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# COMPACT ENDOMORPHISMS OF INFINITELY DIFFERENTIABLE LIPSCHITZ ALGEBRAS

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ABSTRACT. Let $X$ be a perfect compact plane set and $0 < \alpha \leq 1$. The Lipschitz algebra of order $\alpha$, $\mathrm{Lip}\,(X,\alpha)$ is the algebra of all complex-valued functions $f$ on $X$ for which

$$p_\alpha(f) = \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} : z, w \in X, z \neq w\right\} < \infty.$$

Denote by $\mathrm{Lip}^{\,\infty}(X,\alpha)$ the algebra of functions $f$ on $X$ whose derivatives of all orders exist and $f^{(n)} \in \mathrm{Lip}\,(X,\alpha)$ for all $n$. Let $(M_n)$ be a sequence of positive numbers satisfying $M_0 = 1$ and $M_{n+m}/M_n M_m \geq \binom{n+m}{n}$ for all nonnegative integers $m$, $n$, and let

$$\mathrm{Lip}\,(X,M,\alpha) = \left\{f \in \mathrm{Lip}^{\,\infty}(X,\alpha) : \|f\| = \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_\alpha}{M_k} < \infty\right\},$$

where $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$. In this paper we study the endomorphisms of this kind of Lipschitz algebra. When $\mathrm{Lip}\,(X,M,\alpha)$ is a natural Banach function algebra, every nonzero endomorphism $T$ of $\mathrm{Lip}\,(X,M,\alpha)$ has the form $Tf = f \circ \varphi$, for some self-map $\varphi$ of $X$. First we give some sufficient conditions for $\varphi$ to induce an endomorphism of $\mathrm{Lip}\,(X,M,\alpha)$. Then we investigate necessary and sufficient conditions for these endomorphisms to be compact. Finally, we determine the spectra of compact endomorphisms of these algebras.

**1. Introduction and preliminaries.** In this note we investigate endomorphisms of a class of Lipschitz algebras of infinitely differentiable functions. Let $X$ be a perfect compact plane set and $0 < \alpha \leq 1$.

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The Lipschitz algebra $\mathrm{Lip}\,(X,\alpha)$, of order $\alpha$, is the algebra of all complex-valued functions $f$ on $X$ for which

$$p_\alpha(f) = \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} : z, w \in X, z \neq w\right\} < \infty.$$

The algebra $\mathrm{Lip}\,(X,\alpha)$ is a Banach function algebra on $X$, if equipped with the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup_{x\in X}|f(x)|$. It is interesting to note that $\mathrm{Lip}\,(X,1) \subseteq \mathrm{Lip}\,(X,\alpha)$.

The complex-valued function $f$ on $X$ is called complex-differentiable on $X$ if at each point $z_0 \in X$ the limit

$$f'(z_0) = \lim_{\substack{z\to z_0\\ z\in X}} \frac{f(z)-f(z_0)}{z-z_0}$$

exists. The algebra of functions $f$ on $X$ whose derivatives of all orders exist and $f^{(n)} \in \mathrm{Lip}\,(X,\alpha)$ for all $n$, is denoted by $\mathrm{Lip}^{\,\infty}(X,\alpha)$.

We now introduce certain subalgebras of $\mathrm{Lip}^{\,\infty}(X,\alpha)$. Let $M = (M_n)$ be a sequence of positive numbers satisfying $M_0 = 1$ and $M_{n+m}/M_nM_m \geq \binom{n+m}{n}$, where $m$ and $n$ are nonnegative integers. Let

$$\mathrm{Lip}\,(X,M,\alpha) = \left\{f \in \mathrm{Lip}^{\,\infty}(X,\alpha) : \|f\| = \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_\alpha}{M_k} < \infty\right\}.$$

With pointwise addition and multiplication, $\mathrm{Lip}\,(X,M,\alpha)$ is a commutative normed algebra which is not necessarily complete. We call such algebras infinitely differentiable Lipschitz algebras which were first studied in [**8, 11**]. The algebras $\mathrm{Lip}\,(X,M,\alpha)$ are similar to Dales-Davie algebras $D(X,M)$, the algebras of infinitely differentiable functions $f$ on $X$, such that $\|f\| = \sum_{k=0}^{\infty}\|f^{(k)}\|_X/M_k < \infty$, [**3**].

Let $D^1(X)$ be the algebra of continuously differentiable functions on $X$. Then $D^1(X)$ is a normed algebra under the norm $\|f\| = \|f\|_X + \|f'\|_X$ which may not be complete in general.

We now introduce the type of compact sets which we shall consider.

**Definition 1.1.** Let $X$ be a compact plane set which is connected by rectifiable arcs, and let $\delta(z,w)$ be the geodesic metric on $X$, the infimum of the lengths of the arcs joining $z$ and $w$.

(i) $X$ is called regular if for each $z_0 \in X$ there exists a constant $C$ such that for all $z \in X$, $\delta(z, z_0) \leq C|z - z_0|$.

(ii) $X$ is called uniformly regular if there exists a constant $C$ such that, for all $z, w \in X$, $\delta(z, w) \leq C|z - w|$.

As it was proved in [**11**], if $X$ is a finite union of regular sets, then for each $z_0 \in X$ there exists a constant $C$ such that for every $z \in X$ and $f \in D^1(X)$,

$$|f(z) - f(z_0)| \leq C|z - z_0|(\|f\|_X + \|f'\|_X).$$

This inequality is equivalent to the completeness of $D^1(X)$, and the completeness of $D^1(X)$ implies that $\mathrm{Lip}\,(X, M, \alpha)$ is a Banach function algebra on $X$ for every weight sequence $M = (M_n)$ and any $0 < \alpha \leq 1$.

We remark that, for certain compact plane sets $X$ and certain weight sequences $M = (M_n)$, the algebras $\mathrm{Lip}\,(X, M, \alpha)$ are natural, i.e., their maximal ideal spaces are $X$. Now we consider two important subalgebras. We denote by $\mathrm{Lip}_P(X, M, \alpha)$ and $\mathrm{Lip}_R(X, M, \alpha)$ the closed subalgebras of $\mathrm{Lip}\,(X, M, \alpha)$ generated respectively by the polynomials in $z$, and by the rational functions with poles off $X$ which belong to $\mathrm{Lip}\,(X, M, \alpha)$. When $M = (M_n)$ is a nonanalytic sequence, i.e., $\lim_{n\to\infty}(n!/M_n)^{1/n} = 0$, the algebra $\mathrm{Lip}\,(X, M, \alpha)$ includes all the rational functions with poles off $X$. Thus, $\mathrm{Lip}_R(X, M, \alpha)$ is a natural Banach function algebra on $X$, for nonanalytic sequences $M = (M_n)$ and every $0 < \alpha \leq 1$, when $D^1(X)$ is complete. For example, the algebra $\mathrm{Lip}_R(X, M, \alpha)$ is natural when $X$ is a circle, a closed annulus, a closed disc, a compact interval, or certain star-shaped regions. Moreover, the maximal ideal space of $\mathrm{Lip}_P(X, M, \alpha)$ is $\widehat{X}$, the polynomial convex hull of $X$, and $\mathrm{Lip}_R(X, M, \alpha) = \mathrm{Lip}_P(X, M, \alpha)$ if and only if $X = \widehat{X}$. In the case of the circle $X = \{z \in C : |z - z_0| = R\}$ or the annulus $X = \{z \in C : r \leq |z - z_0| \leq R\}$, where $0 < r < R$, the rational functions are dense in $\mathrm{Lip}\,(X, M, \alpha)$ for any weight sequence $M = (M_n)$ and any $0 < \alpha < 1$, that is, $\mathrm{Lip}\,(X, M, \alpha) = \mathrm{Lip}_R(X, M, \alpha)$ [**8**, Theorems 3 and 4]. By assuming $r \to 0$, we can conclude that, whenever $X$ is a closed disc, polynomials are dense in $\mathrm{Lip}\,(X, M, \alpha)$, that is, $\mathrm{Lip}\,(X, M, \alpha) = \mathrm{Lip}_P(X, M, \alpha)$ [**8**, Corollary 1]. Also, whenever $X$ is a regular star-shaped region for which there exists $z_0 \in X$ such that the segment $[z_0, z)$ is contained in the interior of $X$ for all

$z \in X$, then we have $\mathrm{Lip}\,(X, M, \alpha) = \mathrm{Lip}\,_P(X, M, \alpha)$ for any nonanalytic weight sequence $M$ and $0 < \alpha < 1$ [**8**, Theorem 5]. Finally, as it was proved in [**12**], when $X$ is a compact interval, the polynomials are dense in $\mathrm{Lip}\,(X, M, \alpha)$ for nonanalytic weights $M$ and $0 < \alpha < 1$. Thus, for certain types of compact plane sets $X$, $0 < \alpha < 1$ and nonanalytic weights $M = (M_n)$, the algebras $\mathrm{Lip}\,(X, M, \alpha)$ are natural.

In this note we study endomorphisms of $\mathrm{Lip}\,(X, M, \alpha)$ and investigate necessary and sufficient conditions for these endomorphisms to be compact. The endomorphisms of the algebras $D(X, M)$ have already been studied by Feinstein and Kamowitz in [**4, 5, 6, 10**]. However, some parts of the proofs presented here are similar to their works.

