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SPECTRAL THEORY FOR NONLOCAL DISPERSAL WITH PERIODIC OR ALMOST-PERIODIC TIME DEPENDENCE

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ABSTRACT. In applications to spatial structure in biology and to the theory of phase transition, it has proved useful to generalize the idea of diffusion to a nonlocal dispersal with an integral operator replacing the Laplacian. We study the spectral problem for the linear scalar equation

$$u_t(x,t) = \int_\Omega K(x,y)u(y,t)\,dy + h(x,t)u(x,t),$$

and tackle the extra technical difficulties arising because of the lack of compactness for the evolution operator defined by the dispersal. Our aim is firstly to investigate the extent to which the idea of a periodic parabolic principal eigenvalue may be generalized. Secondly, we obtain a lower bound for this in terms of the corresponding averaged spatial problem, and then extend this to the principal Lyapunov exponent in the almost periodic case.

1. Introduction. Recently there has been extensive investigation into a class of models for nonlocal spatial dispersal, in which the dispersal operator D, say, involves an integral operator, for example

$$(Du)(x) = \int_\Omega K(x,y)[u(y) - u(x)]\,dy. \tag{1.1}$$

Such models occur in a number of applications, for example biology and the theory of phase transition, as a generalization of classical diffusion where $D = \Delta$, the Laplacian with a suitable boundary condition. The derivation in the biological context is discussed in [**12, 17, 22**], and for the theory of phase transition, see [**5 , 7, 8**]. The nonlinear theory has

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been investigated in a number of papers, of which a sample, in addition to the above, is [**4, 6, 11, 16, 25**].

We shall consider here aspects of spectral theory for linear evolution problems with nonlocal dispersal; this of course provides a basic technical tool in the nonlinear theory, for example in a discussion of stability. Nonautonomous models have scarcely been considered in the dispersal context, and here we shall focus particularly on the periodic and almost-periodic cases. Consider then the linear evolution equation

$$u_t(x,t) = \int_\Omega K(x,y)u(y,t)\,dy + h(x,t)u(x,t), \tag{1.2}$$

where u_t denotes the derivative of u with respect to time t, with x constant, and $\Omega \subset \mathbf{R}^N$ is a compact spatial region; note that the second term in D in (1.1) has been incorporated into h. It is convenient to abbreviate the notation and write this as

$$u_t = Xu + Hu,$$

where X is the integral operator in (1.2) and H is multiplication by h.

If $D = \Delta$, for the autonomous case (h independent of t) and for the periodic case ($h(x,t) = h(x, t+T)$ for all $x \in \Omega$ and t) there is a well-known theory yielding the existence of a principal eigenvalue (PEV) and eigenfunction (PEF). For the theory, see [**15**]; in applications the idea has been used for example in studying the evolution of diffusion [**18**] and in permanence, see [**2**, Chapter 2]. This has important implications for the study of stability, rate of increase, invasion problems and sub/super solution methods for nonlinear models. The partial differentiation equation (PDE) technique of proof depends critically upon compactness properties for the evolution operator based upon Δ.

Our first objective here is to inquire to what extent these results hold for the nonlocal case (1.2). The PDE technique is not applicable as the evolution operator generated by X does not appear to have compactness properties in convenient spaces. We employ a method based on using the evolution operator generated by the linear operator $(-(\partial/\partial t) + H)$ (this is related to an approach used in [**3**]) together with the compactness of X itself. It is proved that

- *if $N = 1$, reasonable smoothness conditions on h are sufficient to ensure the existence of a* PEV, Theorem 3.1.

However, if $N \geq 2$, then examples show that smoothness is not enough. This interesting issue is further discussed at the end of Section 3.

An upper bound for the growth rate of the solution for a continuous initial condition is provided by the principal Lyapunov exponent λ_L, see Definition 2.3. If a PEV λ exists, then $\lambda_L = \lambda$; in general, it is also shown that

• $\lambda_L = s := \sup_{\lambda \in \sigma} \Re(\lambda)$, *where* σ *is the spectrum of* $(-(\partial/\partial t) + X + H)$ *on the space of* T*-periodic functions*, Theorem 3.2.

If h is almost periodic (AP) only, there appears to be no analogue of a PEV. A partial analogue of the above is provided by using the dynamical spectrum, see Definition 2.2, $\mathcal{S}$ of $(-(\partial/\partial t) + X + H)$. Then $\lambda_L = \lambda_s := \sup_{\lambda \in \mathcal{S}} \lambda$.

Our second main objective is to investigate the influence of time periodicity/almost periodicity on the principal eigenvalue/principal Lyapunov exponent. We show that

• *for the periodic case, when a* PEV *exists it is always larger than or equal to the* PEV *for the associated time-average case*, Theorem 4.1.

This result extends a result of [**19**] for the PDE problem. In the biological context, this inequality shows that, perhaps rather counterintuitively, invasion by a new species, see [**2**, page 220], is always easier in the periodic case. We also show that

• *for the* AP *case, an analogous result holds, viz. that* λ_L *is always larger than or equal to the* PEV *for the time-averaged case*, Theorem 5.1.

An outline of the contents is as follows. In Section 2, the notation is described and some background results proved. This is a preliminary section for the later ones. The question of existence of a PEV for the periodic case is considered in Section 3 and in Section 4 the lower bound for the PEV is established. In Section 5, the AP case is discussed.

2. Definitions and basic properties.

First the notation is described. Some of the spectral theory for the most general case to be considered here (h AP in t) is then outlined. We note that the theory is not as simple as for the well-known case where the dispersal operator is an elliptic partial differential operator.

The following conditions are assumed throughout.

(H1) (a) $\Omega \subset \mathbf{R}^N$ is compact.

(b) $K : \Omega \times \Omega \to \mathbf{R}$ is continuous and

$$K(x, y) \geq 0, \quad x, y \in \Omega. \tag{2.1}$$

(c) $h : \Omega \times \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous, uniform limit of periodic functions (hence, uniformly AP, see [**9**]) in its second argument, t, and uniformly bounded.

Remark. The result in fact holds if h is uniformly AP, but for brevity in the presentation of the proof, we make the stronger assumption (c).

Let $E = C(\Omega)$ be the Banach space of continuous, complex valued functions on Ω with the maximum norm $\|u\| = \max_{x\in\Omega} |u(x)|$. With the ordering induced by the positive cone

$$E_+ = \{u \in E \mid u(x) \geq 0,\ x \in \Omega\},$$

$(E, \|\cdot\|)$ is a complex Banach lattice, and we write $u \geq v$ if $u(x) \geq v(x)$, $x \in \Omega$. The notation $u > v$ if $u(x) > v(x)$ for all $x \in \Omega$ will be adopted. A linear operator $L : E \to E$ is said to be *positive* if $u \geq v \Rightarrow Lu \geq Lv$. For each t, define the (continuous) operators $X, H : E \to E$ as follows.

$$(Xu)(x, t) = \int_\Omega K(x, y)u(y, t)\, dy, \tag{2.2}$$

$$(Hu)(x, t) = h(x, t)u(x, t). \tag{2.3}$$

In writing down the governing equation, we treat $u(\cdot, t)$ as an E-valued function for each t and, for brevity, suppress the t-dependence. We study the equation

$$u_t = Xu + Hu. \tag{2.4}$$

Let $\Phi(s, t)$, $s \leq t$, defined by

$$\Phi(t, s)u_0 = u(t, \cdot; u_0, s), \quad u_0 \in E,$$

be the evolution operator generated on E, where $u(t, x; s, u_0)$ is the solution of (2.4) with $u(s, x; s, u_0) = u_0(x)$. For a given $\lambda \in \mathbf{R}$, define

$$\Phi_\lambda(t, s) = e^{-\lambda(t-s)}\Phi(t, s).$$

In general, the 'spectrum' of the evolution operator and the 'spectrum' of its generator may not be the same, see [**23**, Chapter 2, Example 2.1]. It is the spectrum of the evolution operator which characterizes the asymptotic behavior of solutions. For completeness, this concept is defined here.

