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RELATIVELY BOUNDED EXTENSIONS OF GENERATOR PERTURBATIONS

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ABSTRACT. The Miyadera perturbation theorem provides as a by-product that operators defined on a core for the generator of a C_0-semigroup and satisfying the Miyadera condition have a relatively bounded extension to the domain of the generator. We show that a weakening of the Miyadera condition characterizes relative boundedness with respect to the generator. We also investigate extensions of these results to Hille-Yosida operators. The various conditions we use in the abstract part are illustrated by several examples.

1. Introduction. Additive perturbations of C_0-semigroups typically involve an operator B which is relatively bounded with respect to the generator, A, of the semigroup, T. In important applications one has a good idea of how A operates, but difficulties in determining the precise form of $D(A)$. In these situations, often a dense subspace D of $D(A)$ can be identified which is invariant under T and on which the perturbation can be easily described. The perturbation theorem presented in [**11**] adapts Miyadera's perturbation theorem [**7**] to such a scenario. In its proof, the extension of the perturbation from D to $D(A)$ is obtained as a by-product of the perturbation procedure. The method presented in this paper decouples the extension of the perturbation from the construction of the perturbed semigroup and yields a characterization of A-boundedness by conditions which are similar to but weaker than the Miyadera condition.

In Section 1 we show the characterization mentioned above (Corollary 1.6). Furthermore, we establish the relation between various constants appearing in different versions of relative boundedness (Proposition 1.8). We also present an extension result related to multiplicative perturbations of C_0-semigroups (Theorem 1.12).

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In Section 2, we treat the extension problem for the case that A is a Hille-Yosida operator. In this case, the part $\check{A}$ of the operator A in $\check{X} := \overline{D(A)}$ is the generator of a C_0-semigroup, and the domain $D(\check{B}) = D(\check{A})$ of the perturbation $\check{B}$ is no longer a core for A. Therefore, the question of (unique) extendability of $\check{B}$ to $D(A)$ is not covered by the results of Section 1. We solve the extension problem in two cases. In the first case, $\check{B}$ is assumed to be infinitesimally $\check{A}$-small. In turn, the extension of $\check{B}$ is infinitesimally A-small. In the second case, X is an ordered Banach space, A is resolvent positive, and $\check{B}$ is positive and takes values in an ordered Banach space with a fully regular cone.

In Section 3 we treat examples illustrating several of the results of the previous sections.

1. Perturbations of semigroup generators. In this section let A be the generator of a C_0-semigroup T on a (real or complex) Banach space X. There exist $M \geq 0$, $\omega \in \mathbf{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

A subset D of X will be called *almost invariant under* T if, for all $x \in D$, the set $\{t \in [0,\infty)\,;\, T(t)x \notin D\}$ has Lebesgue measure zero.

1.1 Theorem. *Let $D \subseteq D(A)$ be a dense subspace of X which is almost invariant under T. Let $B_0\colon D \to Y$ be a linear map from D to a Banach space Y such that $B_0T(\cdot)x \in L_{1,\mathrm{loc}}([0,\infty);Y)$ for all $x \in D$. Assume that there exist constants $\alpha, \gamma > 0$ such that*

$$\left\| \int_0^t B_0 T(s)x\, ds \right\| \leq \gamma \|x\| \quad \text{for all } t \in [0,\alpha],\ x \in D. \tag{1.1}$$

Then there exists a uniquely determined A-bounded operator $B\colon D(A) \to Y$ such that, for every $x \in D$, $BT(\cdot)x = B_0T(\cdot)x$ almost everywhere on $[0,\infty)$. For all $t \geq 0$, the operator B satisfies

$$\left\| B \int_0^t T(s)\, ds \right\| = \sup\left\{ \left\| \int_0^t B_0 T(s)x\, ds \right\|;\ x \in D,\ \|x\| \leq 1 \right\}.$$

If B_0 is closable, then B is an extension of B_0.

In general, it cannot be concluded that B is an extension of B_0. This will be illustrated by Example 3.1. However, the set $\{x \in D; B_0 x = Bx\}$ will turn out to be a core for A, see Remarks 1.5. For $x \in D$, the function $BT(\cdot)x$ is continuous; so the assertion shows that $B_0 T(\cdot)x$ is necessarily equivalent to a continuous function. Further, one obtains that

$$\operatorname*{ess\,sup}_{0 \le s \le t} \|Bx - B_0 T(s)x\| \to 0 \quad (t \to 0), \quad \text{for all } x \in D.$$

This shows that B is an extension of B_0 if and only if the preceding statement holds with B replaced by B_0.

Assumption (1.1) is strictly weaker than the *Miyadera condition*

$$\int_0^\alpha \|B_0 T(s)x\| \, ds \le \gamma \|x\| \quad \text{for all } x \in D; \tag{1.2}$$

cf. Example 3.5. Condition (1.2) is needed in the Miyadera perturbation theorem, with $Y = X$ and $\gamma < 1$ (cf. [**4, 7, 11**]).

1.2 *Remarks.* (a) In condition (1.1) as well as in the Miyadera condition (1.2), the function $B_0 T(\cdot)x$ only occurs in an integral. Therefore, it seems natural to require that this function is integrable (rather than continuous, as in Corollary 1.7) and is defined only almost everywhere, i.e., D is almost invariant rather than invariant under T.

(b) Supposing more strongly the invariance of D under T in Theorem 1.1 does not lead to the conclusion that B is an extension of B_0. This is illustrated in Example 3.1 (b).

(c) We note that in Example 3.1 also the Miyadera condition (1.2) is satisfied (since $B_0 T(\cdot)x = 0$ almost everywhere and $B = 0$ on $D(A)$). This means that in the Miyadera perturbation theorem as well, one cannot release the continuity of $B_0 T(\cdot)x$ for $x \in D$ to the weaker requirement that $B_0 T(\cdot)x \in L_1((0,\alpha); X)$ if one wants B to be an extension of B_0.

A function $V\colon [0,\infty) \to L(X,Y)$ (where Y is a Banach space) is called a *cumulative output for T* if

$$V(t)T(s) = V(t+s) - V(s) \tag{1.3}$$

for all $t, s \geq 0$. Equation (1.3) is called the *cumulative output identity* (cf. [**3, 10**]). Setting $t = s = 0$ in (1.3) one concludes $V(0) = 0$. In the following remarks we collect some elementary facts concerning functions satisfying (1.3).

1.3 *Remarks.* (a) Let $\alpha > 0$, and let $V\colon [0, \alpha] \to L(X, Y)$ satisfy (1.3) for all $t, s \geq 0$ such that $t + s \leq \alpha$. Then V extends uniquely to a cumulative output.

