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MARKOV CHAINS: ERGODICITY IN TIME-DISCRETE CASES

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ABSTRACT. The aim of this article is to propose some localization methods for checking ergodicity of time-discrete finite Markov chains. The numerical example allows us to compare these methods in order to find the optimal one. The obtained results are generalized to the n-dimensional case and can be used as a reliable criterion to establish ergodicity. Some explicit theoretical results are given as well.

1. Introduction. Let the matrix $M = (m_{ij})$ be Markov. This means that it is nonnegative and the sum of elements in each matrix column is equal to 1:

$$m_{ij} \geq 0, \quad i,j = 1,\dots,n, \quad \text{and} \tag{1.1}$$

$$\sum_{i=1}^{n} m_{ij} = 1, \quad j = 1,\dots,n. \tag{1.2}$$

If the matrix is Markov, then there exists a critical eigenvalue, equal to 1, that corresponds to the critical eigenvector on the lefthand side: $\mathbf{1} = (1,\dots,1)$, because

$$\mathbf{1}M = \mathbf{1}. \tag{1.3}$$

We call this the Markov property of the matrix M.

Let L be an $(n-1)$-dimensional subspace of the real n-dimensional linear space $\mathbf{R}^n$ consisting of those column vectors x with components which sum to zero:

$$L = \left\{ x \in \mathbf{R}^n : \sum_{i=1}^{n} x_i = 0 \right\}. \tag{1.4}$$

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Moreover, let W be the simplex in the space $\mathbf{R}^n$ consisting of all vectors with nonnegative components with sum equal to 1:

$$W = \left\{ x \in \mathbf{R}^n : x_1 \geq 0, \dots , x_n \geq 0, \ \sum_{i=1}^{n} x_i = 1 \right\}. \tag{1.5}$$

Definition 1.1. A Markov chain, described by the linear difference equation

$$x(t+1) = Mx(t),$$

where M is a Markov matrix and where $t \in \mathbf{N}r$, is called ergodic if for any initial condition $x(0) \in W$ its solution $x(t) = M^t x(0)$ has a limit $\pi \in W$ as $t \to \infty$ and this limit does not depend on the choice of the initial condition.

If the matrix M is considered as an operator, then it maps the simplex W into itself. Since a finite dimensional matrix M determines a continuous operator, this mapping is continuous. The set W is compact and convex. Therefore, by applying Brouwer's fixed point theorem, the matrix M, considered as an operator, has a stationary point (equilibrium state) in W. In general, there exist several such equilibrium states. They form a closed convex set in W. Ergodicity means that there exists only one equilibrium state π and, furthermore, $\|x(t) - \pi\| \to 0$, as $t \to \infty$.

Next we fix the Markov matrix M, and we consider perturbations of the form $M - u\mathbf{1}$, where u is a column vector with components $(u_1, \dots , u_n)$, and $\mathbf{1}$ is the row vector with all components equal to 1; then,

$$M - u\mathbf{1} = \begin{pmatrix} m_{11} - u_1 & \dots & m_{1n} - u_1 \\ \vdots & \ddots & \vdots \\ m_{n1} - u_n & \dots & m_{nn} - u_n \end{pmatrix}. \tag{1.6}$$

These perturbations are admissible in the sense that $(M - u\mathbf{1})|L = M|L$.

2. Localization methods for checking ergodicity of finite discrete Markov chains. If in the subspace L we select an appropriate basis and write the matrix of the operator $M|L$ using this basis, it will be a matrix of order $(n-1)$ and its eigenvalues coincide with those of the matrix M, except 1 counted once. Ergodicity means that the eigenvalues of the new matrix $M|L$ lie in the open unit disc [**2**]. Therefore the following statement is true:

Theorem 2.1. *The Markov chain is ergodic if and only if the operator $M|L$ has a spectrum in the open unit disc, or, in other words, its spectral radius is strictly less than* 1

$$\rho(M|L) < 1. \tag{2.1}$$

If the Markov matrix M is ergodic and the vector u in the space $\mathbf{R}^n$ meets the condition

$$|1 - \mathbf{1}u| < 1, \tag{2.2}$$

then the spectral radius of the perturbed matrix $M - u\mathbf{1}$ is strictly less than 1, *i.e.,*

$$\rho(M - u\mathbf{1}) < 1. \tag{2.3}$$

Conversely, if for some vector $u \in \mathbf{R}^n$ inequality (2.3) *is true for the Markov matrix M, then the matrix M is ergodic, i.e., inequality* (2.1) *is true, and u meets condition* (2.2).

