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ON QUADRATIC FORMS AND GALOIS COHOMOLOGY

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1. Introduction. In the articles [**4-8**], we investigate certain problems that arise naturally in the algebraic theory of quadratic forms and make progress toward their solution. The basic motivation for studying these problems is the interaction between the algebraic theory of quadratic forms over fields and Galois cohomology. In this survey, we discuss some of these problems and results obtained so far.

Throughout, F will denote an arbitrary field of characteristic different from two. We denote by WF the *Witt ring* of equivalence classes of quadratic forms over F and by IF the ideal of even dimensional forms in $W F$. (See [**23**] or [**29**] for terminology from quadratic form theory.) We let $I^nF = (IF)^n$ and denote by $GW\,F$ the associated graded ring, i.e., $GW\,F := \oplus_{i=0}^{\infty}\overline{I}^n F = \oplus_{i=0}^{\infty} I^nF/I^{n+1}F$. We call $GW\,F$ the *graded Witt ring of* F. We let $H^nF := H^n(G_F, \mathbf{Z}/2\mathbf{Z})$, where G_F denotes the absolute Galois group of F and $H^*F := \oplus_{i=0}^{\infty} H^nF$. With the cup product, H^*F is a graded ring, the '*full*' mod 2 *cohomology ring* of F. (See [**28**] for terminology from Galois cohomology theory.) One would like to determine the relationship between the $GW\,F$ and H^*F.

The classical invariants give rise to homomorphisms:

$e_F^0 : W\,F \to H^0F$	dimension mod 2
$e_F^1 : IF \to H^1F$	(signed) discriminant
$e_F^2 : I^2F \to H^2F$	Clifford invariant.

It is natural to ask if this sequence of invariants continues. By analogy with e_F^0, e_F^1, and e_F^2, the general invariant should be a homomorphism $e_F^n : I^nF \to H^nF$ that maps (the class of) the *n-fold Pfister form* $\langle 1, -a_1\rangle \cdots \langle 1, -a_n\rangle$ to $(a_1) \cup \cdots \cup (a_n)$. Here $(a) \in H^1F \simeq \dot{F}/\dot{F}^2$ corresponds to the square class of a. By [**1**] or [**12**], this assignment is indeed a well-defined map on the set of n-fold Pfister forms in $W\,F$.

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Since the n-fold Pfister forms generate I^nF, the question is whether this map extends to a homomorphism on I^nF. If it does, we simply say e_F^n is well-defined.

In [**1**] it was shown that e_F^3 is well-defined. But if $n \geq 4$, it is still not known whether e_F^n is always well-defined. The *Stiefel-Whitney class* of Delzant (cf. [**11**] and [**26**]) gives rise to homomorphisms w_n from I^nF to H^NF, where $N = 2^{n-1}$. These satisfy the formula $w_n = (-1)^{N-n} \cup e_F^n$. It follows that e_F^n is 'stably well-defined'. But it also follows that the w_n cannot be used to classify quadratic forms in general (cf. [**14**]). It is known by [**1**] or [**12**] that e_F^n, when well-defined, factors through $I^{n+1}F$, and so induces a homomorphism $\overline{e}_F^n$ on $\overline{I}^nF$. In fact, $\overline{e}_F^0, \overline{e}_F^1$, and $\overline{e}_F^2$ are isomorphisms, the last by a celebrated result of Merkurjev (cf. [**25**] or [**3**]). It is natural to ask the stronger question of whether $\overline{e}_F^* : GW\,F \to H^*F$ is a well-defined isomorphism. This fundamental problem in the theory of quadratic forms is the motivation for our studies. (Added in proof: Jacob and Rost have now shown that e_F^4 is always well-defined.)

2. Problems. The fundamental problem described above seems to be extremely difficult. Therefore, we have restricted our attention to special cases and modifications. In this section, we formulate some of these and, for convenience, shall number them. Some results proved in [**4**] and further problems will be discussed in the subsequent sections.