In order to show this it is clearly sufficient to extend V (uniquely) to $[0, 2\alpha]$. Let $t \in (\alpha, 2\alpha]$. Then (1.3) implies that the definition $V(t) := V(r) + V(t-r)T(r)$ does not depend on the choice of $r \in (0, \alpha]$ with $t - r \in [0, \alpha]$, and that V thus defined satisfies (1.3) on $[0, 2\alpha]$.

(b) Let $V\colon [0, \infty) \to L(X, Y)$ be a cumulative output for T, and assume that there exists $\alpha > 0$ such that $\sup_{0 \leq t \leq \alpha} \|V(t)\| < \infty$. Then V is exponentially bounded, i.e., there exist constants $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbf{R}$ such that $\|V(t)\| \leq \widetilde{M} e^{\widetilde{\omega} t}$ for all $t \geq 0$.

This statement is proved as in [**3**, Proposition 3.2] or [**10**, Lemma 4.14].

(c) Let $V\colon [0, \infty) \to L(X, Y)$ be a cumulative output for T, and assume that $\text{s-lim}_{t \to 0} V(t) = 0$ $(= V(0))$. Then V is strongly continuous on $[0, \infty)$.

Indeed, writing (1.3) as $V(t) - V(s) = V(t-s)T(s)$, for $0 \leq s \leq t$, we obtain $V(t) - V(s) \to 0$ strongly for fixed s and $t \to s$ as well as for fixed t and $s \to t$.

An operator family $(F(\lambda); \lambda > \theta)$ is called a *resolvent output* for A (cf. [**10**]) if it satisfies the *resolvent output identity*

$$(1.4) \quad (\mu - \lambda)F(\lambda)(\mu - A)^{-1} = F(\lambda) - F(\mu), \quad (\lambda, \mu > \max\{\theta, \omega\}).$$

The concept of a resolvent output and part (a) of the following lemma also apply if A is a closed operator whose resolvent set contains (ω, ∞).

1.4 Lemma. (a) *A family $(F(\lambda); \lambda > \theta)$ of bounded linear operators is a resolvent output for A if and only if there exists an A-bounded operator $B\colon D(A) \to X$ such that $F(\lambda) = B(\lambda - A)^{-1}$ for all $\lambda > \max\{\theta, \omega\}$.*

(b) *Let* $V\colon [0,\infty) \to L(X,Y)$ *be strongly continuous, and assume that there exist constants* $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbf{R}$ *such that* $\|V(t)\| \leq \widetilde{M} e^{\widetilde{\omega} t}$ *for all* $t > 0$. *Define* $F(\lambda) = \lambda \int_0^\infty e^{-\lambda t} V(t)\, dt$ *for* $\lambda > \widetilde{\omega}$. *Then* F *is a resolvent output for* A *if and only if* V *is a cumulative output for* T. *Moreover,*

$$\limsup_{\lambda \to \infty} \|F(\lambda)\| \leq \limsup_{t \to 0+} \|V(t)\|.$$

(c) *Let* $V\colon [0,\infty) \to L(X,Y)$ *be a strongly continuous cumulative output for* T. *Then there exists a uniquely determined* A*-bounded operator* $B\colon D(A) \to Y$ *such that* $V(t) = B \int_0^t T(s)\, ds$ *for all* $t \geq 0$.

Proof. (a) If $F(\lambda) = B(\lambda - A)^{-1}$, for an A-bounded operator $B\colon D(A) \to Y$, then the resolvent output identity follows from the resolvent equation. Conversely, let F be a resolvent output for A. The resolvent output identity (1.4) implies that $F(\lambda)(\mu - A)^{-1} = F(\mu)(\lambda - A)^{-1}$ for all $\lambda, \mu > \max\{\theta, \omega\}$, and therefore the definition $B = F(\lambda)(\lambda - A)$ does not depend on the choice of $\lambda > \max\{\theta, \omega\}$. The boundedness of $F(\lambda)$ implies that B is A-bounded.

(b) follows by multiplying the relation (1.3) by $e^{-\lambda t} e^{-\mu s}$ (where $\lambda, \mu > \max\,\{\omega, \widetilde{\omega}\}$) and integrating over r and t and from the uniqueness properties of the Laplace transform. The inequality follows from Fatou's lemma.

(c) By parts (a) and (b), there exists an A-bounded operator B such that

$$\lambda \int_0^\infty e^{-\lambda t} V(t)\, dt = B(\lambda - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} B\left(\int_0^t T(s)\, ds \right) dt.$$

The uniqueness properties of the Laplace transform imply that $V(t) = B \int_0^t T(s)\, ds$. Since B is A-bounded, one obtains $Bx = \lim_{t \to 0+} (1/t) V(t)x$ for $x \in D(A)$. This shows that B is uniquely determined by V. $\square$

Proof of Theorem 1.1. For $t \in [0, \alpha]$, $x \in D$, we define $V_0(t)x = \int_0^t B_0 T(s) x\, ds$. Then $V_0(t)$ extends to a bounded linear operator $V(t)\colon X \to Y$, and $V\colon [0,\alpha] \to L(X,Y)$ is strongly continuous and bounded.

Let $x \in D$, and let $t \in [0, \alpha)$. Then

$$V_0(t)T(s)x = V_0(t+s)x - V_0(s)x$$

for all $s \in [0, \alpha - t]$ such that $T(s)x \in D$. The set of those s is a set of full measure in $[0, \alpha - t]$, and therefore the strong continuity of V implies

$$V(t)T(s)x = V(t+s)x - V(s)x$$

for all $s \in [0, \alpha - t]$. This shows (1.3) for $t, s \geq 0$ such that $t + s \leq \alpha$.

From Remarks 1.3 and Lemma 1.4 we obtain that there exists a uniquely determined A-bounded operator B such that $V(t) = B\int_0^t T(s)\,ds$ for $t \in [0, \alpha]$. For $x \in D$ one concludes $\int_0^t BT(s)x\,ds = V(t)x = V_0(t)x = \int_0^t B_0T(s)x\,ds$. So $BT(t)x = B_0T(t)x$ for almost every $t \in [0, \alpha]$. Since D is almost invariant under T, this holds for almost every $t \geq 0$. Also, $\int_0^t T(s)\,ds$ is continuous as an operator from X to $D(A)$ (with the graph norm); therefore, $B\int_0^t T(s)\,ds$ is a bounded operator whose norm can be computed on the dense set D.