If condition (2.3) *is fulfilled, the stationary probability distribution π is described by the following formula:*

$$\pi = (I - (M - u\mathbf{1}))^{-1}u. \tag{2.4}$$

To prove (2.1) it suffices to find, among these perturbed matrices, one with a spectral radius strictly less than 1.

In order to estimate the spectral radius of the matrix M of order n we use the special vector space norm described in the following definition.

Definition 2.2. For $\alpha \in [0,1]$ and $M = (m_{ij})_{i,j=1}^n$, written in an appropriate basis, the special α-norm $\|M\|_\alpha$ is described by

$$\|M\|_\alpha = \max_{1\le i\le n}\left\{(1-\alpha)\sum_{j=1}^n |m_{ij}| + \alpha\sum_{j=1}^n |m_{ji}|\right\}. \tag{2.5}$$

According to this definition, for a perturbed matrix with elements $m_{ij} - u_i$ we have:

$$\|M - u\mathbf{1}\|_\alpha = \max_{1\le i\le n}\left\{(1-\alpha)\sum_{j=1}^n |m_{ij} - u_i| + \alpha\sum_{j=1}^n |m_{ji} - u_j|\right\}. \tag{2.6}$$

For a fixed parameter $\alpha \in [0,1]$, the following formulas are known [**6**]:

$$\rho(M) \le \max_{1\le i\le n}\left\{|m_{ii}| + \left(\sum_{j\ne i}|m_{ij}|\right)^{1-\alpha}\left(\sum_{j\ne i}|m_{ji}|\right)^{\alpha}\right\}, \quad 0\le\alpha\le 1, \tag{2.7}$$

and

$$\rho(M) \le \max_{1\le i\le n}\left\{(1-\alpha)\sum_{j=1}^n |m_{ij}| + \alpha\sum_{j=1}^n |m_{ji}|\right\}. \tag{2.8}$$

This last inequality (2.8) is a consequence of (2.7) together with Young's inequality, which reads as follows,

$$uv \le \alpha u^{1/\alpha} + \beta v^{1/\beta}, \quad \alpha,\beta > 0, \quad \alpha+\beta = 1. \tag{2.9}$$

From the latter estimate we get the first criterion of ergodicity.

Proposition 2.3 (First criterion). *Let, for some parameter value* $\alpha \in [0,1]$, *the following* n *conditions be fulfilled*

$$|m_{ii} - u_i| + (1-\alpha)\sum_{j\ne i}|m_{ij} - u_i| + \alpha\sum_{j\ne i}|m_{ji} - u_j| < 1, \qquad i = 1,\dots,n, \tag{2.10}$$

where $u_1, u_2, \dots, u_n$ are some nonnegative numbers. Then the Markov matrix M is ergodic.

The function $v(u,\alpha)$ is defined by the expression in (2.6):

(2.11)
$$v(u,\alpha) = \|M - u\mathbf{1}\|_\alpha = \max_{1\le i\le n}\left\{(1-\alpha)\sum_{j=1}^n |m_{ij}-u_i| + \alpha\sum_{j=1}^n |m_{ji}-u_j|\right\}.$$

The disadvantage of the formulated criterion is the following. Each time we have to apply this criterion we need to find an explicit vector u and an explicit value of the parameter α satisfying (2.10) instead of indicating general conditions under which these objects exist.

Next we propose a second criterion of ergodicity. It is stronger and follows from (2.7).

Proposition 2.4 (Second criterion). *Let for some parameter value $\alpha \in [0,1]$ the following conditions be satisfied:*

$$|m_{ii} - u_i| + \left(\sum_{j\ne i} |m_{ij} - u_i|\right)^{1-\alpha} \left(\sum_{j\ne i} |m_{ji} - u_j|\right)^{\alpha} < 1, \qquad i = 1, \dots, n, \tag{2.12}$$

where again $u_1, u_2, \dots, u_n$ are some nonnegative numbers. Then the Markov matrix M is ergodic.