One natural modification is to use the *quadratic* mod 2 *cohomology ring* $H_q^*F := H^*(\mathrm{Gal}(F_q/F), \mathbf{Z}/2\mathbf{Z})$ of F in place of the 'full' mod 2 cohomology ring. Here F_q denotes the *quadratic closure* (i.e., 2-closure) of F. Indeed one should only expect quadratic form theory over F to give information about 2-extensions of F, viz., about the Galois group $\mathrm{Gal}(F_q/F)$. As before, the assignment $\langle 1, -a_1\rangle \cdots \langle 1, -a_n\rangle \to (a_1) \cup \cdots \cup (a_n)$ is a well-defined map from the set of n-fold Pfister forms in WF to H_q^nF. Here (a) is viewed in $H_q^1F \simeq \dot{F}/\dot{F}^2$. After making the obvious definitions, the modified problem now becomes

PROBLEM 1. *Is $\overline{e}_{q,F}^* : GW\,F \to H_q^*F$ a well-defined isomorphism?*

As noted in [**4**], if $\overline{e}_{q,F}^* : GW\,F \to H_q^*F$ is an isomorphism for all fields F, then so is $\bar{e}_F^* : GW\,F \to H^*F$ and conversely. Of course, Problem 1 above is closely related to the original problem through the inflation maps inf : $H_q^nF \to H^nF$ of group cohomology. In fact, the inflation is an isomorphism for $n = 0, 1, 2$, the last being a consequence of Merkurjev's Theorem (cf. [**25**] or [**3**]). Thus we raise

PROBLEM 2. *Is the inflation map* inf : $H_q^*F \to H^*F$ *an isomorphism?*

As noted in [**4**], if $\bar{e}_{q,F}^* : GW\,F \to H_q^*F$ is an isomorphism for all fields F, then so is inf : $H_q^*F \to H^*F$.

Clearly, if $\bar{e}_F^*$ is an isomorphism then H^*F is generated by H^1F as a ring. Such a conclusion is interesting in its own right. Therefore, we raise

PROBLEM 3. *Is H^*F (respectively, H_q^*F) generated by H^1F (respectively, by H_q^1F) as a ring?*

Note that H^*F generated by H^1F implies that every cohomology class in H^*F is split by an elementary abelian 2-extension.

Problem 3 can be expanded to finding a presentation for H^*F. This is the approach taken by Milnor in [**26**]. Inspired by the *universal Steinberg symbol*, he defined a higher K-theory for fields using generators and relations. He then defined a morphism $h_F^* : k_*F \to H^*F$ of graded rings, where $k_*F := K_*F/2K_*F$. There is also a corresponding morphism $h_{q,F}^* : k_*F \to H_q^*F$ such that $h_F^* = \inf \circ h_{q,F}^*$. Problem 3 is precisely the question of whether these two maps are surjective. It is known that the morphisms h_F^n are isomorphisms for $n \leq 3$. The case $n = 2$ is Merkurjev's Theorem and the case $n = 3$ was recently proven by Rost in [**27**]. (Added in proof: Murkurjev and Suslin also have proved this.) It follows that the same holds for $h_{q,F}^n$ for $n \leq 3$. Hence we raise

PROBLEM 4. *Is $h_F^* : k_*F \to H^*F$ (or $h_{q,F}^* : k_*F \to H_q^*F$) an isomorphism?*

There is an epimorphism $g_F^* : k_*F \to GW\,F$ such that $h_F^* = \overline{e}_F^* \circ g_F^*$ and analogously for the quadratic mod 2 cohomology ring. It follows that Rost's Theorem implies that $\overline{e}_F^3$ and $\overline{e}_{q,F}^3$ are isomorphisms.

3. Basic diagrams. We have solved the problems above for various classes of fields. In many of the proofs we use induction on the degree of a 2-extension and need the following commutative diagram