Definition 2.1. Given $\lambda \in \mathbf{R}$, $\{\Phi_\lambda(t,s)\}_{s,t\in\mathbf{R},s\le t}$ is said to admit an *exponential dichotomy* (ED for short) if there exist $\beta > 0$ and $C > 0$ and continuous projections $P(s) : E \to E$, $s \in \mathbf{R}$, such that for any $s, t \in \mathbf{R}$ with $s \le t$, the following holds:

(1) $\Phi_\lambda(t,s)P(s) = P(t)\Phi_\lambda(t,s)$;

(2) $\Phi_\lambda(t,s)|_{R(P(s))} : R(P(s)) \to R(P(t))$ is an isomorphism for $t \ge s$ (hence, $\Phi_\lambda(s,t) := \Phi_\lambda(t,s)^{-1} : R(P(t)) \to R(P(s))$ is well defined);

(3)

$$\begin{aligned}\|\Phi_\lambda(t,s)(I-P(s))\| &\le Ce^{-\beta(t-s)}, \quad t \ge s\\ \|\Phi_\lambda(t,s)P(s)\| &\le Ce^{\beta(t-s)}, \quad\ \ t \le s.\end{aligned}$$

Definition 2.2. (1) $\lambda \in \mathbf{R}$ is said to be in the *dynamical spectrum*, denoted by $\Sigma(X,H)$, of (2.4) if $\Phi_\lambda(t,s)$ does not admit an ED.

(2) $\lambda_s(X,H) := \sup\{\lambda \in \Sigma(X,H)\}$ is called the *principal dynamical spectrum point* of (2.4).

Definition 2.3.

$$\lambda_L(X,H) := \limsup_{t-s\to\infty} \frac{\ln\|\Phi(t,s)\|}{t-s}$$

is called the *principal Lyapunov exponent* of (2.4).

Proposition 2.4. *Assume that $u_0 \in C(\Omega)$ and $u_0 \ge 0$. Assume also that $p : \Omega\times\mathbf{R} \to \mathbf{R}$ is continuous and uniformly bounded. Suppose that $u : \Omega\times\mathbf{R} \to \mathbf{R}$ is continuous and differentiable in its second argument with u_t continuous on $\Omega\times\mathbf{R}$. Then if u satisfies the following*:

$$\begin{aligned}u_t(x,t) &\ge \int_\Omega K(x,y)u(y,t)\,dy + p(x,t)u(x,t), \quad t \ge s\\ u(x,s) &= u_0(x),\end{aligned} \tag{2.5}$$

then

(1) $K(x,y) \geq 0$, $x, y \in \Omega \Rightarrow u(x,t) \geq 0$, $x \in \Omega, t > s$,

(2) $K(x,y) > 0$, $x, y \in \Omega$, and $u_0 \not\equiv 0 \Rightarrow u(x,t) > 0$, $x \in \Omega$, $t > s$.

Proof. Note first that from (2.5), $v(x,t) = e^{\lambda(t-s)}u(x,t)$ satisfies the inequality

$$v_t(x,t) \geq \int_\Omega K(x,y)v(y,t)\,dy + [p(x,t)+\lambda]v(x,t), \quad t \geq s,$$

with $v(x,s) = u_0(x)$. Therefore, it may be assumed without loss of generality that $p(x,t) > 0$, $x \in \Omega$, $t \in \mathbf{R}$.

(1) Let

$$K_0 = \max_{x\in\Omega} \int_\Omega K(x,y)\,dy \quad \text{and} \quad p_0 = \sup_{\substack{x\in\Omega \\ t\in\mathbf{R}}} p(x,t).$$

Take $\tau = (K_0 + p_0)^{-1}/2$. Suppose that for some x and $t \in [s, s+\tau]$, $u(x,t) < 0$. Then there exist x_1 and $t_1 \in [s, s+\tau]$ such that

$$\min_{\substack{x\in\Omega \\ s\leq t\leq s+\tau}} u(x,t) = u(x_1,t_1) < 0.$$

Integrating (2.5) with respect to t over $[s, t_1]$ and using the mean value theorem for integrals, we deduce that

$$u(x_1,t_1) - u(x_1,s) \geq (t_1 - s)(K_0 + p_0)u(x_1,t_1).$$

But by assumption, $u(x_1,s) = u_0(x_1) \geq 0$. Therefore,

$$u(x_1,t_1)[1 - (t_1 - s)(K_0 + p_0)] \geq 0.$$

Since $(t_1 - s) \leq \tau = (K_0 + p_0)^{-1}/2$, we have $u(x_1,t_1) \geq 0$ which is a contradiction. Therefore $u(x,y) \geq 0$ for all $x \in \Omega$ and $s \leq t \leq s+\tau$. The result follows on repeating the argument with initial times $s+\tau, s+2\tau, \ldots$.

(2) From the result just proved and (2.5), clearly $u_t(x,t) \geq 0$, $x \in \Omega$, $t \geq s$. Also, since $K > 0$ and $u_0 \not\equiv 0$ we see that $u_t(x,s) > 0$, $x \in \Omega$.

Therefore, by continuity and compactness, there exist $\delta > 0$, $t_0 > s$, such that $u(x,t_0) \geq \delta$, $x \in \Omega$, and $u(x,t) > 0$, $x \in \Omega$, $s < t \leq t_0$. If the assertion does not hold, there exist $x_1 \in \Omega$, $t_1 > t_0$, such that $u(x_1,t_1) = 0$. But $u(x_1,t_0) \geq \delta$ and $u_t(x_1,t) \geq 0$ for all $t \geq s$. This yields a contradiction. □

Proposition 2.5. $\lambda_s(X,H) = \lambda_L(X,H)$.

Proof. The proposition may be proved by arguments similar to those in [**24**, Proposition 4.1]. For completeness, we provide a proof here.

First we note that there are $\overline{C}$ and $\omega \in \mathbf{R}$ such that

$$\|\Phi(t,s)\| \leq \overline{C}e^{\omega(t-s)} \tag{2.6}$$

for any $s,t \in \mathbf{R}$ with $s \leq t$.

Next, suppose that $\lambda_s(X,H), \lambda_L(X,H) > -\infty$. By (2.6), $\lambda_s(X,H) < \infty$. Hence, for $\varepsilon > 0$ and $\lambda^* = \lambda_s(X,H) + \varepsilon$, there is a $C > 0$ such that

$$\|e^{-\lambda^*(t-s)}\Phi(t,s)\| \leq C$$

that is,

$$\|\Phi(t,s)\| \leq Ce^{\lambda^*(t-s)}$$

for $s \leq t$. It then follows that

$$\lambda_L(X,H) \leq \lambda^* = \lambda_s(X,H) + \varepsilon.$$

By taking $\varepsilon \to 0$, we have $\lambda_L(X,H) \leq \lambda_s(X,H)$. Conversely, since $\lambda_L(X,H) < \infty$, for any $\varepsilon > 0$,

$$e^{-(\lambda_L(X,H)+\varepsilon)(t-s)}\|\Phi(t,s)\| \to 0 \quad \text{as } t-s \to \infty.$$

This implies that $\lambda_L(X,H) + \varepsilon \in \mathbf{R} \setminus \Sigma(X,H)$ and $\lambda_s(X,H) \leq \lambda_L(X,H) + \varepsilon$. Since ε is arbitrary, $\lambda_s(X,H) \leq \lambda_L(X,H)$. Therefore, $\lambda_s(X,H) = \lambda_L(X,H)$.

Now if $\lambda_s(X,H) = -\infty$ or $\lambda_L(X,H) = -\infty$, by the above arguments, for any $M > 0$, $\lambda_L(X,H) \leq -M$ or $\lambda_s(X,L) \leq -M$. Therefore, $\lambda_L(X,H) = -\infty$ or $\lambda_s(X,H) = -\infty$. □

The above result implies that investigating the properties of $\lambda_s(X,H)$, $\lambda_L(X,H)$, is equivalent to investigating the properties of $\lambda_L(X,H)$, $\lambda_s(X,H)$, which may be an easier problem.