Using condition (2.12) the following function serves as an estimate of the spectral radius of a Markov matrix M:

$$\omega(u,\alpha) = \max_{1\le i\le n}\left\{ |m_{ii} - u_i| + \left(\sum_{j\ne i} |m_{ij} - u_i|\right)^{1-\alpha} \times \left(\sum_{j\ne i} |m_{ji} - u_j|\right)^{\alpha}\right\}. \tag{2.13}$$

Here $u = (u_1, \dots, u_n)$ and $\alpha \in [0,1]$.

Thus, in order to check ergodicity of the Markov matrix M we have to estimate the spectral radius of the perturbed matrix $M - u\mathbf{1}$ and to convince ourselves that the conditions (2.2) and (2.3) are true.

We have given two different methods to do this by minimizing the functions $v(u,\alpha)$ and $\omega(u,\alpha)$ over the parameter α and over the vector $u = (u_1, \dots, u_n)$.

In order to examine the sharpness of the estimation for $\rho(M|L)$ we compute the eigenvalues of the matrix M in the subspace L and compare them with the minimum values of the functions $v(u,\alpha)$ and $\omega(u,\alpha)$. The results of these two methods are illustrated by the next example.

Example $n = 4$.

$$M = \begin{pmatrix} 0.1 & 0 & 0.24 & 0.3 \\ 0.2 & 0.34 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.05 & 0.5 \\ 0.6 & 0.36 & 0.61 & 0.1 \end{pmatrix}.$$

The four eigenvalues of the matrix M are:

$$\lambda_1 = 1; \quad \lambda_2 = -0.5134; \quad \lambda_3 = -0.0461; \quad \lambda_4 = 0.1495.$$

First method. (i) optimal parameter $\alpha = 0.5087$.

(ii) estimate of the spectral radius $\rho = 0.5272$.

(iii) optimal vector $u = (0.1205; 0.3407; 0.4905; 0.3110)$.

In this case the perturbed matrix has the form:

$$M - u\mathbf{1} = \begin{pmatrix} -0.0205 & -0.1205 & 0.1195 & 0.1795 \\ -0.1407 & -0.0007 & -0.2407 & -0.2407 \\ -0.3905 & -0.1905 & -0.4405 & 0.0095 \\ 0.289 & 0.049 & 0.299 & -0.211 \end{pmatrix}$$

The eigenvalues of this matrix are

$$\lambda_1 = -0.5134; \quad \lambda_2 = 0.1495; \quad \lambda_3 = -0.0461; \quad \lambda_4 = -0.2627.$$

Since the spectral radius ρ is the modulus of the largest eigenvalue, we have $\rho(M - u\mathbf{1}) = 0.5134$. This is in agreement with (2.7). It also shows that condition (2.3) is satisfied. Thus, we have

$$\rho(M - u\mathbf{1}) \leq \|M - u\mathbf{1}\|_\alpha = v(u,\alpha) < 1.$$

Second method. (i) optimal parameter $\alpha = 0.6103$,

(ii) estimate of the spectral radius $\rho = 0.5236$

(iii) optimal vector $u = (0.0045; 0.4362; 0.5731; 0.2514)$.

In this case the perturbed matrix has the form:

$$M - u\mathbf{1} = \begin{pmatrix} 0.0955 & -0.0045 & 0.2355 & 0.2955 \\ -0.2362 & -0.0962 & -0.3362 & -0.3362 \\ -0.4731 & -0.2731 & -0.5231 & -0.0731 \\ 0.3486 & 0.1086 & 0.3586 & -0.1514 \end{pmatrix}.$$

Its eigenvalues are:

$$\lambda_1 = -0.5134; \quad \lambda_2 = 0.1495; \quad \lambda_3 = -0.0461; \quad \lambda_4 = -0.2652.$$

Here the spectral radius of the perturbed matrix is given by $\rho(M - u\mathbf{1}) = 0.5134$. This is in agreement with (2.3). Thus, we get

$$\rho(M - u\mathbf{1}) \le \omega(u, \alpha) < 1.$$

Remark. As we see both methods give the same three of the four eigenvalues for the perturbed matrices. Hence, it looks as if three of the eigenvalues of the perturbed matrix $M - u\mathbf{1}$ do not depend on the vector u and that the fourth eigenvalue of the perturbed matrix does not depend on the entries of the Markov matrix M.