(D)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & I^{n+1}F \arrow[r] \arrow[d] & I^{n+1}E \arrow[r] \arrow[d] & I^{n+1}F \arrow[r] \arrow[d] & I^{n+2}F \arrow[r] \arrow[d] & I^{n+2}E \arrow[d] \\
I^{n-1}F \arrow[r] \arrow[d] & I^nF \arrow[r] \arrow[d] & I^nE \arrow[r] \arrow[d] & I^nF \arrow[r] \arrow[d] & I^{n+1}F \arrow[r] & I^{n+1}E \\
H_q^{n-1}F \arrow[r] \arrow[d] & H_q^nF \arrow[r] \arrow[d] & H_q^nE \arrow[r] \arrow[d] & H_q^nF \arrow[d] & & \\
0 & 0 & 0 & 0 & &
\end{tikzcd}$$

where $E = F(\sqrt{d})$ is a quadratic extension of F. The bottom sequence of this diagram is always exact. It arises from the *exact triangle of cohomology*

$$\begin{tikzcd}
 & H_q^*E \arrow[dr, "\mathrm{cor}_{E/F}"] & \\
H_q^*F \arrow[ur, "\mathrm{res}_{E/F}"] & & H_q^*F \arrow[ll, "\cup(d)"']
\end{tikzcd}$$

where $\mathrm{res}_{E/F}$ is the ordinary *restriction map* in group cohomology and $\mathrm{cor}_{E/F}$ is the ordinary *corestriction map* in group cohomology (cf. [**1**]). In both of these diagrams, H^*F can replace H_q^*F. The other sequences of (D) are only known to be zero sequences. They arise from the exact triangle

$$\begin{tikzcd}
 & WE \arrow[dr, "s^*"] & \\
WF \arrow[ur, "i_{E/F}"] & & WF, \arrow[ll, "\otimes\langle 1,-d\rangle"]
\end{tikzcd}$$

where $i_{E/F}$ is induced by the inclusion of fields and s^* is induced by the F-linear functional $s : E \to F$ given by $\sqrt{d} \to 1$ and $1 \to 0$ (cf. [**1**] and [**12**]). The middle row of (D) is exact by [**4**] when $n \leq 3$. The non-obvious vertical arrows in (D) represent the various invariants e_q, when well-defined.

This method of proof is used in [**4**] to show

PROPOSITION 1. *Assume for every finite 2-extension K of F that $\overline{e}^{i}_{q,K}$ is a well-defined isomorphism for each $i < n$ and that $e^n_{q,K}$ is well-defined.*

(1) *If, in addition, for every quadratic extension L of such K, the sequence*

$$I^{n+1}K \to I^{n+1}L \to I^{n+1}K$$

is exact, then $\overline{e}^{n}_{q,F}$ is a monomorphism.

(2) *If, in addition, for every quadratic extension L of such K, the sequence*

$$I^{n+1}K \to I^{n+1}L \to I^{n+1}K \to I^{n+2}K$$

is exact, then $\overline{e}^{n}_{q,F}$ is an isomorphism.

The corresponding proposition holds with $\overline{e}$ replacing $\overline{e}_q$, but in (2), we must assume the hypotheses hold for all finite extensions K of F. The need for these stronger hypotheses is one reason for using the cohomology H^*_q instead of H^*.

One would like to apply the above proposition inductively. The next step is to prove that $e^4_{q,F}$ is well-defined.

4. Nilpotency. We have only been able to solve the problems in §2 for classes of fields satisfying certain conditions. In [**4**], we investigated fields satisfying quadratic forms theoretic conditions. We have

THEOREM 1. *Suppose that* $I^4F(\sqrt{-1}) = 0$. *Then*

(1) $\overline{e}^*_{q,F} : GW\ F \to H^*_q F$ *is a well-defined isomorphism;*

(2) $g^*_F : k_*F \to GW\ F$ *and* $h^*_{q,F} : k_*F \to H^*_q F$ *are isomorphisms;*

(3) $(-1)\cup : H^n_q F \to H^{n+1}_q F$ *is an epimorphism for* $n = 3$ *and an isomorphism for all* $n \geq 4$.

(4) $I^{n+1}F = 2I^nF$ *is torsion-free for all* $n \geq 3$.

In [**4**] the maps in (2) are only shown to be surjective. For the injectivity, we need Rost's result. Observe that, when F is not formally real, (3) and (4) simply say that $H^n_q F = 0$ and $I^nF = 0$ for all $n \geq 4$.

As before we need a stronger hypothesis for the H^*-theory than the H^*_q-theory.