Assume now that B_0 is closable. Let $x \in D$. Then $BT(\cdot)x$ is continuous. There exists a sequence (t_n) in $(0, \infty)$, $t_n \to 0$ $(n \to \infty)$, such that $B_0T(t_n)x = BT(t_n)x$ for all $n \in \mathbf{N}$. Then $x_n := T(t_n)x \to x$, $B_0x_n \to Bx$ $(n \to \infty)$, and this implies $B_0x = Bx$. □

1.5 *Remarks.* (a) Let $D \subseteq D(A)$ be dense in X and almost invariant under T. Then D is a core for A.

Indeed, since D is almost invariant, $\widehat{D} := \bigcup_{t \geq 0} T(t)D$ is contained in the A-closure of D. Since $\widehat{D}$ is invariant under T, one obtains that $\widehat{D}$ is a core for A (cf. [**4**, II, Proposition 1.7]).

(b) Let additionally B_0 and B be as in Theorem 1.1, and define $D_0 := \{x \in D;\, B_0x = Bx\}$. Then D_0 is a core for A.

Indeed, $D_0 \subseteq D(A)$ is dense in X and almost invariant under T, by the assertion of Theorem 1.1. Therefore, part (a) above shows that D_0 is a core for A.

1.6 Corollary. *Let $D \subseteq D(A)$ be a dense subspace of X such that $T(t)D \subseteq D$ for all $t \geq 0$, and let $B_0\colon D \to Y$ be a linear map from D to a Banach space Y such that $B_0T(\cdot)x$ continuous for all $x \in D$.*

Then B_0 extends to an A-bounded operator $B\colon D(A) \to Y$ if and only if there exist constants $\alpha, \gamma > 0$ such that (1.1) *holds.*

Proof. Assume that $B\colon D(A) \to Y$ is an A-bounded extension of B_0. Let $\lambda \in \rho(A)$. Then $B(\lambda - A)^{-1}$ is a bounded operator and, for any $x \in D(A)$,

$$\begin{aligned}\int_0^t BT(s)x\,ds &= B(\lambda - A)^{-1}\int_0^t (\lambda - A)T(s)x\,ds \\ &= B(\lambda - A)^{-1}\left(\lambda\int_0^t T(s)x\,ds - (T(t)x - x)\right).\end{aligned} \tag{1.5}$$

This shows that for any $\alpha > 0$ we can find $\gamma > 0$ such that (1.1) holds.

Assume now that there exist $\alpha, \gamma > 0$ such that(1.1) holds. Then Theorem 1.1 implies that there exists an A-bounded operator $B\colon D(A) \to Y$ such that $BT(\cdot)x = B_0T(\cdot)x$ almost everywhere on $[0, \infty)$ for all $x \in D$. The continuity of both of these functions implies $Bx = BT(0)x = B_0T(0)x = B_0x$. Therefore, $B \supseteq B_0$. □

1.7 Corollary. *Let D_0 be a dense subspace of X, $T(t)D_0 \subseteq D_0$ for all $t \geq 0$. Assume that $D := \operatorname{lin}\{\int_0^t T(s)x\,ds;\, t > 0,\ x \in D_0\} \subseteq D_0$, and let $B_0\colon D \to Y$ be a linear map from D to a Banach space Y.*

Then B_0 extends to a (*unique*) *A-bounded operator $B\colon D(A) \to Y$ if and only if there exist constants $\alpha, \gamma > 0$ such that*

$$\left\| B_0 \int_0^t T(s)x\,ds \right\| \leq \gamma\|x\| \quad \textit{for all } t \in [0, \alpha], \quad x \in D_0. \tag{1.6}$$

As illustrated in Example 3.2, the invariance of D_0 under T cannot be dropped as an assumption.

Proof. The necessity follows from the A-boundedness of B.

For the sufficiency we apply Corollary 1.6. The set D is a subset of $D(A)$, is dense in X and invariant under T. (This implies that D is a core for A, and therefore the A-bounded extension B, if it exists, is unique.) It remains to show that $B_0T(\cdot)x$ is continuous for all $x \in D$ and that (1.1) is valid (with α, γ as above).

Let $r \in (0, \alpha]$. For $x \in D_0$, we define $V_0(r)x := B_0 \int_0^r T(u)x\,du$ and notice that (1.6) implies that $V_0(r)$ extends to an operator $V(r) \in L(X, Y)$. Let $x \in D_0$, $x_r := \int_0^r T(u)x\,du$. Then $B_0T(t)x_r = B_0T(t)\int_0^r T(u)x\,du = V(r)T(t)x$ is continuous as a function of t. The invariance of D_0 under T implies $D = \operatorname{lin}\{\int_0^t T(s)x\,ds\,;\, 0 < t \le \alpha,\ x \in D_0\}$, and therefore we obtain the continuity of $B_0T(\cdot)x$ for all $x \in D$.

In order to show (1.1) it is sufficient to show

$$\int_0^t B_0T(s)x\,ds = B_0\int_0^t T(s)x\,ds \quad \text{for all } x \in D,\ t \in (0, \alpha], \tag{1.7}$$

which in turn follows if we show (1.7) for all $x_r := \int_0^r T(u)x\,du$ ($x \in D_0$, $0 < r \le \alpha$). The latter follows from the computation

$$\begin{aligned} B_0\int_0^t T(s)x_r\,ds &= B_0\int_0^r T(u)\int_0^t T(s)x\,ds\,du \\ &= V(r)\int_0^t T(s)x\,ds \\ &= \int_0^t V(r)T(s)x\,ds \\ &= \int_0^t B_0\int_0^r T(u)T(s)x\,du\,ds \\ &= \int_0^t B_0T(s)x_r\,ds. \qquad \square \end{aligned}$$

The A-boundedness of an operator $B: D(A) \to Y$ can be expressed in different ways. In the following proposition we provide the relation between various numbers connected with this notion. We recall that the *A-bound* of B (or *relative bound* of B with respect to A) is defined as the infimum of the numbers $b \ge 0$ for which there exists $a \ge 0$ such that

$$\|Bx\| \le a\|x\| + b\|Ax\| \quad \text{for all } x \in D(A). \tag{1.8}$$

We call B *infinitesimally A-small* if the A-bound of B equals 0.