We sometimes require the following condition on $X$ which is called the $(*)$-condition.

$(*)$ *There exists a constant $C$ such that for every $z, w \in X$ and $f \in D^1(X)$,*

$$|f(z) - f(w)| \leq C|z - w|(\|f\|_X + \|f'\|_X).$$

For example, every uniformly regular set satisfies the $(*)$-condition. The completeness of $D^1(X)$ is also concluded from the $(*)$-condition.

In general, if $T$ is a unital endomorphism of a unital commutative semi-simple Banach algebra $B$ with the maximal ideal space $\mathcal{M}(B)$, then there exists a continuous map $\varphi : \mathcal{M}(B) \to \mathcal{M}(B)$ such that $\widehat{Tf} = \hat{f} \circ \varphi$ for all $f \in B$. In fact, $\varphi$ is equal to the adjoint $T^*$ restricted to $\mathcal{M}(B)$. In this case we say $\varphi$ induces $T$. If a Banach function algebra $B$ on a compact Hausdorff space $X$ is natural, then every nonzero endomorphism $T$ of $B$ has the form $Tf = f \circ \varphi$ for a self-map $\varphi$ of $X$. Hence, when the Banach function algebra $\mathrm{Lip}\,(X, M, \alpha)$ is natural, every nonzero endomorphism $T$ of $\mathrm{Lip}\,(X, M, \alpha)$ has the form $Tf = f \circ \varphi$ for some continuous self-map $\varphi$ of $X$. This leads us to ask when a map $\varphi : X \to X$ induces an endomorphism of $\mathrm{Lip}\,(X, M, \alpha)$. In other words, under what conditions $\varphi$ satisfies $f \circ \varphi \in \mathrm{Lip}\,(X, M, \alpha)$ whenever $f \in \mathrm{Lip}\,(X, M, \alpha)$. Since $\mathrm{Lip}\,(X, M, \alpha)$ contains the coordinate map $z$, if $\varphi$ induces an endomorphism then $\varphi \in \mathrm{Lip}\,(X, M, \alpha)$.

We now impose an additional condition on an infinitely differentiable map $\varphi$:

We call $\varphi$ *analytic* if

$$\sup_n \left(\frac{\|\varphi^{(n)}\|_X}{n!}\right)^{1/n} < \infty.$$

*Remark* 1.2. The above condition on $\varphi$ is equivalent to the analyticity of $\varphi$ in a neighborhood of $X$. Using Cauchy's estimate, one can show that every analytic function $\varphi$ on a neighborhood of $X$ satisfies the following condition

$$\sup_n \left(\frac{\|\varphi^{(n)}\|_\alpha}{n!}\right)^{1/n} < \infty.$$

We say that $\varphi$ is Lipschitz analytic if it satisfies this condition. Indeed, Lipschitz analyticity and analyticity of $\varphi$ are equivalent to the analyticity of $\varphi$ in a neighborhood of $X$. Moreover, by a direct computation, one can also show that Lipschitz analyticity and analyticity of $\varphi$ are equivalent, when $X$ satisfies the $(*)$-condition. Moreover, for nonanalytic weights $M = (M_n)$, every (Lipschitz) analytic map $\varphi$ is in $\operatorname{Lip}(X, M, \alpha)$.

If $\operatorname{Lip}(X, M, \alpha)$ is a natural Banach function algebra on $X$ and $\varphi(X) \subseteq \operatorname{int} X$, then by the functional calculus theorem, $f \circ \varphi \in \operatorname{Lip}(X, M, \alpha)$ for all $f \in \operatorname{Lip}(X, M, \alpha)$. So, in this case, $\varphi$ induces an endomorphism of $\operatorname{Lip}(X, M, \alpha)$, without any additional conditions. In this paper we determine conditions for $\varphi$ to induce an endomorphism of $\operatorname{Lip}(X, M, \alpha)$ in more general cases. We also investigate necessary and sufficient conditions for the induced endomorphism to be compact. We then determine the spectra of compact endomorphisms of $\operatorname{Lip}(X, M, \alpha)$.

**2. Endomorphisms of** $\operatorname{Lip}(X, M, \alpha)$**.** We will frequently use the following equality for higher derivatives of composite functions which is known as Faá di Bruno's formula,

$$(f \circ \varphi)^{(n)} = \sum_{m=0}^{n} f^{(m)}(\varphi(z)) \sum_a \frac{n!}{a_1! a_2! \cdots a_n!} \prod_{k=1}^{n} \left(\frac{\varphi^{(k)}(z)}{k!}\right)^{a_k}$$

where, here and henceforth, the inner sum $\sum_a$ is taken over nonnegative integers $a_1, a_2, \ldots, a_n$ satisfying $a_1 + a_2 + \cdots + a_n = m$ and $a_1 + 2a_2 + \cdots + na_n = n$.

In the next theorem we give some sufficient conditions for $\varphi$ to induce an endomorphism of $\operatorname{Lip}(X, M, \alpha)$. We need the following important lemma which is easy to verify and will be used in the sequel frequently.

**Lemma 2.1** [**2**, Lemma 1.5]. *Let $K, X$ be two compact plane sets and $K \subseteq \operatorname{int} X$. There exists a finite union of uniformly regular sets in $\operatorname{int} X$ containing $K$, namely $Y$ and a constant $C$ such that for every analytic function $f$ in $\operatorname{int} X$ and any $z, w \in K$, $|f(z) - f(w)| \leq C|z - w|(\|f\|_Y + \|f'\|_Y)$.*

**Theorem 2.2.** *Let $X$ be a perfect compact plane set, $0 < \alpha \leq 1$, and let $M = (M_n)$ be a nonanalytic weight sequence. Then an analytic self-map $\varphi$ of $X$ induces an endomorphism of $\operatorname{Lip}(X, M, \alpha)$ if any one of the following conditions is satisfied:*

(i) *$X$ satisfies the $(*)$-condition and $\|\varphi'\|_\alpha < 1$.*

(ii) *$\varphi \in \operatorname{Lip}(X, 1)$ and $\|\varphi'\|_\alpha < 1$.*

(iii) *$X$ has nonempty interior and $\varphi(X) \subseteq \operatorname{int} X$.*

*Proof.* First we show that $f \circ \varphi \in \operatorname{Lip}^\infty(X, \alpha)$, for all $f \in \operatorname{Lip}(X, M, \alpha)$. By the $(*)$-condition, every infinitely differentiable function is in $\operatorname{Lip}^\infty(X, \alpha)$. If (ii) is satisfied, then for every $f \in \operatorname{Lip}(X, \alpha)$ and for all $z, w \in X$ with $\varphi(z) \neq \varphi(w)$, we have

$$\begin{aligned} \frac{|f \circ \varphi(z) - f \circ \varphi(w)|}{|z - w|^\alpha} &= \frac{|f(\varphi(z)) - f(\varphi(w))|}{|\varphi(z) - \varphi(w)|^\alpha} \left| \frac{\varphi(z) - \varphi(w)}{z - w} \right|^\alpha \\ &\leq p_\alpha(f)(p_1(\varphi))^\alpha, \end{aligned}$$

whence $f \circ \varphi \in \operatorname{Lip}(X, \alpha)$. Thus, $f^{(m)} \circ \varphi \in \operatorname{Lip}(X, \alpha)$ for every $f \in \operatorname{Lip}^\infty(X, \alpha)$ and for all $m$. Using Faá di Bruno's formula, one can show that $(f \circ \varphi)^{(n)}$ is in $\operatorname{Lip}(X, \alpha)$, for every $f \in \operatorname{Lip}^\infty(X, \alpha)$ and for all $n$, that is, $f \circ \varphi \in \operatorname{Lip}^\infty(X, \alpha)$.

By condition (iii), using Lemma 2.1, we obtain a finite union of uniformly regular sets $Y$ in $\operatorname{int} X$ containing $\varphi(X)$ and a constant

$C$ such that for every $f \in \operatorname{Lip}^{\infty}(X,\alpha)$ and for all $z,w \in X$ with $\varphi(z) \neq \varphi(w)$, we have

$$\begin{aligned}\frac{|f\circ\varphi(z) - f\circ\varphi(w)|}{|z-w|^{\alpha}} &= \frac{|f(\varphi(z)) - f(\varphi(w))|}{|\varphi(z)-\varphi(w)|}\,\frac{|\varphi(z)-\varphi(w)|}{|z-w|^{\alpha}}\\ &\leq C(\|f\|_Y + \|f'\|_Y)p_{\alpha}(\varphi).\end{aligned}$$

Again, using Faá di Bruno's formula, one can show that $f\circ\varphi \in \operatorname{Lip}^{\infty}(X,\alpha)$ for every $f \in \operatorname{Lip}^{\infty}(X,\alpha)$.