Proposition 2.6. *$\lambda_L(X,H)$ is continuous in H with respect to the topology of uniform convergence, that is, if $h_n(x,t)\to h(x,t)$ as $n\to\infty$ uniformly in $x\in\Omega$ and $t\in\mathbf{R}$, then $\lambda_L(X,H_n)\to\lambda_L(X,H)$, where $H_n u = h_n u$ and $Hu = hu$.*

Proof. First, let $\Phi^{\pm\varepsilon}(t,s)$ be the evolution operators generated by (2.4) with H being replaced by $H\pm\varepsilon$. It is clear that

$$\Phi^{\pm\varepsilon}(t,s)=\Phi_{\pm\varepsilon}(t,s)=e^{\pm\varepsilon(t-s)}\Phi(t,s).$$

Therefore,

$$\lambda_L(X,H\pm\varepsilon)=\lambda_L(X,H)\pm\varepsilon. \tag{2.7}$$

Next, for given h_1, h_2 with $h_1\le h_2$, let $\Phi^i(t,s)$, $i=1,2$, be the evolution operators generated by (2.4) with $Hu=H_iu:=h_iu$. We claim that

$$\|\Phi^1(t,s)\|\le\|\Phi^2(t,s)\|. \tag{2.8}$$

In fact, for any given $u_0\in E$ with $u_0\ge 0$, by Proposition 2.4 (1) with $p=h_i$, $\Phi^i(t,s)u_0\ge 0$ for $s\le t$ and $i=1,2$. Let $v(x,t)=\Phi^2(t,s)u_0-\Phi^1(t,s)u_0$. Then $v(x,t)$ satisfies

$$\begin{aligned} v_t &= \int_\Omega K(x,y)v(y,t)\,dy+h_2(x,t)v(x,t)+(h_2(x,t)-h_1(x,t))\Phi^1(t,s)\,u_0\\ &\ge \int_\Omega K(x,y)v(y,t)dy+h_2(x,t)v(x,t)\end{aligned}$$

with $v(x,s)=0$. By Proposition 2.4 (1) with $p=h_2$, $v(x,t)\ge 0$ which implies (2.8) and this in turn gives

$$\lambda_L(X,H_1)\le\lambda_L(X,H_2). \tag{2.9}$$

The proposition then follows from (2.7) and (2.9). $\square$

3. The periodic case. Our objective is to inquire to what extent the well-known PDE theory for the existence of a PEV when h is periodic extends to the nonlocal dispersal case (2.4). It will be proved that, under the assumed smoothness condition, that is, h is Lipschitz in x, the results extend if the dimension $N = 1$. If $N > 1$, in general a smoothness condition is not enough; this observation raises questions which are discussed at the end of this section. We also show that even when a PEV does not exist, the principal Lyapunov exponent $\lambda_L = s(\widetilde{X}, \widetilde{H})$, $s(\widetilde{X}, \widetilde{H}) := \sup_{\lambda \in \sigma(\widetilde{X}, \widetilde{H})} \Re(\lambda)$, where $\widetilde{X}, \widetilde{H}$ are X, $-(\partial/\partial t) + H$, respectively, restricted to a space of periodic functions and $\sigma(\widetilde{X}, \widetilde{H})$ is the spectrum of $\widetilde{H} + \widetilde{X}$.

Since the problem is linear and we shall be discussing the spectrum, we may assume $h(x,t) \leq 0$, $x \in \Omega$, $t \in \mathbf{R}$, without loss of generality, since only a shift in the spectrum is involved. By a solution of equation (2.4), or related equations, we shall mean a function $u \in C(\Omega \times \mathbf{R})$ which is continuously differentiable in the second variable with u_t continuous on $\Omega \times \mathbf{R}$.

Set

$$\widetilde{E} = \{u \in C(\Omega \times \mathbf{R}) \mid u(x, t+T) = u(x,t)\},$$

equipped with the sup norm. Let $\widetilde{H}$, $\widetilde{X} : \widetilde{E} \to \widetilde{E}$ be the linear operators defined as follows:

$$(\widetilde{H}u)(x,t) = -u_t(x,t) + h(x,t)u(x,t),$$

with domain

$$\mathcal{D}(\widetilde{H}) = \{u \in \widetilde{H} \mid u \text{ is } C^1 \text{ in } t \text{ and } u_t \in \widetilde{E}\},$$

and

$$(\widetilde{X}u)(x,t) = \int_\Omega K(x,y)u(y,t)\,dy.$$

The governing equation (2.4) restricted to $\widetilde{E}$ becomes, in this notation,

$$(\widetilde{H} + \widetilde{X})u = 0, \tag{3.1}$$

and our objective is to study the spectrum of $(\widetilde{H} + \widetilde{X})$. Denote by $\rho(\widetilde{X}, \widetilde{H})$, $\sigma(\widetilde{X}, \widetilde{H})$ its *resolvent set* and *spectrum*, respectively:

$s(\widetilde{X}, \widetilde{H}) = \sup_{\mu \in \sigma(\widetilde{X}, \widetilde{H})} \Re(\mu)$ will be called the *principal spectrum point* ($s(\widetilde{X}, \widetilde{H})$ is defined to be $-\infty$ if $\sigma(\widetilde{X}, \widetilde{H}) = \varnothing$). As usual, $\lambda \in \mathcal{C}$ is said to be an *eigenvalue* of $(\widetilde{H} + \widetilde{X})$ if there is a nontrivial solution $\phi \in \widetilde{E}$, an *eigenfunction*, of the equation

$$(\widetilde{H} + \widetilde{X})\phi = \lambda\phi. \tag{3.2}$$

An eigenvalue λ is called the *principal eigenvalue* (PEV) if there is exactly one corresponding *principal eigenfunction* ϕ, with $\phi(x,t) \geq 0$, $x \in \Omega$, $t \in \mathbf{R}$, and the inequality $\Re(\mu) \leq \lambda$, $\mu \in \sigma(\widetilde{X}, \widetilde{H})$, holds. Obviously, $\lambda = s(\widetilde{X}, \widetilde{H})$ (if λ exists).

We use $\rho(\widetilde{H})$, $\sigma(\widetilde{H})$, and $s(\widetilde{H})$ for $\rho(\widetilde{X}, \widetilde{H})$, $\sigma(\widetilde{X}, \widetilde{H})$, and $s(\widetilde{X}, \widetilde{H})$, respectively, when $K(x,y) \equiv 0$.

The following additional conditions are imposed in this section.

(H2) (a) $K(x,y) > 0$, $x, y \in \Omega$.

(b) For each x and t, $x \in \Omega$, $t \in \mathbf{R}$, $h(x,t) \leq 0$ and $h(x,t) = h(x, t+T)$.

Our main results of this section are stated as follows.

Theorem 3.1. *Assume* (H1) *and* (H2) *and take* $N = 1$. *Then* $\lambda = s(\widetilde{X}, \widetilde{H})$ *is the* PEV *of* (3.2) *and is an isolated point of* $\sigma(\widetilde{X}, \widetilde{H})$. *Furthermore,* $\phi(x,t) > 0$, $x \in \Omega$, $t \in \mathbf{R}$.

Theorem 3.2. *Assume that* (H1) *and* (H2) *hold. Then* $\lambda_s(X, H) = \lambda_L(X, H) = s(\widetilde{X}, \widetilde{H})$.

The PEV λ and PEF are useful in providing estimates of rates of growth, and for the application of sub/super solution methods. Theorem 3.1 shows the PEV of (3.2) exists when $N = 1$. Observe that the maximum growth rate for the initial value problem is indeed measured by $\lambda_L (= \lambda_s)$, the principal Lyapunov exponent. It may be shown that $\lambda_L = \lambda$, and of course $\lambda = s(\widetilde{X}, \widetilde{H})$, the principal spectrum point (if λ exists). However, as mentioned above, when $N > 1$, in the dispersal case (as opposed to the case of classical diffusion) a PEV may not exist. Theorem 3.2 shows that in general, we always have $\lambda_L = s(\widetilde{X}, \widetilde{H})$, thus providing a partial analogy.