THEOREM 2. *Suppose that* $I^4K = 0$ *for every extension* K *of* $F(\sqrt{-1})$ *of odd degree. Then the inflation map* $\inf : H^*_q F \to H^*F$ *is an isomorphism and all four conclusions of Theorem* 1 *hold. In particular,* $\overline{e}^*_F : GW\ F \to H^*F$ *is a well-defined isomorphism.*

If every anisotropic form over K has dimension at most eight then by [**10**], we have $I^4K = 0$. In particular, using the Tsen-Lang Theorem, we have the following

APPLICATION. *Let* F *be a field of transcendence degree at most three over an algebraically closed field or over a real closed field or of transcendence degree at most two over a finite field. Then all the conclusions of Theorem* 2 *hold.*

Clearly, the important condition in Theorem 1 is the bound on the *nilpotence degree*, $n(F)$ of the ideal $IF(\sqrt{-1})$, i.e., the least $n \leq \infty$ such that $I^nF(\sqrt{-1}) = 0$. (The definition of $n(F)$ really depends only on -1 being totally negative in F, and hence -1 can be replaced by any totally negative element in F.) This number is invariant under finite 2-extensions by [**1**] or [**15**]. It is known that $n(F)$ can increase under finite algebraic extensions. However, it has been conjectured that it cannot increase by more than one. Recently, Baeza proved the analog of this conjecture in the characteristic two case.

Theorem 1 also gives some further credence to the conjecture that $n(F) \leq n$ implies that I^nF is torsion-free. By [**17**], the nilpotence of $IF(\sqrt{-1})$ can be interpreted in terms of WF alone. More precisely, $n(F) \leq n$ if and only if $I^nF = 2I^{n-1}F$ and I^nF contains no non-trivial torsion Pfister forms. Unfortunately, it is still not known whether I^nF containing no non-trivial torsion Pfister forms is equivalent to I^nF being torsion-free. Theorem 1 does let us answer this problem in the following special case.

COROLLARY. $n(F) \leq 4$ *if and only if* $I^4F = 2I^3F$ *is torsion-free.*

This problem is itself a special case of the important open problem of whether the torsion in I^nF is generated by torsion n-fold Pfister forms. This is not even known for $n = 3$.

A major tool in studying torsion in the Witt ring is the well-known Pfister Local-Global Principle. This principle says that the kernel of the total signature map on the Witt ring is just the torsion subgroup. There is a corresponding principle for the graded Witt ring (cf. [**2**] or [**16**]). In contrast to the non-graded case, it is unknown whether the kernel is trivial for pythagorean fields. There is also a corresponding principle for the cohomology rings (cf. [**2**]). But there we also have a corresponding open question about pythagorean fields. (Unfortunately, the proof in [**19**] is incorrect.)

5. Cohomological dimension. In [**6**], we solve our problems for fields satisfying certain cohomological conditions. In such an approach, the difficulty is interpreting cohomological conditions within the theory of quadratic forms. However, if this difficulty can be overcome, the advantages are immense. For example, some major problems in the algebraic theory of quadratic forms concern the relationship between the theory over F and the theory over finitely generated field extensions of F. Since the relationship between the cohomological 2-dimension of a field and its finitely generated field extensions is known (cf. [**28**]), one would obtain a potent tool for studying quadratic forms. In particular, as the cohomological 2-dimension of local and global fields is known, information could be ascertained about quadratic forms over function fields over such fields.

The key result in [**6**] is

THEOREM 3. *Let ρ and σ be two 4-fold Pfister forms over F such that $e_F^4(\rho) = e_F^4(\sigma)$. Then $\rho \simeq \sigma$.*

This result is especially interesting. Recall that 4-fold Pfister forms are not norm forms of any algebras. Thus Theorem 3 represents the first classification of Pfister forms by their associated cohomology class that cannot be related to algebras. This gives some new evidence that the problems listed above may indeed have positive solutions. The proof of this theorem uses generic methods (cf. [**21, 22, 9, 29**]) together with the algebraic techniques developed in [**4**].

COROLLARY. *The cohomological 2-dimension of the Galois group* $\mathrm{Gal}(F_q/F)$ *is at most three if and only if $I^4F = 0$.*

Together with Theorem 1, the corollary yields

THEOREM 4. *Suppose that the cohomological 2-dimension of the Galois group* $\mathrm{Gal}(F_q/F(\sqrt{-1}))$ *is at most three. Then all four conclusions of Theorem 1 hold.*

The same corollary also yields the following two results in the H^*-theory.