Now we show that $f\circ\varphi \in \operatorname{Lip}(X,M,\alpha)$ for all $f \in \operatorname{Lip}(X,M,\alpha)$. By using Faá di Bruno's formula we show that the series

$$\sum_{n=0}^{\infty} \|(f\circ\varphi)^{(n)}\|_{\alpha}/M_n$$

converges.

$$\begin{aligned}(2.1)\quad &\sum_{n=0}^{\infty} \frac{\|(f\circ\varphi)^{(n)}\|_{\alpha}}{M_n}\\ &\leq \sum_{n=0}^{\infty} \frac{1}{M_n} \sum_{m=0}^{n} \|f^{(m)}\circ\varphi\|_{\alpha} \sum_{a} \frac{n!}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_{\alpha}}{k!}\right)^{a_k}\\ &= \sum_{m=0}^{\infty} \|f^{(m)}\circ\varphi\|_{\alpha} \sum_{n=m}^{\infty} \frac{1}{M_n} \sum_{a} \frac{n!}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_{\alpha}}{k!}\right)^{a_k}.\end{aligned}$$

As mentioned in Remark 1.2, the analytic self-map $\varphi$ is Lipschitz analytic. Put $\rho = \limsup_n (\|\varphi^{(n)}\|_{\alpha}/n!)^{1/n} < \infty$, and define

$$h(\lambda) = \sum_{k=1}^{\infty} \frac{\|\varphi^{(k)}\|_{\alpha}}{k!}\lambda^{k-1}, \quad |\lambda| < 1/\rho.$$

The function $h$ is analytic in $|\lambda| < 1/\rho$ and $h(\varepsilon) = \sum_{k=1}^{\infty} (\|\varphi^{(k)}\|_{\alpha}/\varepsilon k!)\varepsilon^k < \infty$ for every $\varepsilon$, $0 < \varepsilon < 1/\rho$.

Since $\lim_{n\to\infty}(n!/M_n)^{1/n} = 0$, for every $0 < \varepsilon < 1/\rho$ there is a constant $B > 0$ such that $(M_m/m!)(n!/M_n) < ((n-m)!/M_{n-m}) < B\varepsilon^{n-m}$, for $n \geq m$.

Therefore, it follows from inequality (2.1) that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\|(f\circ\varphi)^{(n)}\|_\alpha}{M_n}\\
&\leq\sum_{m=0}^{\infty}\|f^{(m)}\circ\varphi\|_\alpha\frac{m!}{M_m}\sum_{n=m}^{\infty}\frac{M_m}{m!}\,\frac{n!}{M_n}\sum_a\frac{1}{a_1!a_2!\cdots a_n!}\prod_{k=1}^{n}\left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k}\\
&\leq B\sum_{m=0}^{\infty}\frac{\|f^{(m)}\circ\varphi\|_\alpha}{M_m}m!\sum_{n=m}^{\infty}\varepsilon^{n-m}\sum_a\frac{1}{a_1!a_2!\cdots a_n!}\prod_{k=1}^{n}\left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k}\\
&= B\sum_{m=0}^{\infty}\frac{\|f^{(m)}\circ\varphi\|_\alpha}{M_m}m!\sum_{n=m}^{\infty}\frac{\varepsilon^n}{n!}\sum_a\frac{n!}{a_1!a_2!\cdots a_n!}\prod_{k=1}^{n}\left(\frac{\|\varphi^{(k)}\|_\alpha}{\varepsilon k!}\right)^{a_k}.
\end{aligned}$$

By applying formula B3, [**1**, page 823], we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\|(f\circ\varphi)^{(n)}\|_\alpha}{M_n}&\leq B\sum_{m=0}^{\infty}\frac{\|f^{(m)}\circ\varphi\|_\alpha}{M_m}\left(\sum_{k=1}^{\infty}\frac{\|\varphi^{(k)}\|_\alpha}{\varepsilon k!}\varepsilon^k\right)^m\\
&= B\sum_{m=0}^{\infty}\frac{\|f^{(m)}\circ\varphi\|_\alpha}{M_m}(h(\varepsilon))^m.
\end{aligned}\tag{2.2}$$

By either of the conditions (i) or (ii) we have $h(0)=\|\varphi'\|_\alpha<1$, so we can choose $0<\varepsilon<1/\rho$ such that $h(\varepsilon)<1$. Hence,

$$\sum_{n=0}^{\infty}\frac{\|(f\circ\varphi)^{(n)}\|_\alpha}{M_n}\leq B\sum_{m=0}^{\infty}\frac{\|f^{(m)}\circ\varphi\|_\alpha}{M_m}\leq BB_1\sum_{m=0}^{\infty}\frac{\|f^{(m)}\|_\alpha}{M_m}<\infty,$$

where $B_1=\max\{1,C^\alpha(\|\varphi\|_X+\|\varphi'\|_X)^\alpha\}$ if (i) is satisfied ($C$ is obtained from the $(*)$-condition), and $B_1=\max\{1,(p_1(\varphi))^\alpha\}$ whenever condition (ii) is satisfied.

With assumption (iii), using Lemma 2.1, we obtain a compact set $Y$ with $\varphi(X)\subseteq Y\subseteq \operatorname{int}X$ and a constant $C$ such that for all $z,w\in X$ with $\varphi(z)\neq\varphi(w)$, we have

$$\begin{aligned}
\frac{|f^{(m)}(\varphi(z))-f^{(m)}(\varphi(w))|}{|z-w|^\alpha}&=\frac{|f^{(m)}(\varphi(z))-f^{(m)}(\varphi(w))|}{|\varphi(z)-\varphi(w)|}\frac{|\varphi(z)-\varphi(w)|}{|z-w|^\alpha}\\
&\leq Cp_\alpha(\varphi)(\|f^{(m)}\|_Y+\|f^{(m+1)}\|_Y).
\end{aligned}$$

Whence,

$$p_\alpha(f^{(m)} \circ \varphi) \leq C p_\alpha(\varphi)(\|f^{(m)}\|_Y + \|f^{(m+1)}\|_Y).$$

Let $\Gamma$ be a finite number of closed paths (polygons) in $\operatorname{int} X \setminus Y$ with length $l$ that surrounds $Y$, and let $\delta = \operatorname{dist}(Y, \Gamma) = \inf\{|z - \zeta| : z \in Y, \zeta \in \Gamma\}$. Since every $f \in \operatorname{Lip}(X, M, \alpha)$ is analytic in $\operatorname{int} X$, it follows from Cauchy's estimate that $\|f^{(m)}\|_Y \leq (1/2\pi) l (m!/\delta^{m+1}) \|f\|_X$. Therefore,

$$\begin{aligned}
\|f^{(m)} \circ \varphi\|_\alpha &= \|f^{(m)} \circ \varphi\|_X + p_\alpha(f^{(m)} \circ \varphi) \\
&\leq \|f^{(m)}\|_Y + C p_\alpha(\varphi)(\|f^{(m)}\|_Y + \|f^{(m+1)}\|_Y) \\
&\leq (1 + C p_\alpha(\varphi))(\|f^{(m)}\|_Y + \|f^{(m+1)}\|_Y) \\
&\leq B_2 \|f\|_X \left( \frac{m!}{\delta^{m+1}} + \frac{(m+1)!}{\delta^{m+2}} \right),
\end{aligned}$$

where $B_2 = (1/2\pi) l (1 + C p_\alpha(\varphi))$.

Now from inequality (2.2) we obtain

$$\begin{aligned}
(2.3) \quad \sum_{n=0}^{\infty} \frac{\|(f \circ \varphi)^{(n)}\|_\alpha}{M_n} & \\
&\leq B B_2 \|f\|_X \sum_{m=0}^{\infty} \left( \frac{m!}{\delta^{m+1}} + \frac{(m+1)!}{\delta^{m+2}} \right) \frac{(h(\varepsilon))^m}{M_m} \\
&\leq B B_2 \|f\|_X \sum_{m=0}^{\infty} \frac{m!}{M_m} \frac{(h(\varepsilon))^m}{\delta^m} \left( \frac{1}{\delta} + \frac{m+1}{\delta^2} \right) < \infty,
\end{aligned}$$

since $\lim_{m\to\infty} (m!/M_m)^{1/m} = 0$. Therefore, the theorem follows. □

We note that if $X$ satisfies the $(*)$-condition, $\operatorname{Lip}(X, M, \alpha)$ is complete. However, under conditions (ii) and (iii), these algebras are not necessarily complete.

The following example which is similar to an example in [**5**] shows that if $\varphi(X) \nsubseteq \operatorname{int} X$ and $\|\varphi'\|_1 > 1$, then $\varphi$ need not induce an endomorphism of $\operatorname{Lip}(X, M, 1)$.

**Example 2.3.** Let $M = (M_n)$ be a weight sequence such that $n^2 M_n/M_{n+1} \to \infty$ as $n \to \infty$. Then the map $\varphi(z) = 1/2(1+z^2)$ on the closed unit disc $\overline{\mathbf{D}}$ does not induce an endomorphism of $\mathrm{Lip}\,(\overline{\mathbf{D}}, M, 1)$.

*Proof.* Let $B > 0$. Choose an integer $N > 1$ such that $n^2 M_{n-1}/M_n > B$, for all $n \geq N$. Choose $A \geq 1$ such that $\sum_{n=0}^{N-1} A^n/M_n < (1/2)(A^N/M_N)$. Then we have $\sum_{n=N}^{\infty} A^n/M_n > (1/2)\sum_{n=0}^{\infty} A^n/M_n$.