The proof of Theorem 3.1 is based on a result by Bürger [**1**, Theorem 2.2 and Remark 2.1], and for convenience we give the part of this result needed here translated into the current notation. Let

$$r_\alpha = r(\widetilde{X}(\widetilde{H} - \alpha)^{-1})$$

where $r(\cdot)$ denotes the spectral radius.

Theorem 3.3. *Assume that*

(1) $\widetilde{X}$ *is positive and bounded and* $\widetilde{X} : \widetilde{F} \to \widetilde{E}$ *is compact, where* $\widetilde{F} = \mathcal{D}(\widetilde{H})$ *with the graph norm.*

(2) $\widetilde{H}$ *is closed with dense domain and generates a positive continuous semi-group of contractions.*

(3) $r_\alpha > 1$ *for some* $\alpha > s(\widetilde{H})$.

Then there is a unique α_0, $> s(\widetilde{H})$, *with* $r_{\alpha_0} = 1$ *and* $\alpha_0 = s(\widetilde{X}, \widetilde{H})$. *Further,* α_0 *is an isolated eigenvalue of* $(\widetilde{X} + \widetilde{H})$ *of finite multiplicity with a positive eigenfunction.*

In the following lemmas we verify that the conditions in Theorem 3.3 hold.

Lemma 3.4. $\widetilde{H}$ *generates a positive continuous semi-group of contractions on* $\widetilde{E}$. $\widetilde{H}$ *is closed with dense domain.*

Proof. Let $\phi(s) : \widetilde{E} \to \widetilde{E}$, $s \in \mathbf{R}$, be defined by

$$(\phi(s)u)(x,t) = \exp\left\{ \int_{t-s}^{t} h(x,\xi)\, d\xi \right\} u(x, t-s),$$

and let $U(s, x, t; u)$ be the solution of

$$\frac{\partial U}{\partial s} = -\frac{\partial U}{\partial t} + h(x,t)U$$

with $U(0, x, t; u) = u(x,t) \in \widetilde{E}$.

Then by direct computation, we have

$$\Big\{u \in \widetilde{E} \mid \lim_{s\to 0+} \frac{\phi(s)u - u}{s} \quad \text{exists}\Big\} = \mathcal{D}(\widetilde{H})$$

and

$$U(s,x,t;u) = (\phi(s)u)(x,t)$$

for any $u \in \mathcal{D}(\widetilde{H})$. Hence $\{\phi(s)\}_{s\in\mathbf{R}^+}$ is a continuous semi-group on $\widetilde{E}$ with generator $\widetilde{H}$.

By the definition of $\phi(s)$, for any $u \in \widetilde{E}$ with $u(x,t) \geq 0$, $\phi(s)u \geq 0$ for any $s \geq 0$. Moreover, since $h(x,t) \leq 0$, we have

$$\|\phi(s)u\|_{\widetilde{E}} \leq \|u\|_{\widetilde{E}}$$

for $s \geq 0$. Therefore, $\{\phi(s)\}_{s\in\mathbf{R}^+}$ is a positive continuous semi-group of contractions on $\widetilde{E}$ with generator $\widetilde{H}$. It follows from [**23**, Chapter 1, Corollary 2.5] that $\widetilde{H}$ is closed with dense domain. $\square$

Lemma 3.5. *$\widetilde{X} : \widetilde{F} \to \widetilde{E}$ is positive and compact, where $\widetilde{F} = \mathcal{D}(\widetilde{H})$ with the graph norm.*

Proof. From (H2)(a), the positivity is obvious. Let $\{u_n\}$ be a sequence in the unit ball of $\widetilde{F}$, and let

$$v_n(x,t) = \int_\Omega K(x,y)u_n(y,t)\,dy.$$

Then $|\partial u_n/\partial t(x,t)| \leq 1$, $n \geq 1$, $x \in \Omega$, $t \in \mathbf{R}$, and there is a constant $M > 0$ such that $|(\partial v_n/\partial t)(x,t)| \leq M$, $n \geq 1$, $x \in \Omega$, $t \in \mathbf{R}$. Also, from the uniform continuity of K, given $\varepsilon > 0$ there is a $\delta > 0$, such that

$$|v_n(x_1,t) - v_n(x_2,t)| < \varepsilon,$$
$$x_1, x_2 \in \Omega, \quad |x_1 - x_2| < \delta, \quad n \geq 1, \quad t \in \mathbf{R}.$$

It follows that the sequence $\{v_n\}$ is equicontinuous and the compactness then follows from the Arzela-Ascoli theorem. $\square$

To investigate the spectrum of $\widetilde{X} + \widetilde{H}$, take

$$\overline{E} = \{u \in C(\mathbf{R}) \mid u(t+T) = u(t)\},$$

and let $C(\mathbf{R})$ have the sup norm. For a given $x_0 \in \Omega$, let $\overline{H}(x_0)$ be the linear operator on $\overline{E}$ defined by

$$(\overline{H}(x_0)u)(t) = -\frac{du}{dt}(t) + h(x_0, t)\, u(t)$$

with $\mathcal{D}(\overline{H})$ the set $C^1(\mathbf{R}) \subset \overline{E}$ of functions with continuous first derivatives. Denote by $\rho(\overline{H}(x_0))$ and $\sigma(\overline{H}(x_0))$ the resolvent set and spectrum, respectively, of $\overline{H}(x_0)$. Define

$$\lambda(x_0) = \frac{1}{T}\int_0^T h(x_0, s)\, ds,$$

and note that by (H2) (b), $\lambda(x_0) \le 0$.

Lemma 3.6. (1) *For fixed $x_0 \in \Omega$ and $\lambda \in \mathbf{R}$, $\overline{H}(x_0)u - \lambda u = 0$ has a nontrivial solution $u \in \overline{E}$ if and only if $\lambda = \lambda(x_0)$.*

Choose any $\delta > 0$. Then there are constants $M_1, M_2 > 0$ such that the following hold for any $x_0 \in \Omega$.

(2)

$$\|(\overline{H}(x_0) - \lambda)^{-1}\| \ge \frac{M_1}{|\lambda - \lambda(x_0)|} \tag{3.3}$$

for $\lambda \in \mathbf{R}$ with $0 < |\lambda - \lambda(x_0)| \le \delta$.

(3)

$$\|(\overline{H}(x_0) - \lambda)^{-1}\| \le \frac{M_2}{|\Re(\lambda) - \lambda(x_0)|} \tag{3.4}$$

for $\lambda \in \mathbf{C}$ with $0 < |\Re(\lambda) - \lambda(x_0)| \le \delta$.

Proof. (1) This follows from the Floquet theory for periodic ordinary differential equations.

In preparation for the proofs of (2) and (3), we first note that by the Fredholm alternative, see [**13**, Chapter IV, Lemma 1.1, Theorem 1.1], for any $x_0 \in \Omega$ and $\lambda \in \mathbf{C}$ with $\Re(\lambda) \ne \lambda(x_0)$, and for any $v \in \overline{E}$, the equation

$$[\overline{H}(x_0) - \lambda]u = v$$

has a unique solution $u \in \overline{E}$. Denote it by $[\overline{H}(x_0) - \lambda]^{-1}v$. We show that $[\overline{H}(x_0) - \lambda]^{-1}v = u(\cdot; v)$, where $u(\cdot; v)$ is defined by

$$u(t;v) = -\int_{-\infty}^{t} \exp\left\{ \int_s^t [h(x_0,\tau) - \lambda]\, d\tau \right\} v(s)\, ds \quad \text{if} \quad \Re(\lambda) > \lambda(x_0) \tag{3.5}$$

and

$$u(t;v) = \int_t^{\infty} \exp\left\{ -\int_t^s [h(x_0,\tau) - \lambda]\, d\tau \right\} v(s)\, ds \quad \text{if} \quad \Re(\lambda) < \lambda(x_0). \tag{3.6}$$

By direct computation, we have that $u(t;v)$ is a solution of

$$[\overline{H}(x_0) - \lambda]u = v.$$

We claim that $u(\cdot; v) \in \overline{E}$, i.e., $u(t+T; v) = u(t; v)$. We prove the claim for the case $\Re(\lambda) > \lambda(x_0)$. It can be proved similarly for the case $\Re(\lambda) < \lambda(x_0)$. By (3.5),

$$\begin{aligned} u(t+T;v) &= -\int_{-\infty}^{t+T} \exp\left\{ \int_s^{t+T} [h(x_0,\tau) - \lambda]\, d\tau \right\} v(s)\, ds \\ &= -\int_{-\infty}^{t} \exp\left\{ \int_{s+T}^{t+T} [h(x_0,\tau) - \lambda]\, d\tau \right\} v(s+T)\, ds \\ &= -\int_{-\infty}^{t} \exp\left\{ \int_s^t [h(x_0,\tau) - \lambda]\, d\tau \right\} v(s)\, ds \\ &= u(t;v). \end{aligned}$$

Hence $u(\cdot; v) \in \overline{E}$. It then follows from the uniqueness of the solutions of $[\overline{H}(x_0) - \lambda]u = v$, $[\overline{H}(x_0) - \lambda]^{-1}v = u(t; v)$.