1.8 Proposition. *Assume that $B\colon D(A) \to Y$ is A-bounded with relative bound β. Then*

$$\beta \le \limsup_{\lambda\to\infty} \|B(\lambda - A)^{-1}\| \le \limsup_{t\to 0} \left\| B \int_0^t T(s)\, ds \right\| \le (M+1)\beta.$$

In particular, B is infinitesimally A-small if and only if

$$\lim_{t\to 0} \left\| B \int_0^t T(s)\, ds \right\| = 0.$$

Proof. The first inequality follows from

$$\|Bx\| \le \lambda \|B(\lambda - A)^{-1}\| \|x\| + \|B(\lambda - A)^{-1}\| \|Ax\| \qquad (x \in D(A)). \tag{1.9}$$

Defining $V(t) = B\int_0^t T(s)\, ds$, $F(\lambda) = \lambda \int_0^\infty e^{-\lambda t} V(t)\, dt$ $(= B(\lambda - A)^{-1})$, we obtain the second inequality from Lemma 1.4 (b). For the proof of the last inequality let $b > \beta$, $a \ge 0$ be such that (1.8) holds. Then

$$\begin{aligned}\left\| B \int_0^t T(s)\, ds \right\| &\le a \left\| \int_0^t T(s)\, ds \right\| + b \left\| A \int_0^t T(s)\, ds \right\| \\ &\le aM \int_0^t e^{s\omega}\, ds + b\, \|T(t) - I\| \\ &\le aM \int_0^t e^{s\omega}\, ds + b(Me^{t\omega} + 1).\end{aligned}$$

Since the right-hand side of the last estimate tends to $b(M+1)$ as $t \to 0$ we obtain the last estimate. □

1.9 *Remark.* In the notation of Corollary 1.6, B is infinitesimally A-small if and only if

$$\sup \left\{ \left\| \int_0^t B_0 T(s) x\, ds \right\| ;\ x \in D,\ \|x\| \le 1 \right\} \to 0 \quad (t \to 0). \tag{1.10}$$

Condition (1.10) is satisfied if B_0 is an *infinitesimally small Miyadera perturbation* of A (cf. [**6**]), i.e., if for any $\gamma > 0$ there exists some $\alpha > 0$ such that

$$\int_0^\alpha \|B_0 T(t)x\| \, dt \le \gamma \|x\| \quad (x \in D).$$

In partial analogy to semigroups, cumulative outputs are strongly continuous if they satisfy an appropriate measurability and integrability condition.

1.10 Proposition. *Let $V\colon [0,\infty) \to L(X,Y)$ be a cumulative output for T. Assume that there exist $0 \le \alpha < \beta$ such that $V(\cdot)x \in L_1((\alpha,\beta);Y)$ for all $x \in X$.*

Then V is strongly continuous.

Proof. (Compare [**8**, Theorem 2.2].) Let $x \in X$. First we note that the equation

$$V(t+s)x = V(t)x + V(s)T(t)x$$

implies that $V(\cdot)x$ is integrable over the interval $(t+\alpha, t+\beta)$, for $t \ge 0$, and therefore $V(\cdot)x \in L_{1,\mathrm{loc}}([\alpha,\infty);Y)$. Integrating

$$V(t)x = V(t+s)x - V(s)T(t)x$$

over s from α to β we obtain

$$\begin{aligned}(\beta-\alpha)V(t)x &= \int_\alpha^\beta V(t+s)x\,ds - \int_\alpha^\beta V(s)T(t)x\,ds \\ &= \int_{t+\alpha}^{t+\beta} V(u)x\,du - \int_\alpha^\beta V(s)T(t)x\,ds.\end{aligned}$$

Since the strong integral $\int_\alpha^\beta V(s)\,ds$ is a bounded linear operator (cf. [**5**, Theorem 3.8.2]), it follows that $V(\cdot)x$ is continuous on $[0,\infty)$. □

1.11 *Remark.* Using Proposition 1.10 one could prove the result of Theorem 1.1 under the following weakening of condition (1.1): Suppose that there exist $\alpha > 0$ and a locally integrable function $\gamma\colon (0,\alpha] \to \mathbf{R}$ such that

$$\left\| \int_0^t B_0 T(s)x\,ds \right\| \le \gamma(t)\|x\| \quad \text{for all } t \in [0,\alpha],\ x \in D.$$

We conclude this section with another application of Proposition 1.10 which is related to multiplicative rather than additive perturbations of the generator A (cf. [**4**, III.3(d)], [**9**]).

Theorem 1.12. *Let $D \subseteq D(A)$ be a dense subspace of X such that $T(t)D \subseteq D$ for all $t \ge 0$. Let $C\colon D \to Y$ be a linear map from D to a Banach space Y such that $CT(\cdot)x$ is Borel measurable for each $x \in D$. Assume that there exist a constant $\alpha > 0$ and a function $\gamma : (0,\alpha] \to \mathbf{R}_+$ such that $\gamma \in L_{1,\mathrm{loc}}(0,\alpha]$ and*

$$\|C(T(t)x - x)\| \le \gamma(t)\|x\| \quad \textit{for all } t \in (0,\alpha],\ x \in D. \tag{1.11}$$

Then there exists a uniquely determined A-bounded operator $B\colon D(A) \to Y$ such that

$$B\int_0^t T(s)x\,ds = C(T(t)x - x) \quad \textit{for all } x \in D,\ t \ge 0.$$

For all $t \ge 0$, the operator B satisfies

$$\left\| B\int_0^t T(s)\,ds \right\| = \sup\left\{\|C(T(t)x - x)\|\,;\, x \in D,\ \|x\| \le 1\right\}.$$

If C is closable (with closure $\overline{C}$), then $B = \overline{C}A$.

Proof. We set $V_0(t)x = C(T(t)x - x)$ for $t \ge 0$, $x \in D$. Then, for every $x \in D$, $V_0(\cdot)x$ is Borel measurable and $V_0(t+r)x = V_0(r)x + V_0(t)T(r)x$ for all $t, r \ge 0$. For $t \in [0,\alpha]$, $V_0(t)$ can be extended to a bounded linear operator $V(t)$ on X. Applying Remark 1.3, Proposition 1.10 and Lemma 1.4 (c) we obtain a uniquely determined A-bounded operator B such that $\int_0^t BT(s)x\,ds = V(t)x = C(T(t)x - x)$ for all $x \in D$, $t \ge 0$.

Let C be closable. The continuity of the operator $B\int_0^t T(s)\,ds$ shows that then $T(t)x - x \in D(\overline{C})$ and

$$B\int_0^t T(s)x\,ds = \overline{C}(T(t)x - x) \tag{1.12}$$

for all $x \in X$, $t \geq 0$. Let $x \in D(A)$. Then, dividing (1.12) by t and taking $t \to 0$ we conclude that $Ax \in D(\overline{C})$ and $Bx = \overline{C}Ax$. □

2. Perturbations of Hille-Yosida operators. Now let A be a Hille-Yosida operator in the Banach space X (cf. [**1**, Section 3.5]). Then the part $\check{A}$ of A in $\check{X} = \overline{D(A)}$ is the generator of a C_0-semigroup T on $\check{X}$. As before there exist $M \geq 0$, $\omega \in \mathbf{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

2.1 Proposition. *Let Y be a Banach space, and let $\check{B}\colon D(\check{A}) \to Y$ be an $\check{A}$-bounded operator. Then the following properties are equivalent.*

(i) *The operator $\check{B}$ has an A-bounded extension $B\colon D(A) \to Y$ satisfying*

$$B(\lambda - A)^{-1} \to 0 \quad \textit{strongly} \quad (\lambda \to \infty). \tag{2.1}$$

(ii) *The limit*

$$Bx := \lim_{\lambda\to\infty} \check{B}\lambda(\lambda - \check{A})^{-1}x \tag{2.2}$$

exists for all $x \in D(A)$.