We now consider the function $f_A(z) = \exp(A(z-1))$. Then $\|f_A\|_1 = \|f_A\|_X + p_1(f_A) \leq 2A$ and, for each $n \geq 0$, we have $f_A^{(n)}(z) = A^n f_A(z)$. Hence, $\|f_A^{(n)}\|_1 = A^n \|f_A\|_1$ and

$$\|f_A\|_{\mathrm{Lip}\,(\overline{\mathbf{D}},M,1)} = \|f_A\|_1 \sum_{n=0}^{\infty} \frac{A^n}{M_n} \leq 2A \sum_{n=0}^{\infty} \frac{A^n}{M_n}. \tag{2.4}$$

Using Faá di Bruno's formula, we get

$$\begin{aligned}
(f_A \circ \varphi)^{(n)}(z) &= \sum_{m=0}^{n} f_A^{(m)}(\varphi(z)) \sum_{a} \frac{n!}{a_1! a_2! 2^{a_2}} (\varphi'(z))^{a_1} \\
&= \sum_{k=0}^{l} f_A^{(n-k)}(\varphi(z)) \frac{n!}{(n-2k)!k!2^k} (\varphi'(z))^{n-2k} \\
&= f_A(\varphi(z)) \sum_{k=0}^{l} A^{n-k} \frac{n!}{(n-2k)!k!2^k} z^{n-2k},
\end{aligned}$$

where $l = n/2$ if $n$ is even, and $l = (n-1)/2$ if $n$ is odd. Using the mean value theorem, we obtain an increasing sequence $\{t_m\}$ with $t_m > m$ for each $m$, such that

$$\begin{aligned}
&p_1((f_A \circ \varphi)^{(n-1)}) \\
&\qquad \geq (f_A \circ \varphi)^{(n)}\left(1 - \frac{1}{t_m}\right) \\
&\qquad = f_A\left(\varphi\left(1 - \frac{1}{t_m}\right)\right) \sum_{k=0}^{l} A^{n-k} \frac{n!}{(n-2k)!k!2^k} \left(1 - \frac{1}{t_m}\right)^{n-2k},
\end{aligned}$$

for each $n > 1$ and for all $m$. Letting $m \to \infty$, we get

$$\begin{aligned}
p_1((f_A \circ \varphi)^{(n-1)}) &\geq f_A(\varphi(1)) \sum_{k=0}^{l} A^{n-k} \frac{n!}{(n-2k)!k!2^k} \\
&= \sum_{k=0}^{l} \frac{n!A^{n-k}}{(n-2k)!k!2^k} \\
&> \frac{n!A^{n-1}}{2(n-2)!} > \frac{(n-1)^2}{2} A^{n-1}, \quad n > 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|f_A \circ \varphi\|_{\mathrm{Lip}\,(\overline{\mathbf{D}},M,1)} &= \sum_{n=0}^{\infty} \frac{\|(f_A \circ \varphi)^{(n)}\|_1}{M_n} \geq \sum_{n=N}^{\infty} \frac{p_1((f_A \circ \varphi)^{(n)})}{M_n} \\
&\geq \frac{1}{2} \sum_{n=N}^{\infty} \frac{n^2 A^n}{M_n} = \frac{A}{2} \sum_{n=N}^{\infty} \frac{n^2 M_{n-1}}{M_n} \frac{A^{n-1}}{M_{n-1}} \\
&> B\frac{A}{2} \sum_{n=N}^{\infty} \frac{A^{n-1}}{M_{n-1}} > \frac{1}{4} BA \sum_{n=0}^{\infty} \frac{A^n}{M_n} \\
&\geq \frac{1}{8} B \|f_A\|_{\mathrm{Lip}\,(\overline{\mathbf{D}},M,1)}.
\end{aligned}$$

In the last inequality, we have used inequality (2.4).

Since $B > 0$ was arbitrary this shows that $\varphi$ cannot induce a bounded endomorphism of $\mathrm{Lip}\,(\overline{\mathbf{D}}, M, 1)$ and so $\varphi$ does not induce an endomorphism. □

By imposing an extra condition on the sequence $M = (M_n)$, a slightly modified argument of Theorem 2.2 (i) and (ii) shows that $\|\varphi'\|_\alpha \leq 1$ is sufficient for $\varphi$ to induce an endomorphism of $\mathrm{Lip}\,(X, M, \alpha)$.

**Theorem 2.4.** *Let $X$ be a perfect compact plane set, $0 < \alpha \leq 1$, and let $M = (M_n)$ be a nonanalytic weight sequence satisfying $\sup_n (n^2 M_{n-1})/M_n < \infty$. Let $\varphi$ be a nonconstant analytic self-map of $X$ with $\|\varphi'\|_\alpha \leq 1$. Then $\varphi$ induces an endomorphism of* $\mathrm{Lip}\,(X, M, \alpha)$ *if either* (i) *$X$ satisfies the* $(*)$*-condition, or* (ii) $\varphi \in \mathrm{Lip}\,(X, 1)$.

*Proof.* As mentioned in the proof of Theorem 2.2, $f\circ\varphi \in \mathrm{Lip}^{\infty}(X,\alpha)$, for all $f \in \mathrm{Lip}\,(X, M, \alpha)$. Let $K = \sup_n (n^2 M_{n-1})/M_n < \infty$. Then $(M_m/m!)(n!/M_n) \le (K/m)^{n-m}$ for $m \le n$.

By Remark 1.2, the analytic self-map $\varphi$ is Lipschitz analytic. Hence, there exists $B > 0$ such that $\|\varphi^{(k)}\|_\alpha/k! \le B^k$ for all $k$. Since inequality (2.1) in the proof of Theorem 2.2 still holds, we have

$$\sum_{n=0}^{\infty} \frac{\|(f\circ\varphi)^{(n)}\|_\alpha}{M_n} \le \sum_{m=0}^{\infty} \|f^{(m)}\circ\varphi\|_\alpha \times \sum_{n=m}^{\infty} \frac{1}{M_n} \sum_a \frac{n!}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k} = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{m=0}^{[BK]} \|f^{(m)}\circ\varphi\|_\alpha \sum_{n=m}^{\infty} \frac{1}{M_n} \sum_a \frac{n!}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k},$$

and

$$\Sigma_2 = \sum_{m=[BK]+1}^{\infty} \|f^{(m)}\circ\varphi\|_\alpha \sum_{n=m}^{\infty} \frac{1}{M_n} \sum_a \frac{n!}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k}.$$

(The symbol $[x]$ denotes the greatest integer less than or equal to $x$).

First we estimate $\Sigma_1$.

$$\begin{aligned}\Sigma_1 &= \sum_{m=0}^{[BK]} \|f^{(m)}\circ\varphi\|_\alpha \sum_{n=m}^{\infty} \frac{1}{M_n} \sum_a \frac{n!}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k} \\ &\le \sum_{m=0}^{[BK]} \|f^{(m)}\circ\varphi\|_\alpha \sum_{n=m}^{\infty} \frac{1}{M_n} \sum_a \frac{n!}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} B^{ka_k} \\ &= \sum_{m=0}^{[BK]} \|f^{(m)}\circ\varphi\|_\alpha \sum_{n=m}^{\infty} \frac{B^n n!}{M_n} \sum_a \frac{1}{a_1!a_2!\cdots a_n!},\end{aligned}$$

since $\sum_{k=1}^{n} ka_k = n$.

The inner sum $\sum_a 1/(a_1!a_2!\cdots a_n!)$ is taken over the nonnegative integers $a_1, a_2, \ldots, a_n$ satisfying $a_1 + a_2 + \cdots + a_n = m$ and $a_1 + 2a_2 +$

$\cdots + na_n = n$. So the number of terms of this inner sum does not exceed $(m+1)^n$ so that $\sum_a 1/(a_1!a_2!\cdots a_n!) \le (m+1)^n$. Hence,

$$\Sigma_1 \le \sum_{m=0}^{[BK]} \|f^{(m)} \circ \varphi\|_\alpha \sum_{n=m}^{\infty} \frac{n!}{M_n} B^n (m+1)^n.$$

The inner sum is finite, since $\lim_{n\to\infty}((n!/M_n)B^n(m+1)^n)^{1/n} = 0$. So $\Sigma_1$ is finite.

We next show that $\Sigma_2$ is finite. The argument is similar to that of Theorem 3.1 in [**5**].

$$\begin{aligned}
\Sigma_2 = & \sum_{m=[BK]+1}^{\infty} \|f^{(m)} \circ \varphi\|_\alpha \sum_{n=m}^{\infty} \frac{1}{M_n} \sum_a \frac{n!}{a_1!a_2!\cdots a_n!} \\
& \times \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k} \\
= & \sum_{m=[BK]+1}^{\infty} \|f^{(m)} \circ \varphi\|_\alpha \frac{m!}{M_m} \sum_{n=m}^{\infty} \frac{M_m}{m!} \frac{n!}{M_n} \\
& \times \sum_a \frac{1}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k} \\
\le & \sum_{m=[BK]+1}^{\infty} \frac{\|f^{(m)} \circ \varphi\|_\alpha}{M_m} m! \sum_{n=m}^{\infty} (\frac{K}{m})^{n-m} \\
& \times \sum_a \frac{1}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_\alpha}{k!}\right)^{a_k} \\
= & \sum_{m=[BK]+1}^{\infty} \frac{\|f^{(m)} \circ \varphi\|_\alpha}{M_m} m! \sum_{n=m}^{\infty} \frac{1}{n!} \\
& \times \sum_a \frac{n!}{a_1!a_2!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{\|\varphi^{(k)}\|_\alpha K^{k-1}}{k!m^{k-1}}\right)^{a_k},
\end{aligned}$$

since $a_2 + 2a_3 + \cdots + (n-1)a_n = n - m$.

