots(x-t+1) + p_0$ and we have $t \leq k$ as stated. If $u < k$, then there exists a subset $I \subseteq \{1, \ldots, t\}$ of size $|I| \geq t/2$ such that $P(-i+1)$ takes the same value for all $i \in I$. By the theory of interpolation, the degree of $P$ must be at least $|I|$, i.e., $u \geq |I| \geq t/2$. On the other hand, $u < k$ and $u|k$ imply $u \leq k/2$. From the last two inequalities we get $t \leq k$ in this case, too.    □

*Proof of Theorem* 2.5. We shall bound the number of those solutions of equation (1.1), for which there exists a solution $(y_1, \ldots, y_n) \neq$

$(x_1, \ldots, x_n)$ with $x_i = y_i$ for some $i \in \{1, \ldots, n\}$. Now equation (1.1) means that

(4.8) $$\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = \mu_1 \varepsilon_1$$

and

(4.9) $$\beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_n y_n = \mu_2 \varepsilon_2$$

where $\mu_1$ and $\mu_2$ are elements of norm $mD^k$ and $\varepsilon_1, \varepsilon_2$ are units in the $\mathbf{Z}$-module generated by $\beta_1, \ldots, \beta_n$. By Lemma 3.4 both $\mu_1$ and $\mu_2$ can be chosen from a set having at most $\Psi(k, n, mD^k)$ elements. Consider fixed values of $\mu_1$ and $\mu_2$. Denote again by $\sigma_1, \ldots, \sigma_k$ the isomorphic embeddings of $K$ into $\mathbf{C}$, choosing their order such that the matrix $B$ in (4.5) has nonzero determinant. Using (4.7), equation (4.8) leads to equation (4.6). This means that

(4.10) $$x_i = \sum_{j=1}^{n} \gamma_{ij} \sigma_j(\mu_1) \sigma_j(\varepsilon_1).$$

Similarly, using equation (4.9) we can show that

(4.11) $$y_i = \sum_{j=1}^{n} \gamma_{ij} \sigma_j(\mu_2) \sigma_j(\varepsilon_2).$$

One can easily check that $\gamma_{ij} \neq 0$ for at least two indices $j \in \{1, \ldots, n\}$. Thus, without loss of generality we may assume that $\gamma_{i1}, \ldots, \gamma_{iN}$ are nonzero and $\gamma_{i,N+1} = \cdots = \gamma_{in} = 0$, for some $2 \leq N \leq n$. Now subtracting equations (4.10) and (4.11) we get

(4.12) $$\sum_{j=1}^{N} \left( \gamma_{ij} \sigma_j(\mu_1) \sigma_j(\varepsilon_1) - \gamma_{ij} \sigma_j(\mu_2) \sigma_j(\varepsilon_2) \right) = 0.$$

This is a homogeneous unit equation consisting of $2N$ terms. We shall bound the number of solutions of this equation. First we count the nondegenerate solutions of (4.12). Dividing the equation by the last term we obtain

(4.13)
$$\sum_{j=1}^{N-1} \left( \frac{\gamma_{ij} \sigma_j(\mu_1)}{\gamma_{in} \sigma_N(\mu_2)} \frac{\sigma_j(\varepsilon_1)}{\sigma_N(\varepsilon_2)} - \frac{\gamma_{ij} \sigma_j(\mu_2)}{\gamma_{iN} \sigma_N(\mu_2)} \frac{\sigma_j(\varepsilon_2)}{\sigma_N(\varepsilon_2)} \right) + \frac{\sigma_N(\mu_1)}{\sigma_N(\mu_2)} \frac{\sigma_N(\varepsilon_1)}{\sigma_N(\varepsilon_2)} = 1,$$

which is an inhomogeneous unit equation having $2N - 1$ terms. We easily see that all solutions to this equation are contained in the subgroup