(2) We next prove (3.3) for the case $\lambda > \lambda(x_0)$; the case $\lambda < \lambda(x_0)$ may be proved similarly. Note that

$$\|[\overline{H}(x_0) - \lambda]^{-1}\| = \sup_{v \in \overline{E}, \|v\|=1} \|[\overline{H}(x_0) - \lambda]^{-1}v\|.$$

By (3.5), since the exponential is positive, we have

$$\|[\overline{H}(x_0)-\lambda]^{-1}\| = \|[\overline{H}(x_0)-\lambda]^{-1}v^*\|,$$

where $v^*(t)\equiv 1$ and

$$[\overline{H}(x_0)-\lambda]^{-1}v^* = -\int_{-\infty}^{t} \exp\left\{\int_s^t [h(x_0,\tau)-\lambda]\,d\tau\right\} ds.$$

Let n_s be the largest integer less than or equal to $(t-s)/T$. Then

$$\begin{aligned}\left|\int_{-\infty}^{t} \exp\left\{\int_s^t [h(x_0,\tau)-\lambda]\,d\tau\right\} ds\right| \\ = \int_{-\infty}^{t} \exp\left\{\int_s^{t-n_sT} [h(x_0,\tau)-\lambda]\,d\tau\right\} \\ \exp\left\{\int_{t-n_sT}^{t} [h(x_0,\tau)-\lambda]\,d\tau\right\} ds.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\int_{-\infty}^{t} \exp\left\{\int_s^t [h(x_0,\tau)-\lambda]\,d\tau\right\} ds\right| \\ &\geq M_1\int_{-\infty}^{t} \exp\left\{\int_{t-n_sT}^{t} [h(x_0,\tau)-\lambda]\,d\tau\right\} ds \\ &= M_1\int_{-\infty}^{t} \exp\left\{[\lambda(x_0)-\lambda]n_sT\right\} ds \\ &\geq M_1\int_{-\infty}^{t} \exp\left\{[\lambda(x_0)-\lambda]\left(\frac{t-s}{T}\right)T\right\} ds \\ &= \frac{M_1}{|\lambda-\lambda(x_0)|},\end{aligned}$$

where

$$M_1 = \inf_{\substack{t-n_sT\geq s\geq t-(n_s+1)T \\ x_0\in\Omega \\ 0<|\lambda-\lambda(x_0)|<\delta}} \left(\exp\left\{\int_s^{t-n_sT} [h(x_0,\tau)-\lambda]\,d\tau\right\}\right).$$

The length of the integration range is less than T, and since the integrand is independent of s, $M_1 > 0$ and (3.3) follows.

(3) We note that from (3.5)

$$\|[\overline{H}(x_0) - \lambda]^{-1}\| \le \|[\overline{H}(x_0) - \Re(\lambda)]^{-1} v^*\|$$

where $v^*(t) \equiv 1$. A very similar argument yields (3.4), and we omit the details.

Lemma 3.7. (1) $\{\lambda(x_0) | x_0 \in \Omega\} \subset \sigma(\widetilde{H})$.

(2) $\mathbf{C} \setminus \{\lambda | \inf_{x_0 \in \Omega} \lambda(x_0) \le \Re(\lambda) \le \sup_{x_0 \in \Omega} \lambda(x_0)\} \subset \rho(\widetilde{H})$.

Proof. (1) Given $\lambda = \lambda(x_0)$ for some $x_0 \in \Omega$, if $\lambda \in \rho(\widetilde{H})$, then for any $\bar{v} \in \widetilde{E}$,

$$-\frac{du}{dt} + [h(x,t) - \lambda(x_0)]u = v$$

has a unique solution $u \in \overline{E}$, where $v(x,t) = \bar{v}(t)$. This implies that for any $\bar{v} \in \overline{E}$,

$$-\frac{du}{dt} + [h(x_0,t) - \lambda(x_0)]u = \bar{v}$$

has a solution $u \in \overline{E}$. Then by the Fredholm alternative, see [**13**, Chapter IV, Lemma 1.1, Theorem 1.1], $\overline{H}(x_0)u - \lambda(x_0)u = 0$ has no nontrivial solution in $\overline{E}$, which contradicts Lemma 3.6 (1). Hence, $\lambda \in \sigma(\widetilde{H})$.

(2) Take any $\lambda \in \mathcal{C}$ with $\Re(\lambda) > \sup_{x_0 \in \Omega} \lambda(x_0)$ or $\Re(\lambda) < \inf_{x_0 \in \Omega} \lambda(x_0)$. By Lemma 3.6 (3), $\lambda \in \rho(\overline{H}(x_0))$. Also, for each $x_0 \in \Omega$,

$$([\widetilde{H} - \lambda]^{-1}u)(x_0,t) = ([\overline{H}(x_0) - \lambda]^{-1}u)(x_0,t).$$

Hence, also $\lambda \in \rho(\widetilde{H})$. □

Lemma 3.8. *Take $N = 1$. There is an $\alpha > s(\widetilde{H})$ such that*

$$r(\widetilde{X}(\widetilde{H} - \alpha)^{-1}) > 1.$$

Proof. First note that $\lambda(x_0)$ is continuous in x_0, and it follows from Lemma 3.7 that there is an $x_0 \in \Omega$ such that $\lambda(x_0) = s(\widetilde{H})$. In the rest of this proof, x_0 is the x_0 for which $\lambda(x_0) = s(\widetilde{H})$. By Lemma 3.6, $[\overline{H}(x) - \alpha]^{-1}$ exists for any $x \in \Omega$ and $\alpha > \lambda(x_0)$, and there is an $M_1 > 0$ such that if $\alpha \neq \lambda(x)$ then

$$\|[\overline{H}(x) - \alpha]^{-1}\| \geq \frac{M_1}{|\alpha - \lambda(x)|} \tag{3.7}$$

for all $x \in \Omega$ and α in a neighborhood of $\lambda(x)$. Note also that

$$\lambda(x) = \frac{1}{T}\int_0^T h(x,s)\,ds. \tag{3.8}$$

From (H1)(c), there is an $M_3 > 0$ such that

$$|h(x,s) - h(x_0,s)| \leq M_3|x - x_0|, \quad x \in \Omega,\ s \in \mathbf{R}.$$

Therefore,

$$\begin{aligned} |\lambda(x) - \lambda(x_0)| &\leq \frac{1}{T}\int_0^T |h(x,s) - h(x_0,s)|\,ds \\ &\leq M_3|x - x_0|. \end{aligned} \tag{3.9}$$

From (3.7) and (3.9),

$$\begin{aligned} \|(\overline{H}(x) - \alpha)^{-1}\| &\geq \frac{M_1}{|\alpha - \lambda(x)|} \\ &\geq \frac{M_1}{|\alpha - \lambda(x_0)| + M_3|x - x_0|}. \end{aligned} \tag{3.10}$$

Noting that, by (H2)(a), $\min_{(x,y)\in\Omega\times\Omega} K(x,y) > 0$, we deduce that for $u(x,t) \equiv 1$, there is a constant $M_4 > 0$ such that

$$\begin{aligned} |(\widetilde{X}(\widetilde{H} - \alpha)^{-1}u)(x,t)| &= \left|\int_\Omega K(x,y)((\widetilde{H} - \alpha)^{-1}u)(y,t)\,dy\right|, \\ &\geq M_4 \int_\Omega \frac{dy}{|\alpha - \lambda(x_0)| + M_3|y - x_0|} \end{aligned}$$

from (3.10). The righthand side of this inequality tends to infinity as $\alpha \to \lambda(x_0)$. This completes the proof. □

Proof of Theorem 3.1. Under the assumptions of Theorem 3.1, (1)–(3) of Theorem 3.3 are satisfied. In fact, (1) follows from (H1) and Lemma 3.5, (2) is a consequence of Lemma 3.4, and (3) follows Lemma 3.8. The assertions of Theorem 3.1 then follow except for the claims that the eigenfunction, ϕ say, is strictly positive and unique.