PROPOSITION 2. *The cohomological 2-dimension of the Galois group of G_F is at most three if and only if $I^4L = 0$ for every odd degree extension L of F.*

And hence

THEOREM 5. *Suppose that the cohomological* 2*-dimension of* $G_{F(\sqrt{-1})}$ *is at most three. Then all the conclusions of Theorem* 2 *hold.*

As seen earlier, Theorem 2 is immediately applicable to fields of small transcendence degree over real-closed, algebraically closed, or finite fields, but not to transcendental extensions of p-adic fields or number fields. Theorem 5 does apply, so we have the following

APPLICATION. *Let F be a field of transcendence degree at most one over a local or global field. Then all the conclusions of Theorem* 2 *hold.*

Combining the application above with work of Kato (cf. [**20**]), we establish a *weak Hasse Principle* (i.e., a local-global principle for hyperbolicity) for quadratic forms with trivial invariants over the function field of a curve over a number field. Precisely, we have

THEOREM 6. *Let k be a global field and F an algebraic function field in one variable over k with exact constant field k. For any place v of k, let F_v be the free composite of F and the completion k_v of k with respect to v. Then the natural map*

$$I^3F \to \prod_v I^3F_v$$

is an injection.

Theorem 6 means that, over such a field F, two forms ϕ and ψ are isometric if and only if they have the same dimension, same discriminant, same Clifford invariant and become isometric over F_v for all v.

The *Pythagoras number* of a field F is the least n such that every totally positive element in F can be written as a sum of n squares in F. In his appendix to [**20**], Colliot-Thélène showed that the pythagoras number of the function field of a curve over a number field is at most seven. It now seems reasonable to hope that further results about quadratic forms over the function field of curves over a number field can be obtained, e.g., the u-invariant of such a field (cf. [**13**]).

Although, at present, we cannot characterize $cd_2 G_F \leq 4$ in terms of quadratic forms, we do have

THEOREM 7. *Suppose that, for every odd degree extension K of F, every anisotropic form in $I^3 K$ has dimension at most sixteen (e.g., F is a C_4-field). Then*

(1) $cd_2 G_F \leq 4$.

(2) $\bar{e}_F^*; GW\,F \longrightarrow H^* F$ *is a well-defined isomorphism.*

(3) *The inflation map* $\inf : H_q^* F \longrightarrow H^* F$ *is an isomorphism.*

(4) $g_F^* : k_* F \longrightarrow GW\,F$ *and* $h_F^* : k_* F \longrightarrow H^* F$ *are isomorphisms.*

This theorem, of course, depends on arithmetic information. It does have the

APPLICATION. *Let F be a field of transcendence at most four over an algebraically closed field or a field of transcendence degree at most three over a finite field. Then all the conclusions of Theorem 7 hold.*

In [**4**] the maps in (4) are only shown to be surjective. For the injectivity, we need Rost's result. That weaker version of Theorem 7 holds for fields F of transcendence degree four over a real-closed field, except that we do not know whether e_F^4 is well-defined.

6. Abstract Witt rings. Results stated so far have been about a field F. We now look at $W\,F$ as an abstract ring. Let $\mathcal{W}$ denote the category of abstract Witt rings. (See [**24**] for terminology.) For various classes $\mathcal{C}$ of abstract Witt rings, we investigate whether $GW\,F \simeq H_q^* F$, for fields F such that $W\,F$ lies in $\mathcal{C}$. These classes are built inductively from Witt rings of fields studied above. Given such an F with $W\,F$ lying in $\mathcal{C}$, one constructs a 2-extension K of F such that K *realizes* an abstract Witt ring R lying in $\mathcal{C}$, i.e., $R \simeq WK$ in $\mathcal{W}$ and is simpler than the initial Witt ring $W\,F$. A relative version of Proposition 1 then reduces questions about $W\,F$ to questions about WK.