(iii) *The limit*

$$\operatorname*{s-lim}_{\lambda\to\infty} \check{B}(\mu - \check{A})^{-1}\lambda(\lambda - A)^{-1} \tag{2.3}$$

exists for some (all) $\mu \in \rho(A)$.

If one (and then all) of these properties are satisfied, then the A-bounded extension B of $\check{B}$ satisfying (2.1) is given by (2.2), and therefore is uniquely determined.

Proof. (i) $\Rightarrow$ (ii). For $x \in D(A)$, the hypothesis implies

$$Bx = \lim_{\lambda\to\infty} B(\lambda - A)^{-1}(\lambda - A)x = \lim_{\lambda\to\infty} \check{B}(\lambda - \check{A})^{-1}\lambda x.$$

(ii) $\Rightarrow$ (i). The uniform boundedness theorem implies that the operator B defined by (2.2) is A-bounded. Also, $B \supseteq \check{B}$. Choose $\mu \in \rho(A)$. Then, for all $x \in D(A)$,

$$B(\lambda - A)^{-1}(\mu - A)x = (\mu - \lambda)\check{B}(\lambda - \check{A})^{-1}x + Bx \to 0 \quad (\lambda \to \infty).$$

From $(\mu - A)D(A) = X$ we obtain that $B(\lambda - A)^{-1} \to 0$ strongly as $\lambda \to \infty$.

(ii) $\Leftrightarrow$ (iii). This equivalence is immediate from the equation $\check{B}(\mu - \check{A})^{-1}\lambda(\lambda - A)^{-1} = \check{B}\lambda(\lambda - \check{A})^{-1}(\mu - A)^{-1}$ together with the fact that, for $\mu \in \rho(A)$, the mapping $\mu - A\colon D(A) \to X$ is bijective. $\square$

2.1 Extensions of infinitesimally small perturbations. A linear operator $B\colon D(A) \to Y$ is infinitesimally A-small if and only if $\|B(\lambda - A)^{-1}\| \to 0$ as $\lambda \to \infty$. (Sufficiency follows from (1.9). For necessity assume that M is such that $\|(\lambda - A)^{-1}\| \le M/\lambda$ for large λ, and let a, b be such that (1.8) holds. Then the inequality

$$\|B(\lambda - A)^{-1}x\| \le a\frac{M}{\lambda}\|x\| + b(1 + M)\|x\| \quad (x \in X)$$

shows $\limsup_{\lambda\to\infty} \|B(\lambda - A)^{-1}\| \le b(1 + M)$.)

2.2 Theorem. *Let Y be a Banach space, and let $\check{B}\colon D(\check{A}) \to Y$ be an infinitesimally $\check{A}$-small operator.*

Then there exists an infinitesimally A-small extension $B\colon D(A) \to Y$ of B_0.

Proof. We set $\check{F}(\lambda) := \check{B}(\lambda - \check{A})^{-1}$ $(\lambda > \omega)$. Then

$$\check{F}(\lambda)(\alpha - \lambda)(\alpha - A)^{-1} = \check{F}(\lambda) - \check{F}(\alpha) \quad (\alpha, \lambda > \omega), \tag{2.4}$$

i.e., $\check{F}$ is a resolvent output for $\check{A}$ (see Lemma 1.4), and by hypothesis, $\|\check{F}(\lambda)\| \to 0$ as $\lambda \to \infty$. Next we show that, for $\lambda > \omega$, the operator norm limit

$$\lim_{\alpha\to\infty} \check{F}(\lambda)\alpha(\alpha - A)^{-1}$$

exists. Let $\alpha, \beta > \omega$. Then

$$\begin{aligned}\check{F}(\lambda)(\alpha - \lambda)(\alpha - A)^{-1} &- \check{F}(\lambda)(\beta - \lambda)(\beta - A)^{-1}\\ &= \operatorname*{s\text{-}lim}_{\mu\to\infty} \check{F}(\lambda)\mu(\mu - \check{A})^{-1}\\ &\quad \times \big((\alpha - \lambda)(\alpha - A)^{-1} - (\beta - \lambda)(\beta - A)^{-1}\big)\\ &= \operatorname*{s\text{-}lim}_{\mu\to\infty} \big(\check{F}(\beta) - \check{F}(\alpha)\big)\mu(\mu - A)^{-1},\end{aligned}$$

by (2.4). This implies

$$\begin{aligned}\|\check{F}(\lambda)(\alpha - \lambda)(\alpha - A)^{-1} &- \check{F}(\lambda)(\beta - \lambda)(\beta - A)^{-1}\|\\ &\le \widehat{M}\|\check{F}(\beta) - \check{F}(\alpha)\|,\end{aligned}$$

where $\widehat{M} := \liminf_{\mu\to\infty} \|\mu(\mu - A)^{-1}\|$. Using $\lim_{\alpha\to\infty} \|\check{F}(\alpha)\| = 0$ we obtain the existence of

$$F(\lambda) := \lim_{\alpha\to\infty} \check{F}(\lambda)\alpha(\alpha - A)^{-1} = \lim_{\alpha\to\infty} \check{F}(\lambda)(\alpha - \lambda)(\alpha - A)^{-1},$$

and $\|F(\lambda)\| \le \widehat{M}\|\check{F}(\lambda)\| \to 0$ as $\lambda \to \infty$. Now the assertions follow from Proposition 2.1 and $B(\lambda - A)^{-1} = F(\lambda)$, $\lambda > \omega$. $\square$

2.2 Extensions in ordered Banach spaces. We assume additionally that X is an ordered Banach space with a generating positive cone X_+. Recall that the positive cone Y_+ of an ordered Banach space Y is *fully regular* if any norm-bounded monotone increasing sequence in Y is convergent.