For the positivity, note first that for every t, there is an x_0 such that $\phi(x_0, t) > 0$. For, otherwise, for some t_0, $\phi(x, t_0) = 0$, $x \in \Omega$, and by uniqueness for the initial value problem, $\phi(x, t) = 0$, $x \in \Omega$, $t \in \mathbf{R}$, in which case ϕ is not an eigenfunction. It follows from Proposition 2.4 (2) with $p(x, t) = h(x, y) - \lambda$ that $\phi(x, t) > 0$, $x \in \Omega$, $t \in \mathbf{R}$.

The uniqueness is proved by a contradiction argument: suppose there is another eigenfunction ψ. Then one can choose $a \in \mathbf{R}$ with $a \neq 0$ such that $\omega = \phi - a\psi$ and

$$\omega(x, t) \geq 0, \quad x \in \Omega, \quad t \in \mathbf{R} \quad \text{and} \quad \omega(x_0, t_0) = 0$$

for some $x_0 \in \Omega$, $t_0 \in \mathbf{R}$. But this contradicts the conclusion of the previous paragraph and so yields uniqueness. □

To show Theorem 3.2, we first show

Lemma 3.9. $s(\widetilde{X}, \widetilde{H}) \geq s(\widetilde{H})$.

Proof. We prove the lemma by contradiction. Assume that $s(\widetilde{X}, \widetilde{H}) < s(\widetilde{H})$. Let $\lambda_0 = s(\widetilde{H})$. Then $\lambda_0 I - (\widetilde{X} + \widetilde{H})$ is invertible and

$$\begin{aligned} \lambda_0 I - \tilde{H} &= \lambda_0 I - (\tilde{X} + \tilde{H}) + \tilde{X} \\ &= \Big(\lambda_0 I - (\tilde{X} + \tilde{H})\Big)\Big(I + \big(\lambda_0 I - (\tilde{X} + \tilde{H})\big)^{-1} \tilde{X}\Big). \end{aligned}$$

By Lemma 3.7, $s(\widetilde{H}) \in \sigma(\widetilde{H})$. We then must have $-1 \in \sigma((\lambda_0 I - (\widetilde{X} + \widetilde{H}))^{-1}\widetilde{X})$. By Lemma 3.5, $\widetilde{X}$ is compact. Hence $(\lambda_0 I - (\widetilde{X} + \widetilde{H}))^{-1}\widetilde{X}$ is compact and -1 is then an isolated eigenvalue of $(\lambda_0 I - (\widetilde{X} + \widetilde{H}))^{-1}\widetilde{X}$.

Let $u_0 \in \widetilde{E}$ be a nontrivial solution of

$$\Big(I + \big(\lambda_0 I - (\widetilde{X} + \widetilde{H})\big)^{-1} \widetilde{X}\Big) u_0 = 0. \tag{3.11}$$

It follows that

$$(\lambda_0 I - \widetilde{H})u_0 = 0,$$

i.e.,

$$\frac{\partial u_0(x,t)}{\partial t} = (h(x,t) - \lambda_0)u_0(x,t). \tag{3.12}$$

This implies that

$$u_0(x,t) = u_0(x,0)e^{\int_0^t (h(x,\tau)-\lambda_0)\,d\tau}$$

for every $x \in \Omega$.

Let $v_0(x,t) = |u_0(x,t)|$. Clearly $v_0(x,t)$ is also a nontrivial solution of (3.12) and thus is a nontrivial solution of (3.11).

Note that $\widetilde{X}$ is positive. By [**10**, Theorem 1.1], $(\lambda_0 I - (\widetilde{X} + \widetilde{H}))^{-1}$ is positive. Hence, $(\lambda_0 I - (\widetilde{X} + \widetilde{H}))^{-1}\widetilde{X}$ is positive. Therefore,

$$0 = \Big(I + \big(\lambda_0 I - (\widetilde{X} + \widetilde{H})\big)^{-1}\widetilde{X}\Big)v_0 \geq v_0 \geq 0.$$

This implies that $v_0 \equiv 0$ and hence $u_0 \equiv 0$. This is a contradiction. The lemma then follows. □

Proof of Theorem 3.2. To clarify the proof, we shall slightly contract the notation and put $\lambda_s = \lambda_s(X,H)$, $\lambda_L = \lambda_L(X,H)$ and $s = s(\widetilde{X}, \widetilde{H})$. From Proposition 2.5, $\lambda_s = \lambda_L$.

We claim that $s \leq \lambda_s$. For any λ with $\Re(\lambda) > \lambda_s$,

$$\|\Phi_\lambda(t,s)\| = \|e^{-\lambda(t-s)}\Phi(t,s)\| \longrightarrow 0$$

as $t - s \to \infty$ exponentially. It then follows that, for any $v \in C(\Omega \times \mathbf{R})$ with $v(x,t+T) = v(x,t)$,

$$u(x,t) = -\int_{-\infty}^{t} \Phi_\lambda(t,s)v(x,s)\,ds$$

is the unique periodic solution of (3.1) with period T. Therefore $\lambda \in \rho(\widetilde{X}, \widetilde{H})$, and the claim follows.

We next prove that, for any $\varepsilon > 0$, $\lambda_L \leq s+\varepsilon$. Consider the equation

$$-u_t + Xu + Hu - \lambda_\varepsilon u = v, \tag{3.13}$$

where $\lambda_\varepsilon = s+\varepsilon$ and $v \in \widetilde{E}$. Since $\lambda_\varepsilon \in \rho(\widetilde{X}, \widetilde{H})$, equation (3.13) has exactly one solution $u \in \widetilde{E}$. Rewrite (3.13) as

$$\left(X - \left[\frac{\partial}{\partial t} - H + \lambda_\varepsilon\right]\right)u = v. \tag{3.14}$$

By Lemma 3.4, $\widetilde{H} = -\partial/(\partial t) + H$ generates a positive continuous semi-group of contractions on $\widetilde{E}$. By Lemma 3.9, $\lambda_\varepsilon \in \rho(\widetilde{H})$. Hence $\lambda_\varepsilon - \widetilde{H} = \partial/(\partial t) - H + \lambda_\varepsilon$ is invertible and

$$\left(\frac{\partial}{\partial t} - H + \lambda_\varepsilon\right)^{-1}$$

and

$$X\left(\frac{\partial}{\partial t} - H + \lambda_\varepsilon\right)^{-1}$$

are positive. Equation (3.13) can then be rewritten as

$$\left(X\left[\frac{\partial}{\partial t} - H + \lambda_\varepsilon\right]^{-1} - I\right)\left(\frac{\partial}{\partial t} - H + \lambda_\varepsilon\right)u = v. \tag{3.15}$$

From [**1**, Theorem 2.2 (ii)], if $\alpha = s$,

$$r\left(X\left[\frac{\partial}{\partial t} - H + \alpha\right]^{-1}\right) = 1.$$

By Lemma 3 of [**1**], $r(\cdot)$ above is a strictly decreasing continuous function of α. Hence,

$$r\left(X\left[\frac{\partial}{\partial t} - H + \lambda_\varepsilon\right]^{-1}\right) < 1.$$

Therefore, by [**21**, Proposition 4.1.1], $I - X((\partial/\partial t) - H + \lambda_\varepsilon)^{-1}$ is invertible and has a positive inverse. Now (3.13) can be rewritten as

$$u = \left(\frac{\partial}{\partial t} - H + \lambda_\varepsilon\right)^{-1}\left(X\left[\frac{\partial}{\partial t} - H + \lambda_\varepsilon\right]^{-1} - I\right)^{-1} v. \tag{3.16}$$

By the positivity of $((\partial/\partial t) - H + \lambda_\varepsilon)^{-1}$ and $[I - X((\partial/\partial t) - H + \lambda_\varepsilon)^{-1}]^{-1}$, if $v \leq 0$, then $u \geq 0$.