First we shall show this in the four-dimensional case, where we have

$$(2.14) \quad \det(M - u\mathbf{1} - \lambda I)$$
$$= \det \begin{pmatrix} m_{11} - u_1 - \lambda & m_{12} - u_1 & m_{13} - u_1 & m_{14} - u_1 \\ m_{21} - u_2 & m_{22} - u_2 - \lambda & m_{23} - u_2 & m_{24} - u_2 \\ m_{31} - u_3 & m_{32} - u_3 & m_{33} - u_3 - \lambda & m_{34} - u_3 \\ m_{41} - u_4 & m_{42} - u_4 & m_{43} - u_4 & m_{44} - u_4 - \lambda \end{pmatrix}$$
$$= \lambda^4 - (\operatorname{trace} M - \operatorname{trace}(u\mathbf{1}))\lambda^3$$
$$+ \frac{1}{2}((\operatorname{trace} M - \operatorname{trace}(u\mathbf{1}))^2 - \operatorname{trace}((M - u\mathbf{1})^2))\lambda^2$$
$$- \frac{1}{6}((\operatorname{trace} M - \operatorname{trace}(u\mathbf{1}))^3$$
$$- 3(\operatorname{trace} M - \operatorname{trace}(u\mathbf{1}))\operatorname{trace}((M - u\mathbf{1})^2)$$
$$+ 2\operatorname{trace}((M - u\mathbf{1})^3))\lambda + \det(M - u\mathbf{1}).$$

Using the Markov properties of the matrix M:

$$\begin{aligned} m_{11}+m_{21}+m_{31}+m_{41}&=1;\\ m_{12}+m_{22}+m_{32}+m_{42}&=1;\\ m_{13}+m_{23}+m_{33}+m_{43}&=1;\\ m_{14}+m_{24}+m_{34}+m_{44}&=1, \end{aligned}$$

and the fact that for any Markov matrix M:

$$\begin{gathered} (u\mathbf{1})M = u\mathbf{1}, \quad \operatorname{trace}(M(u\mathbf{1})) = \operatorname{trace}(u\mathbf{1}), \quad \text{and} \\ \operatorname{trace}(u\mathbf{1}v\mathbf{1}) = \operatorname{trace}(u\mathbf{1})\operatorname{trace}(v\mathbf{1}), \end{gathered} \tag{2.15}$$

together with

$$\begin{aligned} \det M = {}& \frac{1}{6}(\operatorname{trace} M)^3 - \frac{1}{2}\operatorname{trace} M \operatorname{trace}(M^2) + \frac{1}{3}\operatorname{trace}(M^3) \\ & - \frac{1}{2}(\operatorname{trace} M)^2 + \frac{1}{2}\operatorname{trace}(M^2) - 1 + \operatorname{trace} M, \end{aligned} \tag{2.16}$$

we obtain the following characteristic polynomial, see equality (2.18) and (2.19) of Proposition 2.5,

$$\begin{aligned} &\lambda^4 - (\operatorname{trace} M - \operatorname{trace}(u\mathbf{1}))\lambda^3 \\ &+ \frac{1}{2}((\operatorname{trace} M)^2 - \operatorname{trace}(M^2) + 2(1-\operatorname{trace} M)\operatorname{trace}(u\mathbf{1}))\lambda^2 \\ &- \frac{1}{6}((\operatorname{trace} M)^3 - 3\operatorname{trace} M \operatorname{trace}(M^2) + 2\operatorname{trace}(M^3))\lambda \\ &+ \frac{1}{2}((\operatorname{trace} M)^2 - \operatorname{trace}(M^2) - 2\operatorname{trace} M + 2)\operatorname{trace}(u\mathbf{1})\lambda \\ &+ (1-\operatorname{trace}(u\mathbf{1}))\det M = (\lambda - 1 + \operatorname{trace}(u\mathbf{1}))\frac{\det(\lambda I - M)}{\lambda - 1}. \end{aligned} \tag{2.17}$$

The zeros of this polynomial are the eigenvalues of the matrix $M - u\mathbf{1}$. One of these eigenvalues is $1 - \operatorname{trace}(u\mathbf{1})$, the others are eigenvalues of the matrix M.