2.3 Theorem. *Assume that the Hille-Yosida operator A is resolvent positive. Let Y be an ordered Banach space with fully regular cone Y_+. Let $\check{B}: D(\check{A}) \to Y$ be a positive operator. Then there exists an A-bounded extension $B: D(A) \to Y$ of $\check{B}$ satisfying* (2.1), *and B is positive.*

Proof. We show that the limit (2.3) exists. We define

$$\check{F}(\lambda) = \check{B}(\lambda - \check{A})^{-1} \quad (\lambda > \omega).$$

As in the proof of Theorem 2.2, $\check{F}$ satisfies the resolvent output identity (2.4). Since $\lambda \mapsto (\lambda - \check{A})^{-1}$ is decreasing (by the resolvent equation) we obtain that $\lambda \mapsto \check{F}(\lambda)$ is decreasing. From the resolvent output identity (2.4) we therefore obtain that, for all $\mu > \omega$, the function

$$(\mu, \infty) \ni \lambda \longmapsto (\lambda - \mu)\check{F}(\mu)(\lambda - \check{A})^{-1}$$

is increasing. The equation

$$\begin{aligned}\check{F}(\mu)(\lambda - A)^{-1} &= \operatorname*{s\text{-}lim}_{\alpha\to\infty} \check{F}(\mu)\alpha(\alpha - \check{A})^{-1}(\lambda - A)^{-1} \\ &= \operatorname*{s\text{-}lim}_{\alpha\to\infty} \check{F}(\mu)(\lambda - \check{A})^{-1}\alpha(\alpha - A)^{-1}\end{aligned}$$

shows that

$$(\mu, \infty) \ni \lambda \longmapsto (\lambda - \mu)\check{F}(\mu)(\lambda - A)^{-1}$$

is increasing as well. Since $\{(\lambda - \mu)\check{F}(\mu)(\lambda - A)^{-1}; \lambda > \mu\}$ is bounded and the cone Y_+ is fully regular, the limit

$$F(\mu)x = \lim_{\lambda\to\infty} (\lambda - \mu)\check{F}(\mu)(\lambda - A)^{-1}x = \lim_{\lambda\to\infty} \lambda\check{F}(\mu)(\lambda - A)^{-1}x$$

exists for all $x \in X_+$. Since X_+ is generating, we obtain the existence of the limit in (2.3). Now Proposition 2.1 shows the existence of B.

From equation (2.2) we obtain that B inherits positivity from $\check{B}$. □

2.4 Theorem. *Assume that the norm on X is additive on X_+, and that the Hille-Yosida operator A is resolvent positive and dissipative. Let $\check{B}: D(\check{A}) \to X$ be a positive linear operator and $B: D(A) \to X$ the extension of $\check{B}$ whose existence was shown in Theorem* 2.2, *and assume that* $\limsup_{\lambda\to\infty} \|\check{B}(\lambda - \check{A})^{-1}\| < 1$.

Then $A + B$ is a resolvent positive Hille-Yosida operator.

Proof. Note that the additivity of the norm on X_+ implies that the cone X_+ is fully regular. For $x \in X$, the dissipativity of A and (2.2) imply

$$\begin{aligned}\|B(\mu - A)^{-1}x\| &= \lim_{\lambda\to\infty} \|\lambda\check{B}(\lambda - A)^{-1}(\mu - A)^{-1}x\| \\ &= \lim_{\lambda\to\infty} \|\check{B}(\mu - \check{A})^{-1}\lambda(\lambda - A)^{-1}x\| \\ &\le \|\check{B}(\mu - \check{A})^{-1}\| \limsup_{\lambda\to\infty} \lambda\|(\lambda - A)^{-1}x\| \\ &\le \|\check{B}(\mu - \check{A})^{-1}\|\|x\|.\end{aligned}$$

Hence $\limsup_{\mu\to\infty} \|B(\mu - A)^{-1}\| < 1$, and the statement follows from [**10**, Theorem 1.4]. $\square$

3. Examples. In the first example we illustrate that, in Theorem 1.1 it cannot be concluded that B is an extension of B_0.

3.1 Example. (a) Let $1 \le p < \infty$, $X := L_p(\mathbf{R})$, let T be the C_0-semigroup of right translations on $L_p(\mathbf{R})$, $T(t)f = f(\cdot - t)$, and let A be the generator of T. Let

$$D := \{f \in D(A);\ f \text{ Lipschitz-continuous, } f \text{ right and left differentiable at } 0\},$$

Then $D \subseteq D(A)$ is dense in $L_p(\mathbf{R})$. We define $B_0 : D \to \mathbf{K}$ by

$$B_0 f := f'_+(0) - f'_-(0)$$

(where $f'_+(0)$, $f'_-(0)$ are the right and left derivatives of f at 0, respectively). Let $f \in D$. Then f is differentiable almost everywhere, and therefore $T(t)f$ is differentiable at 0 for almost every $t \ge 0$, i.e., $T(t)f \in D$ and $B_0T(t)f = 0$ for almost every $t \ge 0$. (In particular, D is almost invariant, but not invariant under T.) Therefore, $B = 0$ is the A-bounded operator obtained in Theorem 1.1. However, $B_0 \ne 0$, and therefore B is not an extension of B_0.

(b) We keep the example from part (a), with the modification of defining the smaller set

$$D := \{f \in D(A);\ f \text{ Lipschitz-continuous and right and left differentiable}\}.$$

Then all the properties mentioned in part (a) are still valid, and additionally D is invariant under T.

A variant of the preceding example shows that the invariance of D_0 under the semigroup T cannot be dropped in Corollary 1.7. Recall that a function f is called *regulated* if its right-hand and left-hand limits, $f_+(s)$ and $f_-(s)$, exist for all s.

3.2 Example. Let $X = L_p(\mathbf{R})$, T, and A be as in Example 3.1. Choose $D_0 := \{f \in L_p(\mathbf{R})\,;\ f$ regulated on $\mathbf{R},\ f$ continuous on $[-2,1]\}$, D as in Corollary 1.7, $B_0 f := f'_+(0) - f'_-(0)$ for $f \in D$. Then

$$B_0 \int_0^t T(s)f\,ds = f_+(0) - f_-(0) - f_+(-t) + f_-(-t) \qquad (f \in D_0,\ t > 0).$$

This expression is 0 for $f \in D_0$ and $0 < t \leq 1$ (trivially implying estimate (1.6) for $\alpha = 1$), while it is not 0 for $f = \mathbf{1}_{[-4,-3]} \in D_0$ and $t = 3$.

However, B_0 does not have an A-bounded extension to $D(A)$: For $f \in D$ $(\subseteq D_0 \cap D(A))$ one obtains $(1/t)\int_0^t T(s)f\,ds \to f$ as $t \to 0$, with respect to the graph norm of A. This would imply $B_0 f = 0$ for all $f \in D$, contradicting $B_0 \neq 0$.

The remaining examples will be constructed in the context presented subsequently.