Again from formula B3 [**1**, page 823], we get

$$\Sigma_2 \le \sum_{m=[BK]+1}^{\infty} \frac{\|f^{(m)} \circ \varphi\|_\alpha}{M_m} \left(\sum_{k=1}^{\infty} \frac{\|\varphi^{(k)}\|_\alpha K^{k-1}}{k!m^{k-1}}\right)^m$$

$$\leq \sum_{m=[BK]+1}^{\infty} \frac{\|f^{(m)} \circ \varphi\|_\alpha}{M_m} \left( \|\varphi'\|_\alpha + \sum_{k=2}^{\infty} \frac{B^k K^{k-1}}{m^{k-1}} \right)^m$$

$$= \sum_{m=[BK]+1}^{\infty} \frac{\|f^{(m)} \circ \varphi\|_\alpha}{M_m} \|\varphi'\|_\alpha^m \left( 1 + \frac{1}{\|\varphi'\|_\alpha} \frac{B^2 K}{m - BK} \right)^m$$
$$= \sum_{m=[BK]+1}^{\infty} \frac{\|f^{(m)} \circ \varphi\|_\alpha}{M_m} \left( 1 + \frac{B^2 K}{\|\varphi'\|_\alpha (m - BK)} \right)^m,$$

since $\|\varphi'\|_\alpha \leq 1$. Let $B_2$ be an upper bound of

$$(1 + (B^2 K / \|\varphi'\|_\alpha (m - BK)))^m$$

in $m$. We also have $\|f^{(m)} \circ \varphi\|_\alpha \leq B_1 \|f^{(m)}\|_\alpha$, where $B_1 = \max\{1, C^\alpha (\|\varphi\|_X + \|\varphi'\|_X)^\alpha\}$ if condition (i) is satisfied ($C$ is obtained from the $(*)$-condition), and $B_1 = \max\{1, (p_1(\varphi))^\alpha\}$ whenever (ii) is satisfied. Therefore,

$$\Sigma_2 \leq B_1 B_2 \sum_{m=[BK]+1}^{\infty} \frac{\|f^{(m)}\|_\alpha}{M_m} \leq B_1 B_2 \|f\|_{\mathrm{Lip}\,(X,M,\alpha)} < \infty.$$

So $\varphi$ induces an endomorphism of Lip $(X, M, \alpha)$. □

We may state Theorems 2.2 and 2.4 for homomorphisms between two infinitely differentiable Lipschtiz algebras induced by $\varphi : X \to Y$. For example, we state the following theorem which is similar to Theorem 2.2 (i) and will be used in the sequel.

**Theorem 2.5.** *Let $X$ and $Y$ be perfect compact plane sets satisfying the $(*)$-condition, $0 < \alpha \leq 1$, and let $M = (M_n)$ be a nonanalytic weight sequence. Then, any analytic function $\varphi : X \to Y$ induces a homomorphism* Lip $(Y, M, \alpha) \to$ Lip $(X, M, \alpha)$ *if* $\|\varphi'\|_\alpha < 1$.

**3. Compactness of endomorphism when $X = [0, 1]$.** We recall that Lip $([0, 1], M, \alpha)$ is a Banach function algebra for all weights $M$ and

$0 < \alpha \leq 1$, and it is natural if the weight sequence $M$ is nonanalytic and $0 < \alpha < 1$.

In this section, first we show that for any $0 < a < 1$, $0 \leq b \leq 1 - a$, the operator $T : f(x) \to f(ax + b)$ is a compact endomorphism of $\mathrm{Lip}\,([0,1], M, \alpha)$ for all weights $M = (M_n)$, and $0 < \alpha \leq 1$. Then we show that the condition $\|\varphi'\|_\alpha < 1$ is sufficient for the analytic self-map $\varphi$ to induce a compact endomorphism of $\mathrm{Lip}\,([0,1], M, \alpha)$.

**Theorem 3.1.** *Let $M = (M_n)$ be a weight sequence and $0 < \alpha \leq 1$. Then $(Tf)(x) = f(ax + b)$, $x \in [0,1]$, is a compact endomorphism of $\mathrm{Lip}\,([0,1], M, \alpha)$ for any $0 < a < 1$ and $0 \leq b \leq 1 - a$.*

*Proof.* Let $I = [0,1]$. For every $k \geq 0$, we have $(Tf)^{(k)}(x) = a^k f^{(k)}(ax + b)$. So $\|(Tf)^{(k)}\|_I \leq a^k \|f^{(k)}\|_I$ and $p_\alpha((Tf)^{(k)}) \leq a^k a^\alpha p_\alpha(f^{(k)}) \leq a^k p_\alpha(f^{(k)})$, since $a < 1$. Hence,

$$\sum_{k=0}^{\infty} \frac{\|(Tf)^{(k)}\|_\alpha}{M_k} \leq \sum_{k=0}^{\infty} \frac{a^k \|f^{(k)}\|_\alpha}{M_k} \leq \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_\alpha}{M_k} < \infty.$$

So $Tf \in \mathrm{Lip}\,(I, M, \alpha)$, whence $T$ is an endomorphism of $\mathrm{Lip}\,(I, M, \alpha)$. (It also follows from Theorem 2.2 that $T$ is an endomorphism of $\mathrm{Lip}\,(I, M, \alpha)$ induced by $\varphi(x) = ax + b$, since $\|\varphi'\|_\alpha = a < 1$.)

For the compactness of $T$, let $\{f_n\}$ be a bounded sequence in $\mathrm{Lip}\,(I, M, \alpha)$, with $\|f_n\|_{\mathrm{Lip}\,(I,M,\alpha)} = \sum_{k=0}^{\infty} (\|f_n^{(k)}\|_\alpha / M_k) \leq 1$. Thus, $\|f_n^{(k)}\|_I \leq M_k$ and $p_\alpha(f_n^{(k)}) \leq M_k$ for every $k = 0, 1, 2, \ldots$ and any positive integer $n$. In particular, $\|f_n\|_I \leq M_0$ and $p_\alpha(f_n) \leq M_0$ for all $n$. So, $\{f_n\}$ is a bounded and equicontinuous sequence in $C(I)$. Hence by the Arzela-Ascoli theorem, there exists a subsequence $\{f_{0,n}\}$ of $\{f_n\}$ and a function $g \in C(I)$ with $\|f_{0,n} - g\|_I \to 0$ as $n \to \infty$. We also have $\|f_{0,n}^{(k)}\|_\alpha \leq M_k$ for every $k, n$. Similarly, by using the Arzela-Ascoli theorem for $\{f'_{0,n}\}$, we can find a subsequence $\{f_{1,n}\}$ of $\{f_{0,n}\}$ and $g_1 \in C(I)$ with $\|f'_{1,n} - g_1\|_I \to 0$ as $n \to \infty$. We also have $\|f_{1,n} - g\|_I \to 0$ as $n \to \infty$ and $\|f_{1,n}^{(k)}\|_\alpha \leq M_k$ for any $k$ and $n$, since $\{f_{1,n}\}$ is a subsequence of $\{f_{0,n}\}$. So $g$ is differentiable and $g' = g_1$. We then have $p_\alpha(f_{1,n} - g) \to 0$ as $n \to \infty$, since $p_\alpha(f_{1,n} - g) \leq \|f'_{1,n} - g'\|_I$. Therefore, $\|f_{1,n} - g\|_\alpha \to 0$, $\|f'_{1,n} - g'\|_I \to 0$ as $n \to \infty$ and $\|g\|_\alpha = \lim \|f_{1,n}\|_\alpha \leq M_0$.

By following an inductive argument, we conclude that $g \in C^\infty(I)$, and, moreover, we obtain a nested sequence $\{f_{k,n}\}$ of subsequences of $\{f_n\}$ with the properties $\|f_{k,n}^{(i)} - g^{(i)}\|_\alpha \to 0$ as $n \to \infty$, $\|g^{(i)}\|_\alpha = \lim \|f_{k,n}^{(i)}\|_\alpha \leq M_i$, for each $i = 0, 1, \dots, k-1$ and $\|f_{k,n}^{(k)} - g^{(k)}\|_I \to 0$ as $n \to \infty$.

Since the sequence $\{f_{i,i}\}$ has the following properties:

(i) $\|f_{i,i}^{(k)}\|_\alpha = \|f_{i,i}^{(k)}\|_I + p_\alpha(f_{i,i}^{(k)}) \leq M_k$ for each $k$,

(ii) $\|f_{i,i}^{(k)} - g^{(k)}\|_\alpha \to 0$ as $i \to \infty$ for each $k$.