In equality (2.17) we used assertion (b) and (c) of the following proposition, for $n = 4$.

Proposition 2.5. *For a general $n \times n$ matrix X write*

$$\det(X - \lambda I) = \sum_{j=0}^{n} (-1)^{n-j} a_j(X)\lambda^{n-j}.$$

(a) *Then* $a_0(X) = 1$, $a_1(X) = \text{trace}\,(X)$, $a_2(X) = (1/2)((\text{trace}\,(X))^2 - \text{trace}\,(X^2))$, $a_n(X) = \det(X)$. *In general* $a_j(X)$ *is a linear combination of expressions of the form*

$$\text{trace}\,(X^{j_1})\,\text{trace}\,(X^{j_2})\cdots\text{trace}\,(X^{j_k}),$$

with $j_1 + j_2 + \cdots + j_k = j$, $j_l \in \mathbf{N}$, $1 \le l \le k$, $1 \le k \le j$.

(b) *The following relationship exists between the coefficients* $a_j(M)$, $0 \le j \le n$, *of a Markov matrix* M:

$$\sum_{j=0}^{n} (-1)^j a_j(M) = 0. \tag{2.18}$$

(c) *Let* M *be a Markov matrix, and* u *a vector in* $\mathbf{R}^n$. *Then*

(2.19)
$$a_j(M - u\mathbf{1}) = a_j(M) + \text{trace}\,(u\mathbf{1}) \sum_{k=0}^{j-1} (-1)^{j-k} a_k(M), \quad 1 \le j \le n.$$

(d) *Again, let* M *be a Markov matrix and* u *a vector in* $\mathbf{R}^n$. *Then the following identities are valid*:

(2.20)
$$\begin{aligned}
&a_0(M - u\mathbf{1}) = 1;\\
&a_1(M - u\mathbf{1}) = \text{trace}\ M - \text{trace}\,(u\mathbf{1});\\
&a_2(M - u\mathbf{1}) = \frac{1}{2}((\text{trace}\ M - \text{trace}\,(u\mathbf{1}))^2 - \text{trace}\,((M - u\mathbf{1})^2));\\
&a_n(M - u\mathbf{1}) = \det(M - u\mathbf{1}) = (1 - \text{trace}\,(u\mathbf{1}))\det M.
\end{aligned}$$

Proof. A proof of assertion (a) can be found in [**8**].

(b) By definition we have $\det(M - I) = \sum_{j=0}^{n} (-1)^{n-j} a_j(M)$. If, in addition, M is Markov, then $\det(M - I) = 0$, and the conclusion in (b) follows.

(c) For general $n \in \mathbf{N}$ and an arbitrary vector $u \in \mathbf{R}^n$ we have:

$$\begin{aligned}
\det(M - u\mathbf{1} - \lambda I) &= \frac{\lambda - 1 + \text{trace}\,(u\mathbf{1})}{\lambda - 1} \det(M - \lambda I)\\
&= \frac{\lambda - 1 + \text{trace}\,(u\mathbf{1})}{\lambda - 1} \sum_{j=0}^{n} (-1)^{n-j} a_j(M) \lambda^{n-j}
\end{aligned}$$

(use the following equalities $\det(M-I) = \sum_{j=0}^{n}(-1)^{n-j}a_j(M) = 0$)