$$\Gamma = \left\{ \left( \frac{\sigma_1(\varepsilon_1)}{\sigma_N(\varepsilon_2)}, \frac{\sigma_1(\varepsilon_2)}{\sigma_N(\varepsilon_2)}, \frac{\sigma_2(\varepsilon_1)}{\sigma_N(\varepsilon_2)}, \frac{\sigma_2(\varepsilon_2)}{\sigma_N(\varepsilon_2)}, \ldots, \frac{\sigma_N(\varepsilon_1)}{\sigma_N(\varepsilon_2)} \right) \,\Big|\, \varepsilon_1, \varepsilon_2 \in \Omega_K^* \right\}$$

of $(\mathbf{C}^*)^{2N-1}$. Clearly, this group has rank at most $2r_K$, where $r_K$ is the unit rank of the field $K$. Indeed, if $\eta_1, \ldots, \eta_{r_K}$ denotes a fundamental system of units in $K$ then, the subgroup $\Gamma_0$ of $(\mathbf{C}^*)^{2N-1}$, generated by the vectors

$$\mathbf{a}_j = (\sigma_1(\eta_j), 1, \sigma_2(\eta_j), 1, \ldots, 1, \sigma_N(\eta_j)), \quad j = 1, \ldots, r_K,$$

and

$$\mathbf{b}_i = \left( \frac{1}{\sigma_N(\eta_j)}, \frac{\sigma_1(\eta_j)}{\sigma_N(\eta_j)}, \frac{1}{\sigma_N(\eta_j)}, \frac{\sigma_2(\eta_j)}{\sigma_N(\eta_j)}, \ldots, \frac{\sigma_{N-1}(\eta_j)}{\sigma_N(\eta_j)}, \frac{1}{\sigma_N(\eta_j)} \right),$$
$$j = 1, \ldots, r_K,$$

has rank at most $2r_K$. Further, the factor group $\Gamma/\Gamma_0$ is a torsion group. This means that the solutions of equation (4.13) belong to a subgroup of rank at most $2k - 2$ of $(\mathbf{C}^*)^{2N-1}$. Thus, $\sigma_1(\varepsilon_1)/\sigma_N(\varepsilon_2)$ is contained in a set of at most

$$\exp\left( (12N - 6)^{6N-3}(2k - 1) \right)$$

elements. Fix now such a value. Then using that the Galois group of $K$ is doubly transitive, we see that $(\sigma_l(\varepsilon_1))/(\sigma_j(\varepsilon_2))$ is also fixed for each $j, l \in \{1, \ldots, k\}$. By multiplying the ratios $(\sigma_1(\varepsilon_1))/(\sigma_j(\varepsilon_2))$ for $j \in \{1, \ldots, k\}$ and using that $\prod_{j=1}^k \sigma_j(\varepsilon_2) = \pm 1$ we get that $\varepsilon_1$ may assume at most $2k$ values. Similarly, $\varepsilon_2$ may assume at most $2k$ values. These altogether show that the number of nondegenerate solutions of equation (4.12) is bounded by

$$(4.14) \qquad\qquad \exp\left( (12N - 6)^{6N-2}(4k - 2) \right).$$

Now we have to estimate the number of degenerate solutions of (4.12), too. If $\gamma_{ij}\sigma_j(\mu_1)\sigma_j(\varepsilon_1) - \gamma_{ij}\sigma_j(\mu_2)\sigma_j(\varepsilon_2) = 0$ for all $j \in \{1, \ldots, N\}$ then we get that $\sigma_l(\mu_1)\sigma_l(\varepsilon_1) = \sigma_l(\mu_2)\sigma_l(\varepsilon_2)$ for some $l \in \{1, \ldots, N\}$

and thus $\mu_1\varepsilon_1 = \mu_2\varepsilon_2$. Now subtracting equations (4.8) and (4.9) and using that $\beta_1, \dots, \beta_n$ are linearly independent, we get that $x_j = y_j$ for all $j \in \{1, \dots, n\}$, which is a contradiction. Thus, we must have one of the following two cases:

(i) Equation (4.12) has a minimal vanishing sub-sum, i.e., a sub-sum with no further vanishing sub-sums, which contains both $\sigma_j(\varepsilon_1)$ and $\sigma_l(\varepsilon_2)$ for some $j \neq l$, $j, l \in \{1, \dots, N\}$. Similarly to the case of nondegenerate solutions, we can prove that the number of solutions of (4.12) is bounded by the expression in (4.14).