It follows that $\|g^{(k)}\|_\alpha \leq M_k$ for each $k$. Clearly, $g \in \mathrm{Lip}^{\infty}(I, \alpha)$ but it may not be in $\mathrm{Lip}\,(I, M, \alpha)$. Let $F(x) = g(ax + b)$. For the compactness of $T$, we show that $F \in \mathrm{Lip}\,(I, M, \alpha)$ and that $Tf_{i,i} \to F$ in $\mathrm{Lip}\,(I, M, \alpha)$ as $i \to \infty$. Since, for every $a \in (0, 1)$,

$$\sum_{k=0}^{\infty} \frac{\|F^{(k)}\|_\alpha}{M_k} \leq \sum_{k=0}^{\infty} \frac{a^k \|g^{(k)}\|_\alpha}{M_k} \leq \sum_{k=0}^{\infty} a^k = \frac{1}{1-a},$$

we conclude that $F \in \mathrm{Lip}\,(I, M, \alpha)$. Furthermore, we have $\sum_{k=N+1}^{\infty} a^k < \varepsilon/4$ for arbitrary $\varepsilon > 0$ and some $N$. Also, there exists a $J$ such that $\|f_{i,i}^{(k)} - g^{(k)}\|_\alpha < (\varepsilon M_k)/(2(N+1))$ for every $i \geq J$ and for each $k = 0, 1, 2, \dots, N$. Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\|(Tf_{i,i})^{(k)} - F^{(k)}\|_\alpha}{M_k} &\leq \sum_{k=0}^{\infty} \frac{a^k \|f_{i,i}^{(k)} - g^{(k)}\|_\alpha}{M_k} \\ &= \sum_{k=0}^{N} \frac{a^k \|f_{i,i}^{(k)} - g^{(k)}\|_\alpha}{M_k} \\ &\quad + \sum_{k=N+1}^{\infty} \frac{a^k \|f_{i,i}^{(k)} - g^{(k)}\|_\alpha}{M_k} \\ &\leq \sum_{k=0}^{N} \frac{\varepsilon}{2(N+1)} + 2 \sum_{k=N+1}^{\infty} a^k < \varepsilon.\end{aligned}$$

Hence, $Tf_{i,i} \to F$ in $\mathrm{Lip}\,(I, M, \alpha)$. $\square$

**Theorem 3.2.** *Let $M = (M_n)$ be a nonanalytic weight sequence and $0 < \alpha \leq 1$, and let $\varphi$ be an analytic self-map of $I = [0, 1]$ such that $\|\varphi'\|_\alpha < 1$. Then $\varphi$ induces a compact endomorphism of* $\mathrm{Lip}\,(I, M, \alpha)$.

*Proof.* Let $T$ be the endomorphism induced by $\varphi$. If we set $\|\varphi'\|_\alpha = c$, then $c < 1$. We also have $b = \|\varphi\|_I \leq 1$. We consider two cases $b = \|\varphi\|_I < 1$ and $b = \|\varphi\|_I = 1$.

First, if $b = \|\varphi\|_I < 1$, choose $c'$ such that $c < c' < \min\{1, (c/b)\}$, and let $\varphi_1 = (c'/c)\varphi$. Then $\varphi_1$ is a self-map of $[0, 1]$ and $\|\varphi_1'\|_\alpha = c' < 1$. So by Theorem 2.2 (i), $\varphi_1$ induces an endomorphism of $\mathrm{Lip}\,(I, M, \alpha)$, say $T_1$. Define $\psi(x) = (c/c')x$. By the previous theorem, $\psi$ induces a compact endomorphism $S$ of $\mathrm{Lip}\,(I, M, \alpha)$. Therefore, $T$ is compact, since $T = T_1 \circ S$.

Now assume that $b = \|\varphi\|_I = 1$. Since $\varphi$ is continuous on the compact set $[0, 1]$, there exists $x_1 \in [0, 1]$ such that $\varphi(x_1) = b = \|\varphi\|_I = 1$. If there exists $x_2 \in [0, 1]$ such that $\varphi(x_2) = 0$, then by the mean value theorem there exists $t \in [0, 1]$ such that $\varphi'(t) = 1/(x_1 - x_2)$. Hence,

$$\|\varphi'\|_\alpha \geq \|\varphi'\|_I \geq |\varphi'(t)| = \frac{1}{|x_2 - x_1|} \geq 1.$$

This is in contradiction with $\|\varphi'\|_\alpha < 1$. So if we set $\varphi_2(x) = 1 - \varphi(x)$, then $\|\varphi_2\|_I < 1$ and $\|\varphi_2'\|_\alpha = \|\varphi'\|_\alpha = c < 1$. Hence, by the previous case, $\varphi_2$ induces a compact endomorphism of $\mathrm{Lip}\,(I, M, \alpha)$, say $T_2$.

By considering the operator $(Sf)(x) = f(1 - x)$, we can directly show that $Sf \in \mathrm{Lip}\,(I, M, \alpha)$, for all $f \in \mathrm{Lip}\,(I, M, \alpha)$, so that $S$ is an endomorphism of $\mathrm{Lip}\,(I, M, \alpha)$. We can also see that $T = T_2 \circ S$. Hence, $T$ is compact, by the compactness of $T_2$. □

**4. Compactness of endomorphism for general plane sets $X$.** In this section we investigate the compact endomorphisms of $\mathrm{Lip}\,(X, M, \alpha)$ for more general plane sets $X$. With the same computation as in the proof of Theorem 3.1, one can show the following proposition.

**Proposition 4.1.** *Let $Y$ be a perfect compact plane set satisfying the $(*)$-condition, $0 < \alpha \leq 1$, and let $M = (M_n)$ be a weight sequence. Then the map $\varphi : (1/c)Y \to Y$ by $\varphi(z) = cz$ with $|c| < 1$ induces a compact homomorphism*: $\mathrm{Lip}\,(Y, M, \alpha) \to \mathrm{Lip}\,((1/c)Y, M, \alpha)$.

We now show that the result of Theorem 3.2 holds for more general compact plane sets $X$.

**Theorem 4.2.** *Let $X$ and $Y$ be perfect compact plane sets satisfying the $(*)$-condition, $0 < \alpha \leq 1$, and let $M = (M_n)$ be a nonanalytic weight sequence. Then, any analytic function $\varphi : X \to Y$ induces a compact homomorphism* $\operatorname{Lip}(Y, M, \alpha) \to \operatorname{Lip}(X, M, \alpha)$, *if* $\|\varphi'\|_\alpha < 1$.

*Proof.* Choose $c$ with $\|\varphi'\|_\alpha < c < 1$, and define $\varphi_1 : X \to (1/c)Y$ by $\varphi_1(z) = (1/c)\varphi(z)$. Then $\|\varphi_1'\|_\alpha = (1/c)\|\varphi'\|_\alpha < 1$. So, by Theorem 2.5, $\varphi_1$ induces the homomorphism $T_1 : \operatorname{Lip}((1/c)Y, M, \alpha) \to \operatorname{Lip}(X, M, \alpha)$.

Now define $\varphi_2 : (1/c)Y \to Y$ by $\varphi_2(z) = cz$. Then, by Proposition 4.1, the homomorphism $T_2 : \operatorname{Lip}(Y, M, \alpha) \to \operatorname{Lip}((1/c)Y, M, \alpha)$, $(T_2 f)(z) = f(cz)$, induced by $\varphi_2$, is compact. The composite map $T = T_1T_2$ is the homomorphism induced by $\varphi$, which is therefore compact. $\square$

Note that the circle, the annulus and the closed unit disc are certainly uniformly regular, whence they satisfy the $(*)$-condition. So for these compact plane sets we have

**Corollary 4.3.** *Let $X$ be one of the above compact plane sets, $0 < \alpha \leq 1$, and let $M$ be a nonanalytic weight sequence. Then any analytic self-map $\varphi$ of $X$ induces a compact endomorphism of* $\operatorname{Lip}(X, M, \alpha)$, *if* $\|\varphi'\|_\alpha < 1$.

We now investigate compact endomorphisms when the underlying set $X$ has nonempty interior.

**Theorem 4.4.** *Let $M = (M_n)$ be a nonanalytic weight sequence, $0 < \alpha \leq 1$. Let $X$ be a perfect compact plane set with nonempty interior such that* $\operatorname{Lip}(X, M, \alpha)$ *is a Banach algebra. Suppose that $\varphi$ is an analytic self-map of $X$. If $\varphi$ is either constant or $\varphi(X) \subseteq \operatorname{int} X$, then $\varphi$ induces a compact endomorphism of* $\operatorname{Lip}(X, M, \alpha)$.

*Proof.* When $\varphi$ is constant, it is clear. If $\varphi(X) \subseteq \operatorname{int} X$, then by Theorem 2.2 (iii), $\varphi$ induces an endomorphism of $\operatorname{Lip}(X, M, \alpha)$.