(2.21)

$$\begin{aligned}
&= \frac{\lambda - 1 + \operatorname{trace}(u\mathbf{1})}{\lambda - 1} \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M)(\lambda^{n-j}-1) \\
&= (\lambda - 1 + \operatorname{trace}(u\mathbf{1})) \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M) \sum_{k=0}^{n-j-1}\lambda^k \\
&= \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M) \sum_{k=0}^{n-j-1}\lambda^{k+1} - \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M) \sum_{k=0}^{n-j-1}\lambda^k \\
&\quad + \operatorname{trace}(u\mathbf{1}) \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M) \sum_{k=0}^{n-j-1}\lambda^k \\
&= \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M) \sum_{k=1}^{n-j}\lambda^k - \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M) \sum_{k=0}^{n-j-1}\lambda^k \\
&\quad + \operatorname{trace}(u\mathbf{1}) \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M) \sum_{k=0}^{n-j-1}\lambda^k \\
&= \sum_{k=1}^{n}\sum_{j=0}^{n-k}(-1)^{n-j}a_j(M)\lambda^k - \sum_{k=0}^{n-1}\sum_{j=0}^{n-1-k}(-1)^{n-j}a_j(M)\lambda^k \\
&\quad + \operatorname{trace}(u\mathbf{1}) \sum_{k=0}^{n-1}\sum_{j=0}^{n-1-k}(-1)^{n-j}a_j(M)\lambda^k \\
&= \sum_{k=1}^{n-1}\sum_{j=0}^{n-k}(-1)^{n-j}a_j(M)\lambda^k - \sum_{k=1}^{n-1}\sum_{j=0}^{n-1-k}(-1)^{n-j}a_j(M)\lambda^k \\
&\quad + (-1)^n a_0(M)\lambda^n - \sum_{j=0}^{n-1}(-1)^{n-j}a_j(M) \\
&\quad + \operatorname{trace}(u\mathbf{1}) \sum_{k=0}^{n-1}\sum_{j=0}^{n-1-k}(-1)^{n-j}a_j(M)\lambda^k \\
&= \sum_{k=1}^{n-1}(-1)^k a_{n-k}(M)\lambda^k + (-1)^n a_0(M)\lambda^n - \sum_{j=0}^{n}(-1)^{n-j}a_j(M)
\end{aligned}$$

$$+ (-1)^0 a_n(M) + \operatorname{trace}(u\mathbf{1}) \sum_{k=0}^{n-1} \sum_{j=0}^{n-1-k} (-1)^{n-j} a_j(M)\lambda^k$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^k a_{n-k}(M)\lambda^k + \operatorname{trace}(u\mathbf{1}) \sum_{k=0}^{n-1} \sum_{j=0}^{n-1-k} (-1)^{n-j} a_j(M)\lambda^k$$

Since we also have

$$\det(M - u\mathbf{1} - \lambda I) = \sum_{k=0}^{n} (-1)^k a_{n-k}(M - u\mathbf{1})\lambda^k,$$

from equality (2.21) we obtain, for $0 \le k \le n-1$,

$$\begin{aligned} (-1)^k a_{n-k}(M - u\mathbf{1}) &= (-1)^k a_{n-k}(M) \\ &\quad + \operatorname{trace}(u\mathbf{1}) \sum_{j=0}^{n-1-k} (-1)^{n-j} a_j(M), \end{aligned} \tag{2.22}$$

or, equivalently, the equality in (2.19).

(d) This assertion follows from (a). □

3. Conclusion. The main issue related to a time-discrete finite Markov chain consists in investigating its ergodicity. The methods given in this article are based on using the elements of the Markov matrix $M = (m_{ij})$, see (1.1) and (1.2), as well as the elements of the perturbed matrix $M - u\mathbf{1}$, see (1.6).

In order to establish the ergodicity of the Markov chain M it is required that the absolute values of the noncritical eigenvalues of the matrix M are strictly less than 1 or, equivalently, that, for some vector $u \in \mathbf{R}^n$, the eigenvalues of the perturbed matrix $M - u\mathbf{1}$ have absolute values strictly less than 1, see Theorem 2.1. In the example we first compute the eigenvalues of the given Markov matrix and then we check its ergodicity by two different methods. In order to find the optimal method we finally compare the corresponding two criteria.

The eigenvalues of the matrix depend only on its matrix elements. Since we are only interested in the eigenvalues that are less than 1, we use two localization methods for the matrix eigenvalues.

Unfortunately, both methods have the following disadvantage. We need to find an explicit vector $u \in \mathbf{R}^n$ and an explicit value of the parameter α which satisfy (2.10) and (2.12), respectively, instead of indicating general conditions under which such objects exist. Our numerical example is in agreement with the observation that the second method is stronger than the first one. Some of the above results can be obtained via the following Theorem 3.1. In particular, Theorem 2.1 follows from its Corollary 3.2. The result of Theorem 3.1 was inspired by the result in the four-dimensional case (2.17); a proof of the latter equality was based on Proposition 2.5. However, the proof of the following theorem is in fact independent of Proposition 2.5.