Take $v^*(x,t) \equiv -1$, and let u^* be the (unique) solution of (3.13) given by (3.16) in $\widetilde{E}$ with $v = v^*$. Clearly $u^* \not\equiv 0$, and by the conclusion of the last paragraph $u^* \geq 0$, so there are an s and an x such that $u^*(x,s) > 0$. Since u^* satisfies

$$\begin{aligned} u_t^* &= Xu^* + (H - \lambda_\varepsilon)u^* + 1 \\ &> Xu^* + (H - \lambda_\varepsilon)u^*, \end{aligned}$$

by Proposition 2.4 (2) with $p(x,t) = h(x,t) - \lambda$, $u^* > 0$. From periodicity and the compactness of Ω, there exists $\delta > 0$ such that $u^*(x,t) \geq \delta$, $x \in \Omega$, $t \in \mathbf{R}$.

Fix some $s \in \mathbf{R}$, and define

$$\begin{aligned} \theta(x,t) &= \Phi(t,s)u^*(x,s), \\ \omega(x,t) &= e^{\lambda_\varepsilon (t-s)}u^*(x,t). \end{aligned}$$

Simple calculations show that θ is the solution of

$$\theta_t - X\theta - H\theta = 0 \tag{3.17}$$

with $\theta(x,s) = u^*(x,s)$, and ω is the solution of

$$\omega_t - X\omega - H\omega = e^{\lambda_\varepsilon (t-s)} \tag{3.18}$$

with $\omega(x,s) = u^*(x,s)$. Therefore, $(\omega - \theta)(x,s) = 0$. Also, subtracting (3.17) from (3.18), we see that by Proposition 2.4 (1) applied to $p = \omega - \theta$, $\theta(x,t) \leq \omega(x,t)$, $x \in \Omega$, $t \in \mathbf{R}$. Thus, for all x, s, t with $s \leq t$,

$$0 \leq \Phi(t,s)u^*(x,s) \leq e^{\lambda_\varepsilon (t-s)}u^*(x,t). \tag{3.19}$$

Note next that for any $u_0 \in C(\Omega)$ with $\|u_0\| = 1$,

$$-\frac{u^*(x,t)}{\delta} \leq u_0(x) \leq \frac{u^*(x,s)}{\delta}.$$

Applying again Proposition 2.4 and using (3.19), we conclude that

$$\|\Phi(t,s)\| \leq e^{\lambda_\varepsilon (t-s)}\|u^*\|/\delta.$$

It follows from Definition 2.3 that

$$\lambda_L \leq \lambda_\varepsilon,$$

and since ε is arbitrary, $\lambda_L \leq s$. Together, with the opposite inequality, this proves the result. $\square$

Remark 3.10. By Theorem 3.1, when $N = 1$, the Lipschitz continuity of h ensures the existence of a PEV of (3.2). If $N = 2$, a smoothness condition on h is not quite enough. However, we may prove the following by a slight extension of the above argument. Let Ω have a uniform cone property: there exist $a, b > 0$ such that for any $x \in \Omega$, there is a right circular cone V_x with vertex x, opening a, height b, such that $V_x \subset \Omega$. Assume that $h(\cdot, t) \in C^1(\Omega)$ and $h_x(\cdot, t)$, the partial derivative of h with respect to x, is uniformly Lipschitz. Note that there is an $x_0 \in \Omega$ such that $\lambda(x_0) = \max_{x\in\Omega} \lambda(x)$, where $\lambda(x)$ is defined by (3.8).

Then the conclusions of Theorem 3.1 hold if $x_0 \in \operatorname{Int}(\Omega)$, where $x_0 \in \Omega$ is such that $\lambda(x_0) = \max_{x\in\Omega} \lambda(x)$.

In order to discuss the dimension issue further, let us rewrite the governing equation (2.4) slightly, by replacing X by ρX, where X is fixed and $\rho > 0$ is a parameter, obtaining

$$u_t = \rho X u + H u. \tag{3.20}$$

Here ρ is a dispersal rate, analogous to the diffusion rate for the corresponding reaction-diffusion case. To show that a PEV may *not* exist, let us consider the special case where h is independent of t, that is the stationary case, and the kernel $K \equiv 1$. It is then straightforward to show explicitly, by constructing a counterexample, that in general a PEV does not exist for small $\rho > 0$, even in the following cases.

(1) $N = 1$ and h is continuous.

(2) $N = 2$ and h satisfies the conditions of Remark 3.10 except for the restriction $x_0 \in \operatorname{Int}(\Omega)$.

(3) $N \geq 3$, Ω is the closed unit ball and $h = -r^2$ where r is the distance from the center of Ω.

In each of these, a further condition is needed, and similar remarks of course broadly apply for general K. This condition is that the dispersal rate ρ in (3.20) is large enough. It is not apparent what the implications are in applications, for example in biology. This issue raises interesting questions about the invasion of species, and further investigation is warranted. For information on invasion and its relation to the PEV for classical diffusion, see [**2**, page 220].

4. A bound for the principal eigenvalue. Here it is assumed that a strictly positive PEF ϕ exists for the periodic case. We show that a lower bound for the PEV is the PEV for the stationary case obtained by taking the time average of h. Again it is assumed that this time-averaged problem does have a strictly positive PEF, ψ.

Theorem 4.1. *Assume that* (H1) *and* (H2)(a) *hold and that* $h(x,t)$ *is periodic in* t. *Define*

$$\hat{h}(x) = \frac{1}{T}\int_0^T h(x,t)\,dt, \tag{4.1}$$

and let λ, λ^* *and* ϕ, ψ *be the* PEV*s and* PEF*s for the original problem and the time-averaged autonomous problem, respectively. That is,*

$$-\phi_t(x,t) + \int_\Omega K(x,y)\phi(y,t)\,dy + h(x,t)\phi(x,t) = \lambda\phi(x,t) \tag{4.2}$$

and

$$\int_\Omega K(x,y)\psi(y)\,dy + \hat{h}(x)\psi(x) = \lambda^*\psi(x). \tag{4.3}$$

Then $\lambda \geq \lambda^*$. *Also, if* $\lambda = \lambda^*$, *then*

$$h(x,t) = \hat{h}(x) + g(t).$$

The following corollary follows directly from Theorems 3.2 and 4.1.

Corollary 4.2. *If* $K(x,y) > 0$ *for* $x, y \in \Omega$, *then* $\lambda_s(X,H) = s(\widetilde{X},\widetilde{H}) \geq s(X,\hat{H}) = \lambda^*$, *where* $\hat{H}u = \hat{h}u$.

The proof of the theorem will depend upon a preliminary lemma, and the proof of that depends upon a Jensen inequality, viz. if f is a positive, continuous function defined on $[0,T]$ then,

$$\frac{1}{T}\int_0^T f(t)\,dt \geq \exp\left\{\frac{1}{T}\int_0^T \ln[f(t)]\,dt\right\} \tag{4.4}$$

with equality if and only if f is a constant function.

Lemma 4.3. *Let $w(x,t)$ be a positive, continuous function defined on $\Omega \times [0,T]$. Let*

$$\theta(x,y) = \frac{1}{T}\int_0^T \frac{w(y,t)}{w(x,t)}\,dt.$$

Then either *$w(x,t)$ is independent of x* or *there exists $x^* \in \Omega$ such that*

$$\theta(x^*,y) \geq 1 \quad \textit{for all } y \in \Omega$$

with strict inequality for some y.