We start from the C_0 semigroup of left translations on $Y = C_0(\mathbf{R})$, $S(t)f(a) = f(a+t)$. Let $(S^*(t))_{t\geq 0}$ be the dual semigroup on $X = Y^*$. The dual space Y^* can be identified with $\mathcal{M}(\mathbf{R})$, the space of signed Borel measures of bounded variation, and S^* is the semigroup of right translations. The dual operator A of the generator of S (or equivalently, the weak* generator of S^* [**2**, Corollary 1.4.5]) is given by $Af = -f'$, with

$$D(A) = \{f \in X\,;\ f' \in X\},$$

where f' denotes the distributional derivative of f. The operator A is a Hille-Yosida operator, and $\check{X} = Y^{\odot}$ (the S-sun dual of Y, i.e., the subspace of Y^* consisting of those elements x for which $S^*(\cdot)x$ is strongly continuous) can be identified with $L^1(\mathbf{R})$. Let $T = S^{\odot}$ be the restriction of S^* to $Y^{\odot}$, so

$$T(t)f = f(\cdot - t) \quad (f \in L^1(\mathbf{R})),$$

and $\check{A}f = -f'$, with

$$D(\check{A}) = \{f \in L^1(\mathbf{R})\,;\ f \text{ absolutely continuous},\ f' \in L^1(\mathbf{R})\},$$

is the generator of T. We choose $D := C_c^\infty(\mathbf{R})$, the set of infinitely often differentiable functions on $\mathbf{R}$ with compact support.

Let $g_0 \in L^1_{\mathrm{loc}}(\mathbf{R})$ and define $B_0 \colon D \to \mathbf{K}$ by

$$B_0 f = -\int_{\mathbf{R}} g_0(a) f'(a)\, da = \int_{\mathbf{R}} g_0(a) \check{A} f(a)\, da \quad (f \in D).$$

Notice that B_0 does not change if we add a constant to g_0.

3.3 Example. If $g_0 \in L^\infty(\mathbf{R})$, then the operator B_0 defined above has a unique extension to an $\check{A}$-bounded operator $\check{B}$. However, B_0 has uncountably many A-bounded extensions, corresponding to bounded Borel measurable representatives of g_0. Indeed, choosing a fixed representative g of g_0 one obtains an extension B of B_0 by

$$Bf := -\int_{\mathbf{R}} g(a)\, f'(da) \quad (f \in D(A)) \tag{3.1}$$

(integral with respect to the measure $f' \in \mathcal{M}(\mathbf{R})$). Now, let g_1, g_2 be different representatives of g_0, $g_1(b) \neq g_2(b)$, $g_1(c) = g_2(c)$ for some $b, c \in \mathbf{R}$, $b < c$. Let B_1, B_2 be the extensions of B_0 corresponding to g_1, g_2. The function $\mathbf{1}_{[b,c]}$ belongs to $D(A)$, and $A\mathbf{1}_{[b,c]} = -\mathbf{1}'_{[b,c]} = \delta_c - \delta_b$ (Dirac measure concentrated at c, b, respectively). Therefore, $B_j \mathbf{1}_{[b,c]} = g_j(c) - g_j(b)$, $j = 1, 2$, $B_1 \mathbf{1}_{[b,c]} \neq B_2 \mathbf{1}_{[b,c]}$.

In particular, we obtain that the operator $B_0 = 0$ has uncountably many nonzero A-bounded extensions.

3.4 Example. In order to check under what conditions for $g_0 \in L^1_{\mathrm{loc}}(\mathbf{R})$ the estimates (1.1) and (1.10) are satisfied, we compute

$$\begin{aligned} \int_0^t B_0 T(s) f\, ds &= \int_{\mathbf{R}} g_0(a) T(t) f(a)\, da - \int_{\mathbf{R}} g_0(a) f(a)\, da \\ &= \int g_0(a) f(a-t)\, da - \int g_0(a) f(a)\, da \\ &= \int (g_0(a+t) - g_0(a)) f(a)\, da \quad (f \in D). \end{aligned} \tag{3.2}$$

$$\left| \int_0^t B_0 T(s) f\, ds \right| \le \operatorname*{ess\,sup}_{a \in \mathbf{R}} |g_0(a+t) - g_0(a)|\, \|f\|_1$$

then shows that (1.1) is satisfied for any $g_0 \in L^\infty(\mathbf{R})$ (but also for functions like $g_0(t) = |t|^p$ with $0 < p \leq 1$). Since D is dense in $\check{X} = L^1(\mathbf{R})$, it follows from (3.2) that

$$\sup\left\{\left|\int_0^t B_0T(s)f\,ds\right|;\ f \in D,\ \|f\|_1 \leq 1\right\} = \operatorname*{ess\,sup}_{a\in\mathbf{R}}|g_0(a+t) - g_0(a)|.$$

So (1.10) is satisfied if and only if g_0 has a (not necessarily bounded) uniformly continuous representative. Choosing this representative for the extension of B_0 to $D(A)$ one obtains the unique extension whose existence was shown in Theorem 2.2. Adding nonzero extensions of the zero operator, see Example 3.3, one sees that B_0 has many other A-bounded extensions.

3.5 Example. In order to investigate conditions for the validity of (1.2), we choose $g_0 \in L^1_{\mathrm{loc}}(\mathbf{R})$ having a Borel measure (denoted by g_0') on $\mathbf{R}$ as distributional derivative. Let $t > 0$. Then

$$\begin{aligned}
\int_0^t |B_0T(s)f|\,ds &= \int_0^t \left|\int g_0(a)f'(a-s)\,da\right| ds \\
&= \int_0^t \left|\int f(a-s)\,g_0'(da)\right| ds \\
&\leq \int\!\!\int_0^t |f(a-s)|\,ds\,\mathrm{var}\,(g_0')(da) \\
&= \int_{\mathbf{R}} |f(s)| \int_{s\leq a\leq s+t} \mathrm{var}\,(g_0')(da)\,ds \\
&= \int_{\mathbf{R}} |f(s)|\mathrm{var}\,(g_0')((s,s+t))\,ds \\
&\leq \sup_{s\in\mathbf{R}} \mathrm{var}\,(g_0')((s,s+t))\|f\|_1 \quad (f \in D).
\end{aligned}$$

We are going to show that in fact

$$\sup\left\{\int_0^t |B_0T(s)f|\,ds;\ f \in D,\ \|f\|_1 \leq 1\right\} = \sup_{s\in\mathbf{R}} \mathrm{var}\,(g_0')((s,s+t)) \tag{3.3}$$

(including the statement that the left-hand side can only be finite if g_0' is a measure).