Theorem 3.1. *Let M be an arbitrary $n \times n$ Markov matrix. Then the following identities are valid:*

$$((\lambda-1)I + u\mathbf{1})(\lambda I - M) = (\lambda-1)(\lambda I + u\mathbf{1} - M); \tag{3.1}$$

$$\det((\lambda-1)I + u\mathbf{1}) = (\lambda-1)^{n-1}(\lambda - 1 + \operatorname{trace}(u\mathbf{1})); \tag{3.2}$$

$$(\lambda-1)\det(\lambda I + u\mathbf{1} - M) = (\lambda - 1 + \operatorname{trace}(u\mathbf{1}))\det(\lambda I - M). \tag{3.3}$$

Proof. In (3.1) the Markov property of the matrix M is heavily used: see (2.15). Equality (3.2) follows via mathematical induction. More precisely, developing $\det((\lambda-1)I + u\mathbf{1})$ with respect to the first row shows

$$\det\begin{pmatrix} \lambda-1+u_1 & u_1 & \cdots & u_1 \\ u_2 & \lambda-1+u_2 & \cdots & u_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_n & u_n & \cdots & \lambda-1+u_n \end{pmatrix}$$
$$= (\lambda-1+u_1)\det\begin{pmatrix} \lambda-1+u_2 & \cdots & u_2 \\ \vdots & \ddots & \vdots \\ u_n & \cdots & \lambda-1+u_n \end{pmatrix}$$
$$- u_1\det\begin{pmatrix} u_2 & u_2 & \cdots & u_2 \\ u_3 & \lambda-1+u_3 & \cdots & u_3 \\ \vdots & \vdots & \ddots & \vdots \\ u_n & u_n & \cdots & \lambda-1+u_n \end{pmatrix} + \cdots$$

$$+ (-1)^n \det \begin{pmatrix} u_2 & \lambda - 1 + u_2 & \cdots & u_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & u_{n-1} & \cdots & \lambda - 1 + u_{n-1} \\ u_n & u_n & \cdots & u_n \end{pmatrix}$$

(in the first term we use the induction hypothesis; in the second we use the first row to simplify and calculate the determinant; in the third term we use the second row and in the last term the $n-1$th row)

$$\begin{aligned} &= (\lambda - 1 + u_1)(\lambda - 1)^{n-2}(\lambda - 1 + u_2 + \cdots + u_n) \\ &\quad - u_1 u_2 (\lambda - 1)^{n-2} - \cdots - u_1 u_n (\lambda - 1)^{n-2} \\ &= (\lambda - 1)^{n-1}(\lambda - 1 + \operatorname{trace}(u\mathbf{1})). \end{aligned}$$

Notice that (3.3) is a consequence of (3.1) and (3.2) and the fact that $\det(AB) = \det(A)\det(B)$ for general square matrices A and B. □

As a corollary we have the following; it is in agreement with Theorem 3.1.

Corollary 3.2. *Let M be a Markov matrix and u a vector in $\mathbf{R}^n$. Then*

$$\sigma(M - u\mathbf{1}) \cup \{1\} = \sigma(M) \cup \{1 - \operatorname{trace}(u\mathbf{1})\}. \tag{3.4}$$

Proof. This assertion follows from equality (3.3) together with the remark that 1 is always an eigenvalue of the matrix M. □

As a conclusion for the n-dimensional case we proved that of the n eigenvalues of the perturbed matrix $M - u\mathbf{1}$ there are $n-1$ which do not depend on the vector u. The other eigenvalue, of the perturbed matrix, does not depend on the entries of the Markov matrix M. As we have seen, this was suggested by a numerical example.

The example shows us that the exact estimates of the spectral radius $\rho(M|L)$ and their approximate values are quite close to each other. But generally the second method, using the function $\omega(u,\alpha)$, gives us a better result than the first method, using the function $\upsilon(u,\alpha)$.

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