The major difficulty with this approach is realizing morphisms of abstract Witt rings. Specifically, suppose there is a morphism of abstract Witt rings $WF \to R$. Does there exist a field extension K of F such that $WK \simeq R$ and the given morphism corresponds to $i_{K/F} : WF \to WK$? A classical example is the case $R \simeq \mathbf{Z}$. Then K can be chosen to be a real closure of F relative to the ordering induced by the morphism. Unfortunately, the answer is in general negative. But in the important case when $WF = R \times S$ in the category of abstract Witt rings and the morphism is the projection map, we have a positive result. In [**7**] we extend the work of [**18**] and [**30**] on valuations and use this extension to show

THEOREM 8. *Let $\phi : WF \xrightarrow{\sim} R \times S$ be an isomorphism of abstract Witt rings and let $\pi : R \times S \to R$ be the projection. Assume that R is not basic (i.e., R is a group Witt ring over a subring). Then there exists a 2-extension K of F and an isomorphism $\psi : WK \xrightarrow{\sim} R$ of abstract Witt rings such that the diagram*

$$\begin{array}{ccc} WF & \xrightarrow[\phi]{\sim} & R \times S \\ {\scriptstyle i_{K/F}}\downarrow & & \downarrow{\scriptstyle \pi} \\ WK & \xrightarrow[\psi]{\sim} & R \end{array}$$

is commutative.

It is still open whether this is true without any conditions on R.

Theorem 8 allows induction on the complexity of abstract Witt rings under suitable circumstances. It is used together with

THEOREM 9. *Let K be a commutative semi-simple F-algebra. Let $H_q^n K := \prod H^n(\mathrm{Gal}((K_i)_q/K_i), \mathbf{Z}/2\mathbf{Z})$, where $K := \prod K_i$, each K_i a field extension of F. Assume that the following three conditions hold:*

(i) *$i_{K/F} : H_q^n F \to H_q^n K$ is surjective.*

(ii) $\ker(i_{K/F} : H_q^{n+1} F \to H_q^{n+1} K) = (-1) \cup \ker(i_{K/F} : H_q^n F \to H_q^n K)$.

(iii) *If $\alpha \in \ker(i_{K/F} : H_q^{n+1} F \to H_q^{n+1} K)$ satisfies $(-1) \cup \alpha = 0$ then $\alpha = 0$.*

Let E be a finite 2-extension of F and $L = E \otimes_F K$. Then

(1) $i_{L/E} : H_q^m E \to H_q^m L$ *is surjective for all* $m \geq n$.

(2) $\ker(i_{L/E} : H_q^m F \to H_q^m L) = (-1)^{m-n} \cup \ker(i_{K/F} : H_q^m E \to H_q^m L)$ for all $m > n$

(3) *If* $m > n$ *and* $\alpha \in \ker(i_{L/E} : H_q^m E \to H_q^m L)$ *satisfies* $(-1)^k \cup \alpha = 0$ *then* $\alpha = 0$.

If, in the Theorem, the field F is not formally real then (ii) and (iii) simply mean that

$$i_{K/F} : H_q^{n+1} F \to H_q^{n+1} K \text{ is injective,}$$

and the conclusion becomes

$$i_{L/E} : H_q^m E \to H_q^m L \text{ is an isomorphism for all } m > n.$$

In this case, the theorem may be viewed as a statement about the 'relative cohomological 2-dimension' of K over F.

We obtain

THEOREM 10. *Let $\mathcal{C}$ be the smallest class of abstract Witt rings that contains all abstract Witt rings R such that $I^4 R = 2I^3 R$ is torsion-free and that is closed under finite direct products and the arbitrary group ring operation. If $W F$ lies in $\mathcal{C}$ then $GW F \simeq H_q^* F \simeq k_* F$.*

The class of *elementary* Witt rings is the smallest class of abstract Witt rings containing the Witt rings of $\mathbf{R}$, $\mathbf{C}$, finite fields, and local fields and is closed under finite direct products and the finite group ring operation.

APPLICATION. *If $W F$ is an elementary Witt ring then $GW F \simeq H_q^* F \simeq k_* F$.*

In contrast to Theorem 10 itself, the application can be proven by elementary means, i.e., without using Merkurjev's Theorem or the well-definition of e_F^3 (cf. [**5**]).

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