The inequality "$\leq$" has already been proved above. Assume now that the left-hand side of (3.3) is finite, without loss of generality equal to 1. We show that then, for any interval $(c, c+t) \subseteq \mathbf{R}$ of length t, one obtains

$$\left| \int \phi'(a) g_0(a)\, da \right| \leq \|\phi\|_\infty \quad (\phi \in C_c^\infty(c, c+t)). \tag{3.4}$$

In order to prove (3.4) let $(\rho_k) \subseteq C_c^\infty(\mathbf{R})$ be a δ-sequence. Then

$$\begin{aligned}
&\left| \iint \phi'(b) \rho_k(a-b)\, db\, g_0(a)\, da \right| \\
&\qquad = \left| \iint \phi(b) \rho_k'(a-b)\, db\, g_0(a)\, da \right| \\
&\qquad \leq \int |\phi(b)| \left| \int \rho_k'(a-b) g_0(a)\, da \right| db \\
&\qquad \leq \int_c^{c+t} \left| \int \rho_k'(a-b) g_0(a)\, da \right| db\, \|\phi\|_\infty \\
&\qquad = \int_0^t \left| \int \rho_k'(a-s-c) g_0(a)\, ds \right| db\, \|\phi\|_\infty \\
&\qquad = \int_0^t |B_0 T(s) \rho_k(\cdot - c)|\, ds\, \|\phi\|_\infty \leq \|\phi\|_\infty.
\end{aligned}$$

Taking $k \to \infty$ on the left-hand side of this inequality we obtain (3.4). Inequality (3.4) implies that, on $(c, c+t)$, the distributional derivative of g_0 is a measure of total variation ≤ 1. Since this holds for all $c \in \mathbf{R}$ we obtain $\sup_{c \in \mathbf{R}} \operatorname{var}(g_0')((c, c+t)) \leq 1$.

This shows that (1.2) is satisfied if and only if $\sup_{s \in \mathbf{R}} \operatorname{var}(g_0')((s, s+1)) < \infty$ and that B_0 is infinitesimally Miyadera small, see Remark 1.9, if and only if $\sup_{s \in \mathbf{R}} \operatorname{var}(g')((s, s+t)) \to 0$ as $t \to 0$. The latter is satisfied, in particular, if g_0 has a representative g which is continuously differentiable with bounded derivative.

Thus, choosing a function $g_0 \in C_c(\mathbf{R})$ which is not of bounded variation we obtain an example of an operator B_0 which does not satisfy the Miyadera condition (1.2), but (1.10) is satisfied, i.e., B_0 has an infinitesimally A-small extension, according to Example 3.4.

3.6 Example. We explore conditions on $g_0 \in L^\infty(\mathbf{R})$ under which one can obtain an extension B satisfying (2.1) (which is unique by Proposition 2.1). Let g be a representative of g_0, and let B be the associated extension (3.1) of B_0. g can be identified with an element in X^* by setting $\langle \mu, g\rangle = \int_{\mathbf{R}} g(a)\mu(da)$ for $\mu \in \mathcal{M}(\mathbf{R})$. Then

$$((\lambda - A)^{-1})^* g(a) = \int_0^\infty e^{-\lambda t} g(a+t)\, dt$$

and

$$\begin{aligned} B(\lambda - A)^{-1}\mu &= \langle A(\lambda - A)^{-1}\mu, g\rangle = \langle \mu, \lambda((\lambda - A)^{-1})^* g - g\rangle \\ &= \int_{\mathbf{R}} \left(\int_0^\infty e^{-s} g(a + \lambda^{-1}s)\, ds - g(a)\right) \mu(da). \end{aligned} \tag{3.5}$$

So we see that B_0 has an extension B with (2.1) if and only if

$$g(a) = \lim_{h \to 0+} \int_0^\infty e^{-s} g_0(a + hs)\, ds$$

exists for all $a \in \mathbf{R}$. The extension B is associated with this particular g is uniquely determined by (2.1). This holds in particular if g_0 has a right-continuous representative g.

If g_0 is continuous, but not uniformly continuous, then (1.10) is not satisfied, according to Example 3.4, but (2.1) holds.

It is not difficult to construct a measurable function $g_0 \colon \mathbf{R} \to \{0, 1\}$, $g_0(a) = 0$ $(a \le 0)$, satisfying

$$\liminf_{h \to 0+} \int_0^\infty e^{-s} g_0(hs)\, ds = 0, \qquad \limsup_{h \to 0+} \int_0^\infty e^{-s} g_0(hs)\, ds = 1.$$

This function provides an example of an operator B_0 satisfying (1.1) but for which no extension satisfying (2.1) exists.

However, if g_0 is such that (1.2) holds, then by Example 3.5, we obtain a representative g of uniformly locally bounded variation, which can be assumed to be right continuous. This implies that for the type of example considered here one has that (1.2) implies that B_0 has a uniquely determined A-bounded extension B satisfying (2.1).

3.7 Example. We modify our perturbations in order to illustrate that (1.1) with $\gamma < 1$ is not necessary for the perturbed operator to be also a Hille-Yosida operator. We define

$$C_0 f = \bigg(\int_{\mathbf{R}} g_0(a) \check{A} f(a)\, da \bigg) \delta_0 = (B_0 f)\delta_0 \quad (f \in D),$$

with a function $g_0 \in L^\infty(\mathbf{R})$ possessing a monotone increasing representative g. In the light of Example 3.6, we choose the representative to be right continuous. For $f \in D(A)$, by (2.1) and (3.5),

$$\begin{aligned} Bf &= \lim_{\lambda\to\infty} B(\lambda - A)^{-1}(\lambda - A) f = \lim_{\lambda\to\infty} \lambda B(\lambda - A)^{-1} f \\ &= \lim_{\lambda\to\infty} \lambda \int_{\mathbf{R}} \bigg(\int_0^\infty e^{-s} g(a + \lambda^{-1}s)\, ds - g(a) \bigg) f(a)\, da. \end{aligned}$$

This shows that B and the extension $C = B(\cdot)\delta_0$ of C_0 are positive. A positive perturbation $A+C$ of a resolvent-positive Hille-Yosida operator on an abstract L-space like $\mathcal{M}(\mathbf{R})$ is also a resolvent-positive Hille-Yosida operator, if (cf. [**10**]) and only if (cf. [**12**]) there is some $\lambda > 0$ such that the spectral radius of $C(\lambda - A)^{-1}$ is strictly less than 1. The latter is the case, indeed, as $\|(C(\lambda - A)^{-1})^2\| \le \|B(\lambda - A)^{-1}\delta_0\|\, \|B(\lambda - A)^{-1}\| \to 0$ as $\lambda \to \infty$. Choosing $g = \xi \mathbf{1}_{[1,\infty)}$ with $\xi \ge 1$, we see that (1.1) and (1.2) are satisfied, but only with $\gamma = \xi \ge 1$.

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