Proof. Let

$$\chi(x) = \exp\left(\frac{1}{T}\int_0^T \ln[w(x,t)]\,dt\right), \quad x \in \Omega.$$

Then from the inequality (4.4)

$$\begin{aligned}\theta(x,y) &= \frac{1}{T}\int_0^T \frac{w(y,t)}{w(x,t)}\,dt \\ &\geq \exp\left\{\frac{1}{T}\int_0^T \ln\frac{w(y,t)}{w(x,t)}\,dt\right\} \\ &= \exp\left\{\frac{1}{T}\int_0^T \ln[w(y,t)]\,dt - \frac{1}{T}\int_0^T \ln[w(x,t)]\,dt\right\} \\ &= \frac{\exp\left\{1/T\int_0^T \ln[w(y,t)]\,dt\right\}}{\exp\left\{1/T\int_0^T \ln[w(x,t)]\,dt\right\}} \\ &= \frac{\chi(y)}{\chi(x)}.\end{aligned} \tag{4.5}$$

Now χ is a continuous function defined on the compact set Ω and so is bounded and attains its bounds. Let its least value occur at x_0. Then

$$\theta(x_0, y) \geq 1 \quad \text{for all } y \in \Omega.$$

If χ is not a constant, then the inequality will be strict for some y and the required result (with $x^* = x_0$) is established. Otherwise χ is a constant and so

$$\theta(x, y) \geq 1 \quad \text{for all } (x, y) \in \Omega \times \Omega.$$

If this last inequality is somewhere strict, say at (x_1, y_1), then the theorem is proved with $x^* = x_1$. Suppose therefore that there is equality everywhere, i.e., $\theta(x, y) \equiv 1$. This implies that there is equality in (4.5) and so there is equality in (4.4) with $f(t) = w(y, t)/w(x, t)$ and therefore $w(y, t)/w(x, t)$ is independent of t. Let

$$w(0, t) = \gamma(t) \quad \text{and} \quad \frac{w(y, t)}{w(0, t)} = F(y),$$

then

$$w(y, t) = F(y)\gamma(t)$$

and so

$$1 = \frac{1}{T} \int_0^T \frac{w(y, t)}{w(x, t)}\, dt = \frac{1}{T} \int_0^T \frac{F(y)}{F(x)}\, dt = \frac{F(y)}{F(x)}.$$

Thus, F is a constant and $w(x, t)$ depends only upon t. □

Proof of Theorem 4.1. Following from the arguments in the proof of Theorem 3.1, $\phi(x, t) > 0$ for all $t \in \mathbf{R}$ and $x \in \Omega$ and $\psi(x) > 0$ for all $x \in \Omega$. Then, from equation (4.3),

$$\lambda^* = \hat{h}(x) + \frac{\int_\Omega K(x, y)\psi(y)\, dy}{\psi(x)} \quad \text{for all } x \in \Omega,$$

and from equation (4.2),

$$\lambda = h(x, t) + \frac{\int_\Omega K(x, y)\phi(y, t)\, dy}{\phi(x, t)} - \frac{\phi_t(x, t)}{\phi(x, t)} \quad \text{for all } x, t. \tag{4.6}$$

When integrated over t, this last equation, because of equation (4.1) and ϕ being periodic in t, implies

$$\lambda = \hat{h}(x) + \frac{1}{T}\int_\Omega K(x,y)\int_0^T \frac{\phi(y,t)}{\phi(x,t)}\,dt\,dy \quad \text{for all } x \in \Omega$$

and so

$$\lambda^* - \lambda = \int_\Omega K(x,y)\left\{\frac{\psi(y)}{\psi(x)} - \frac{1}{T}\int_0^T \frac{\phi(y,t)}{\phi(x,t)}\,dt\right\}dy \quad \text{for all } x \in \Omega.$$

Set $w(y,t) = \phi(y,t)/\psi(y)$ to give

$$\lambda^* - \lambda = \int_\Omega K(x,y)\frac{\psi(y)}{\psi(x)}\left\{1 - \frac{1}{T}\int_0^T \frac{w(y,t)}{w(x,t)}\,dt\right\}dy \quad \text{for all } x \in \Omega.$$

From the lemma, we know that there exists an x^* such that when $x = x^*$ the expression within $\{\}$ is nonpositive for all y. Since $K(x,y)$ is nonnegative and ψ is positive, it follows that $\lambda^* \le \lambda$.

If K is strictly positive everywhere, then the lemma implies that either $w(y,t)$ varies with y (and hence $\lambda^* < \lambda$) or $w(y,t)$ depends only upon t (and hence $\lambda = \lambda^*$). In this latter case

$$\phi(y,t) = \psi(y)\gamma(t)$$

and so, from equation (4.6),

$$\begin{aligned}\lambda &= h(x,t) + \frac{\int_\Omega K(x,y)\psi(y)\,dy}{\psi(x)} - \frac{1}{\gamma}\frac{d\gamma}{dt}\\ &= \lambda^* = \hat{h}(x) + \frac{\int_\Omega K(x,y)\psi(y)\,dy}{\psi(x)}\\ &\Longrightarrow\ h(x,t) = \hat{h}(x) + \frac{1}{\gamma}\frac{d\gamma}{dt},\end{aligned}$$

and so h has the required form. $\square$

5. The almost-periodic case. In this section, we consider (2.4) with $h(x,\cdot)$ being almost periodic. Our aim is to obtain a lower bound for the principal dynamic spectrum point λ_s, or equivalently the principal Lyapunov exponent. This provides the natural extension of the previous section from the periodic to the AP case. Define

$$\hat{h}(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h(x,s)\,ds. \tag{5.1}$$

Of course, $\rho(X,\widehat{H})$, $\sigma(X,\widehat{H})$ and $s(X,\widehat{H})$ are defined in the obvious way for the stationary problem on $C(\Omega)$ for the operator $(X+\widehat{H})$, where X is the integral operator and $\widehat{H}$ is multiplication by $\hat{h}$.

Theorem 5.1. *Suppose that* (H1) *and* (H2)(a) *hold. Assume that a* PEV λ *exists for the stationary case with* $\hat{h}$ *defined by* (5.1), *and suppose moreover that* λ *is an isolated point of* $\sigma(X,\widehat{H})$. *Then*

$$\lambda_L(X,H) = \lambda_s(X,H) \geq s(X,\widehat{H}) = \lambda.$$

Proof. By (H1)(c), there are periodic functions $h_n(x,t)$ such that

$$h_n(x,t) \longrightarrow h(x,t) \quad \text{as } n\to\infty$$

uniformly for $x\in\Omega$ and $t\in\mathbf{R}$. Then

$$\frac{1}{t}\int_0^t \big(h(x,s)-h_n(x,s)\big)\,ds \longrightarrow 0 \quad \text{as } n\to\infty$$

uniformly in $x\in\Omega$, that is,

$$\hat{h}_n(x) \longrightarrow \hat{h}(x) \quad \text{as } n\to\infty.$$

By Proposition 2.6,

$$\lambda_s(X,\widetilde{H}_n) \longrightarrow \lambda_s(X,H) \quad \text{as } n\to\infty \tag{5.2}$$

and by Proposition 2.6 and Theorem 3.2,

$$s(X,\hat{H}_n) = \lambda_s(X,\widehat{H}_n) \longrightarrow \lambda_s(X,\widehat{H}) = s(X,\widehat{H}) \quad \text{as } n\to\infty. \tag{5.3}$$

From the assumed condition on λ, by perturbation theory for the spectrum, see [**20**, IV Section 3.5], $\lambda_s(X, \widehat{H}_n) = s(X, \widehat{H}_n)$ is an isolated PEV of $(X + \widehat{H}_n)$ for $n \gg 1$. Hence, by Theorem 4.1,

$$\lambda_s(X, \widetilde{H}_n) \geq \lambda_s(X, \widehat{H}_n) \tag{5.4}$$

for $n \gg 1$. It then follows from (5.2)–(5.4) that

$$\lambda_s(X, H) \geq s(X, \widehat{H}). \qquad \square$$

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