For the compactness of $T$, assume that $\{f_n\}$ is a bounded sequence in $\text{Lip}\,(X, M, \alpha)$ with $\|f_n\|_{\text{Lip}\,(X,M,\alpha)} \leq 1$. We have $\|f_n\|_X + p_\alpha(f_n) = \|f_n\|_\alpha \leq \sum_{k=0}^{\infty}(\|f_n^{(k)}\|\alpha/M_k) \leq 1$. Thus, $\{f_n\}$ is a bounded and equicontinuous sequence in $C(X)$. By the Arzela-Ascoli theorem, $\{f_n\}$ has a subsequence $\{f_{n_j}\}$ which is uniformly convergent on $X$. So it is uniformly Cauchy on $X$, that is, $\|f_{n_j} - f_{n_l}\|_X \to 0$ as $j, l \to \infty$. We show that $\{Tf_{n_j}\}$ converges in $\text{Lip}\,(X, M, \alpha)$. By completeness of $\text{Lip}\,(X, M, \alpha)$, it is enough to show that $\{Tf_{n_j}\}$ is a Cauchy sequence in $\text{Lip}\,(X, M, \alpha)$.

Using the same technique and the same notation as in the proof of Theorem 2.2 (iii), we obtain the following, which is similar to inequality (2.3).

$$\begin{aligned}
\|Tf_{n_j} - Tf_{n_l}\|_{\text{Lip}\,(X,M,\alpha)}& \\
&= \|(f_{n_j} - f_{n_l}) \circ \varphi\|_{\text{Lip}\,(X,M,\alpha)} \\
&= \sum_{n=0}^{\infty} \frac{\|((f_{n_j} - f_{n_l}) \circ \varphi)^{(n)}\|_\alpha}{M_n} \\
&\leq BB_2\|f_{n_j} - f_{n_l}\|_X \sum_{m=0}^{\infty} \frac{m!}{M_m} \frac{(h(\varepsilon))^m}{\delta^m}\left(\frac{1}{\delta} + \frac{m+1}{\delta^2}\right) \\
&= C_1\|f_{n_j} - f_{n_l}\|_X \to 0 \quad \text{as} \quad j, l \to \infty,
\end{aligned}$$

where

$$C_1 := BB_2 \sum_{m=0}^{\infty} \frac{m!}{M_m} \frac{h(\varepsilon)^m}{\delta^m}\left(\frac{1}{\delta} + \frac{m+1}{\delta^2}\right) < \infty,$$

since $\lim((m!/M_m)(h(\varepsilon)^m/\delta^m)((1/\delta) + ((m+1)/\delta^2)))^{1/m} = 0$. This completes the proof of theorem. $\square$

The following are two immediate consequences of Theorem 4.4.

**Corollary 4.5.** *Let $M = (M_n)$ be a nonanalytic weight sequence, $0 < \alpha \leq 1$. Suppose a self-map $\varphi$ is analytic on the closed unit disc $\overline{\mathbf{D}}$. If $\varphi$ is either constant or $\|\varphi\|_{\mathbf{D}} < 1$, then $\varphi$ induces a compact endomorphism of* $\text{Lip}\,(\overline{\mathbf{D}}, M, \alpha)$.

**Corollary 4.6.** *Let $X = \{z \in \mathbf{C} : r \leq |z - z_0| \leq R\}$ for some $z_0 \in \mathbf{C}$ where $0 < r < R$, and let $M = (M_n)$ be a nonanalytic weight sequence,*

$0 < \alpha \leq 1$. *Suppose a self-map* $\varphi$ *is analytic on* $X$. *If* $\varphi$ *is either constant or* $r < \|\varphi\|_X < R$, *then* $\varphi$ *induces a compact endomorphism of* $\mathrm{Lip}\,(X, M, \alpha)$.

**5. Converse problem.** In this section, we give a necessary condition that $\varphi$ induces a compact endomorphism of $\mathrm{Lip}\,(X, M, \alpha)$ when $X$ is the closed unit disc $\overline{\mathbf{D}}$ or the unit circle $\Gamma$. For this, we need the following lemma.

**Lemma 5.1.** *Let* $X$ *be a connected compact plane set,* $0 < \alpha \leq 1$, $M = (M_n)$ *a nonanalytic weight sequence and the Banach function algebra* $\mathrm{Lip}\,(X, M, \alpha)$ *natural. Suppose that* $T$ *is a compact endomorphism of* $\mathrm{Lip}\,(X, M, \alpha)$ *induced by the self-map* $\varphi$ *of* $X$. *If* $z_0$ *is the fixed point of* $\varphi$, *then* $|\varphi'(z_0)| < 1$.

*Proof.* For the function $f(z) = z$, by Theorem 1.2 in [**7**], we have

$$\|T^n f - f(z_0)1\|_{\mathrm{Lip}\,(X,M,\alpha)} \longrightarrow 0,$$

from which we obtain $\|\varphi_n - z_0 1\|_{\mathrm{Lip}\,(X,M,\alpha)} \to 0$, where $\varphi_n$ is the $n$th iterate of $\varphi$. On the other hand,

$$\|\varphi_n - z_0 1\|_{\mathrm{Lip}\,(X,M,\alpha)} \geq \frac{\|\varphi_n'\|_X}{M_1} \geq \frac{|\varphi_n'(z_0)|}{M_1} = \frac{|\varphi'(z_0)|^n}{M_1},$$

which implies $|\varphi'(z_0)| < 1$. $\square$

We recall that the algebras $\mathrm{Lip}\,(\overline{\mathbf{D}}, M, \alpha)$ and $\mathrm{Lip}\,(\Gamma, M, \alpha)$ are Banach function algebras for all weights $M$ and $0 < \alpha \leq 1$. They are also natural if $M$ is nonanalytic and $0 < \alpha < 1$.

**Theorem 5.2.** *Let* $M = (M_n)$ *be a nonanalytic weight sequence and* $0 < \alpha < 1$. *If* $\varphi$ *induces a compact endomorphism of* $\mathrm{Lip}\,(\overline{\mathbf{D}}, M, \alpha)$, *then either* $\varphi(\overline{\mathbf{D}}) \subseteq \mathbf{D}$ *or* $|\varphi'(z)| < 1$ *for all* $z$ *such that* $|\varphi(z)| = 1$.

*Proof.* Let $\varphi(a) = b \in \partial\mathbf{D}$ for some $a \in \partial\mathbf{D}$. We show that $|\varphi'(a)| < 1$. If we define $\psi(z) = \varphi((a/b)z)$, then $\psi$ is a self-map of $\overline{\mathbf{D}}$ and $\|\psi^{(n)}\|_\alpha = \|\varphi^{(n)}\|_\alpha$ for all nonnegative integers $n$. By Faá di Bruno's formula, we have $\|(f \circ \psi)^{(n)}\|_{\mathrm{Lip}\,(\overline{\mathbf{D}},M,\alpha)} = \|(f \circ \varphi)^{(n)}\|_{\mathrm{Lip}\,(\overline{\mathbf{D}},M,\alpha)}$ for every $f \in \mathrm{Lip}\,(\overline{\mathbf{D}}, M, \alpha)$. Therefore, $\psi$ also induces a compact endomorphism of $\mathrm{Lip}\,(\overline{\mathbf{D}}, M, \alpha)$. Since $\psi(b) = \varphi(a) = b$, by Lemma 5.1, $|\psi'(b)| < 1$. On the other hand, $\psi'(b) = (a/b)\varphi'(a)$. Therefore, $|\varphi'(a)| = |\psi'(b)| < 1$. □

**Theorem 5.3.** *Let $M = (M_n)$ be a nonanalytic weight sequence and $0 < \alpha < 1$. If $\varphi$ induces a compact endomorphism of* $\mathrm{Lip}\,(\Gamma, M, \alpha)$, *then $\|\varphi'\|_\Gamma < 1$.*

*Proof.* Using the same argument as in the proof of Theorem 5.2, one can conclude that for all $z \in \Gamma$, $|\varphi'(z)| < 1$. □

Now by giving two examples we show that the converses of Theorems 4.2 and 4.4 are not necessarily true. First we show that there exists a nonconstant self-map $\varphi$ of $\overline{\mathbf{D}}$ with $\varphi(\overline{\mathbf{D}}) \not\subseteq \mathbf{D}$, $\|\varphi\|_{\mathbf{D}} = 1$, so that $\varphi$ induces a compact endomorphism of $\mathrm{Lip}\,(\overline{\mathbf{D}}, M, \alpha)$.

**Example.** Let $M = (M_n)$ be a nonanalytic weight sequence, $0 < \alpha \leq 1$. Let $a, b$ be complex numbers with $|a| < 1$ and $|b| = 1 - |a|$. Consider the map $\varphi(z) = az + b$ for $|z| \leq 1$. Then $\varphi$ is a self-map of $\overline{\mathbf{D}}$ and there exists $z \in \overline{\mathbf{D}}$ such that $|\varphi(z)| = 1$. Either using the same argument as in the proof of Theorem 3.1 or by Corollary 4.3, one can conclude that $\varphi$ induces a compact endomorphism $T(f)(z) = f(az+b)$ of $\mathrm{Lip}\,(\overline{\mathbf{D}}, M, \alpha)$.

We now give another example in which $\|\varphi'\|_\alpha \geq \|\varphi'\|_X > 1$, and still $\varphi$ induces a compact endomorphism.