(ii) Equation (4.12) has both a minimal vanishing sub-sum which contains $\sigma_j(\varepsilon_1)$ and $\sigma_l(\varepsilon_1)$ for some $j \neq l$, $j, l \in \{1, \dots, N\}$, and a minimal vanishing sub-sum which contains $\sigma_u(\varepsilon_2)$ and $\sigma_v(\varepsilon_2)$ for some $u \neq v$, $u, v \in \{1, \dots, N\}$. Further, these vanishing sub-sums contain at most $N$ terms. Thus, we infer again a much better bound than the bound (4.14) on the number of solutions in this case.

Finally, we have $2^{2N-1}$ possibilities for choosing the considered sub-sums, so altogether the number of solutions $(\varepsilon_1, \varepsilon_2)$ of equation (4.12) is bounded by

$$(4.15) \qquad \exp\left((12N - 6)^{6N-1}(4k - 2)\right).$$

Thus (using that $N \leq n$) the number of those solutions of equation (1.1), for which there exists a solution $(y_1, \dots, y_n) \neq (x_1, \dots, x_n)$ with $x_i = y_i$, is bounded by

$$\Psi(k, n, mD^k) \exp\left((12n - 6)^{6n-1}(4k - 2)\right).$$

Thus, the number of those solutions $(x_1, \dots, x_n)$ of equation (1.1), for which there exists another solution $(y_1, \dots, y_n) \neq (x_1, \dots, x_n)$, such that $\prod_{i=1}^{n}(x_i - y_i) = 0$ is bounded by

$$n\Psi(k, n, mD^k) \exp\left((12n - 6)^{6n-1}(4k - 2)\right)$$
$$\leq \Psi(k, n, mD^k) \exp\left(k(12n)^{6n}\right). \qquad \square$$

*Proof of Theorem 2.6.* We start the proof of the present theorem exactly in the same way as the proof of Theorem 2.5. The first difference is that instead of equation (4.12) we get

$$(4.16) \qquad \sum_{j=1}^{N} \left(\gamma_{ij}\sigma_j(\mu_1)\sigma_j(\varepsilon_1) - \gamma_{ij}\sigma_j(\mu_2)\sigma_j(\varepsilon_2)\right) = d \in T.$$

Now divide this equation by $d$ to get an inhomogeneous $S$-unit equation having $2N$ terms. Using Lemma 3.3 we can bound (similarly to the proof of Theorem 2.5) the possibilities for either the values of $(\sigma_u(\varepsilon_1))/d$, or the values of $(\sigma_u(\varepsilon_2))/d$ for some $u$, depending on the vanishing subsums in the unit equation. This bound is given by

$$(4.17) \qquad \exp\left((12N)^{6N}(s+2k-1)\right).$$

Since $d \in \mathbf{Z}$ and $\sigma_u(\varepsilon_1)$ is a unit, thus if $(\sigma_u(\varepsilon_1))/d$ is fixed, then $d$ may assume at most two values and by fixing one of those, $\sigma_u(\varepsilon_1)$ becomes also fixed. Then we can fix $\varepsilon_2$, too. A similar argument works also when first we are able to fix $(\sigma_u(\varepsilon_2))/d$. Thus, for the number of solutions of equation (1.1) for which there exists another solution $(y_1,\ldots,y_n) \neq (x_1,\ldots,x_n)$, such that $x_i - y_i \in T$ for some $i \in \{1,\ldots,n\}$, is bounded by

$$\Psi(k,n,mD^k)\exp\left((s+k)(12n)^{6n+3}\right). \qquad \square$$

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