**Example.** Let $1/2 < r < 1$, and consider the map $\varphi(z) = rz^2$ on $\overline{\mathbf{D}}$. Then $\|\varphi\|_{\mathbf{D}} = r < 1$, so $\varphi(\overline{\mathbf{D}}) \subseteq \mathbf{D}$. Hence, by Corollary 4.5, $\varphi$ induces a compact endomorphism of $\mathrm{Lip}\,(\overline{\mathbf{D}}, M, \alpha)$ for every nonanalytic weight sequence $M = (M_n)$ and any $0 < \alpha \leq 1$, while $\|\varphi'\|_\alpha \geq \|\varphi'\|_{\mathbf{D}} = 2r > 1$.

**6. Spectra of compact endomorphisms.** In this section we determine the spectrum of a compact endomorphism of the Banach function algebras $\mathrm{Lip}\,(X, M, \alpha)$. We denote the spectrum of an operator $T$ by $\sigma(T)$. Kamowitz in [**9**] proved that if $T$ is a nonzero compact endomorphism of a commutative semi-simple Banach algebra $B$ induced by $\varphi : \mathcal{M}(B) \to \mathcal{M}(B)$, then $\cap\varphi_n(\mathcal{M}(B))$ is finite, where $\varphi_n$ is the $n$th iterate of $\varphi$. Moreover, if $\mathcal{M}(B)$ is connected and $B$ is unital, $\cap\varphi_n(\mathcal{M}(B))$ is a singleton. In this case, if $\cap\varphi_n(\mathcal{M}(B)) = \{x_0\}$, then $x_0$ is the fixed point for $\varphi$. In [**2**, Theorem 4.1] we have proved the following result:

Let $B$ be a natural Banach function algebra on a compact plane set $X$ containing the coordinate function $z$ whose elements are analytic on $\mathrm{int}\, X$. Let $T$ be a compact endomorphism of $B$ induced by $\varphi$. If $\varphi(X) \subseteq \mathrm{int}\, X$ and $z_0$ is a fixed point of $\varphi$, then $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbf{N}\} \cup \{0, 1\}$.

Observing the proof of this theorem, we can assume $\varphi(z_0) = z_0 \in \mathrm{int}\, X$ instead of $\varphi(X) \subseteq \mathrm{int}\, X$. So we have the following corollary.

**Corollary 6.1.** *Let $X$ be a perfect compact plane set with nonempty interior such that the Banach function algebra* $\mathrm{Lip}\,(X, M, \alpha)$ *is natural. Let $T$ be a compact endomorphism of* $\mathrm{Lip}\,(X, M, \alpha)$ *induced by a self-map $\varphi$. If $\varphi$ has an interior fixed point $z_0$, then $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbf{N}\} \cup \{0, 1\}$.*

In the above results we need the underlying set $X$ to have a nonempty interior. In the following we determine the spectrum of a compact endomorphism of $\mathrm{Lip}\,(X, M, \alpha)$ for uniformly regular sets $X$ without assuming a nonempty interior for $X$. In general, we have

**Theorem 6.2.** *Let $B$ be a natural Banach function algebra on a perfect compact plane set $X$ containing the coordinate function $z$ and $B \subseteq D^1(X)$. If a self-map $\varphi$ induces a compact endomorphism $T$ of $B$ and $z_0$ is a fixed point of $\varphi$, then $\{\varphi'(z_0)^n : n \in \mathbf{N}\} \cup \{0, 1\} \subseteq \sigma(T)$.*

*Proof.* It is clear that $0, 1 \in \sigma(T)$. We first show that $\varphi'(z_0) \in \sigma(T)$. We may assume that $\varphi'(z_0) \neq 0$. If $\varphi'(z_0) \notin \sigma(T)$, then there exists $g \in B$ such that $g \circ \varphi - \varphi'(z_0) g = z - z_0$. By differentiation at $z_0$ we get a contradiction. Hence, $\varphi'(z_0) \in \sigma(T)$. So $f(\varphi(z)) = \varphi'(z_0) f(z)$ for some nonzero $f \in B$. Therefore, for each positive integer $n$,

$f^n(\varphi(z)) = (\varphi'(z_0))^n f^n(z)$, whence $(\varphi'(z_0))^n \in \sigma(T)$ for all positive integers $n$. $\square$

**Theorem 6.3.** *Let $X$ be a uniformly regular set, $0 < \alpha \leq 1$, $M = (M_n)$ a nonanalytic weight sequence and $\mathrm{Lip}\,(X, M, \alpha)$ natural. Suppose a self-map $\varphi$ induces a compact endomorphism $T$ of $\mathrm{Lip}\,(X, M, \alpha)$ and $z_0$ is the fixed point of $\varphi$. If either $\varphi$ is analytic or there exists a neighborhood $U$ of $z_0$ such that $U \cap X$ is convex, then $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbf{N}\} \cup \{0, 1\}$.*

*Proof.* By Theorem 6.2, $\{(\varphi'(z_0))^n : n \in \mathbf{N}\} \cup \{0, 1\} \subseteq \sigma(T)$. Let $\lambda \in \sigma(T)\backslash\{0, 1\}$. By the compactness of $T$ there exists a nonzero function $f \in \mathrm{Lip}\,(X, M, \alpha)$ such that $Tf = f \circ \varphi = \lambda f$. Then $f(z_0) = 0$, since $\lambda \neq 1$. It suffices to show that $f^{(n)}(z_0) \neq 0$ for some integer $n$. To see this, let $n$ be the smallest integer such that $f^{(n)}(z_0) \neq 0$. By $n$ times differentiation of $f \circ \varphi = \lambda f$, we have $\varphi'(z_0)^n f^{(n)}(\varphi(z_0)) = \lambda f^{(n)}(z_0)$ and so $\lambda = \varphi'(z_0)^n$. Therefore, $\sigma(T)\backslash\{0, 1\} \subseteq \{\varphi'(z_0)^n : n \in \mathbf{N}\}$.

We now show that $f^{(n)}(z_0) \neq 0$ for some integer $n$. When $\varphi$ is analytic we can extend $\varphi$ to an analytic function on a neighborhood $\Omega$ of $X$. By Theorem 5.1, $|\varphi'(z_0)| < 1$, so we can choose $a$ with $|\varphi'(z_0)| < a < 1$. Hence, there exists an $\varepsilon > 0$ such that $|\varphi'(z)| < a$ if $z \in B(z_0, \varepsilon)$. We can choose $\varepsilon$ small enough such that $B(z_0, \varepsilon) \subseteq \Omega$ in the case that $\varphi$ is analytic, and in the other case, $B(z_0, \varepsilon) \subseteq U$. By Theorem 1.2 (ii) in [**7**], there exists a positive integer $N$ such that $\varphi_n(X) \subseteq B(z_0, \varepsilon)$ for all $n \geq N$. Hence for $n > N$ and for all $z \in X$, we have

$$|\varphi_n(z) - z_0| \leq a|\varphi_{n-1}(z) - z_0| \leq a^{n-N}|\varphi_N(z) - z_0| < \varepsilon a^{n-N}.$$

Since $a < 1$, we may choose $m$ with $a^m < |\lambda|$. Fix this number $m$ for the remainder of this proof. If $f^{(n)}(z_0) = 0$ for all $n$, then

$$|f(z)| \leq \frac{C^m}{(m-1)!}|z - z_0|^m \|f^{(m)}\|_X$$

for some $C > 0$ [**3**, Lemma 1.5(iii)]. Since $f \circ \varphi = \lambda f$, we have $f(\varphi_n(z)) = \lambda^n f(z)$ for all $n \in \mathbf{N}$ and $z \in X$. Thus,

$$\begin{aligned}|f(z)| = \frac{|f(\varphi_n(z))|}{|\lambda|^n} &\le \frac{C^m}{(m-1)!|\lambda|^n}|\varphi_n(z) - z_0|^m \|f^{(m)}\|_X \\ &\le \left(\frac{\varepsilon C}{a^N}\right)^m \frac{\|f^{(m)}\|_X}{(m-1)!}\left(\frac{a^m}{|\lambda|}\right)^n.\end{aligned}$$

If $n \to \infty$ we obtain $f(z) = 0$, which is a contradiction. $\square$

We remark that when $X$ is the closed unit disc or the closed unit interval, the map $\varphi$ is not required to be analytic. However, when $X$ is an annulus or the unit circle we do not yet know whether this condition for $\varphi$ is redundant. So we have the following.

**Corollary 6.4.** *Let $X$ be either the closed unit disc or the closed unit interval. Let $M = (M_n)$ be a nonanalytic weight sequence and $0 < \alpha < 1$. If $T$ is a compact endomorphism of* $\mathrm{Lip}\,(X, M, \alpha)$ *induced by the self-map $\varphi$ and $z_0$ is the fixed point of $\varphi$, then $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbf{N}\} \cup \{0, 1\}$.*

**Corollary 6.5.** *Let $X$ be either the annulus or the unit circle. Let $M = (M_n)$ be a nonanalytic weight sequence and $0 < \alpha < 1$. If $T$ is a compact endomorphism of* $\mathrm{Lip}\,(X, M, \alpha)$ *induced by the analytic self-map $\varphi$ and $z_0$ is the fixed point of $\varphi$, then $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbf{N}\} \cup \{0, 1\}$.*

Note that, when $X$ is an annulus and the fixed point of $\varphi$ is an interior point of $X$, by Corollary 6.1, the analycity of $\varphi$ is not required.

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