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A CONSTRUCTIVE PROOF OF CONVERGENCE OF THE EVEN APPROXIMANTS OF POSITIVE PC-FRACTIONS

WILLIAM B. JONES AND W.J. THRON

ABSTRACT. Positive PC-fractions are closely related to the trigonometric moment problem, Szegö polynomials and Wiener filters used in digital signal processing. This paper describes constructive methods for proving convergence and sharp truncation error estimates of the even ordered approximants of a positive PC-fraction. Connections with related problems are described briefly.

1. Introduction. Continued fractions of the form

$$(1.1a)\qquad \delta_0 - \frac{2\delta_0}{1} + \frac{1}{\bar{\delta}_1 z} + \frac{(1-|\delta_1|^2)z}{\delta_1} + \frac{1}{\bar{\delta}_2 z} + \frac{(1-|\delta_2|^2)z}{\delta_2} + \cdots,$$

where

$$(1.1b)\qquad \delta_0 > 0, \quad |\delta_j| < 1, \quad \delta_j \in \mathrm{C}, \quad j = 1, 2, 3, \ldots,$$

called *positive Perron-Carathéodory fractions* (*or* PPC-*fractions*), were introduced in [**4**]. It was shown that the approximants of a PPC-fraction are weak two-point Padé approximants for a pair (L_0, L_∞) of formal power series (f.p.s.)

$$(1.2)\quad L_0 := c_0^{(0)} + \sum_{k=1}^{\infty} c_k z^k, \quad L_\infty := -c_0^{(\infty)} - \sum_{k=1}^{\infty} c_{-k} z^{-k}, \quad c_0 := c_0^{(0)} + c_0^{(\infty)}.$$

We denote by $P_n(z)$ and $Q_n(z)$ the nth numerator and denominator, respectively, of (1.1).

PPC-fractions have been used in [**4**, Theorem 3.3] to solve the trigonometric problem. For that purpose the even approximants

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$f_n(z) := P_{2n}(z)/Q_{2n}(z)$ were shown to converge to a *normalized Carathéodory function* $f(z)$ (i.e., a function analytic in the open unit disk $|z| < 1$, mapping this disk into the closed right-half plane $\mathrm{Re}(z) \geq 0$, and normalized by $f(0) \in (0, \infty)$); conversely every normalized Carathéodory function is the limit of the even approximants of a PPC-fraction [**3**, §3]. The odd-ordered denominators $\rho_n(z) := Q_{2n+1}(z)$ are the monic Szegö polynomials, orthogonal on the unit circle with respect to the measure determined by the trigonometric moment problem. The transfer functions $G_n(z)$ of Wiener filters associated with weakly stationary stochastic processes can be expressed by $G_n(z) = z^{-n}\rho_n(z)$; such filters are very useful in digital signal processing of speech and other phenomena (see, for example, [**6**, **8**]).

The proof of convergence of $\{P_{2n}(z)/Q_{2n}(z)\}$ given in [**4**] is nonconstructive, since it is based on normal families and the Stieltjes-Vitali theorem. Moreover, no information is obtained about speed of convergence or truncation error. In this paper we give a constructive proof of convergence that yields both *a posteriori* and *a priori* truncation error bounds and shows that the convergence is geometric (Theorem 3.1). The method consists of constructing best value regions (Lemma 3.2) and inclusion regions and then estimating the diameter of the latter. This approach has been used to develop much of continued fraction convergence theory [**5**, **4**, **9**]. The resulting *a posteriori* truncation error bounds are shown to be best. Similar results by different methods have been given by Geronimus [**1**, Theorem IV, p. 747]. In §2 we derive some results on the Szegö polynomials that are subsequently used. We conclude this introduction with a brief summary of known results on PPC-fractions that are employed.

Associated with the double sequence of complex coefficients $\{c_k\}_{k=-\infty}^{\infty}$ in (1.2) are the Toeplitz determinants $T_k^{(m)} := \det(c_{m-\mu+\nu})_{\mu,\nu=0}^{k-1}$, for $m = 0, \pm 1, \pm 2, \ldots, k = 1, 2, 3, \ldots$, and $T_0^{(m)} := 1$. Of particular interest are $\Delta_n := T_{n+1}^{(0)}$, $\Theta_n := T_n^{(-1)}$ and $\Phi_n := T_n^{(1)}$, $n \geq 0$, where $\Delta_{-1} := -\Delta_{-2} := 1$. *Jacobi's identities* become

$$\Delta_{n-1}^2 = \Delta_n \Delta_{n-2} + \Theta_n \Phi_n, \quad n = 1, 2, 3, \ldots. \tag{1.3}$$

By the difference equations for continued fractions [**5**, (2.1.6)] we obtain for a PPC-fraction (1.1),

$$P_0 = \delta_0, \; P_1 = -\delta_0, \; Q_0 = Q_1 = 1, \tag{1.4a}$$

$$(1.4b) \qquad \begin{pmatrix} P_{2n}(z) \\ Q_{2n}(z) \end{pmatrix} = \overline{\delta}_n z \begin{pmatrix} P_{2n-1}(z) \\ Q_{2n-1}(z) \end{pmatrix} + \begin{pmatrix} P_{2n-2}(z) \\ Q_{2n-2}(z) \end{pmatrix},$$

$$(1.4c) \qquad \begin{pmatrix} P_{2n+1}(z) \\ Q_{2n+1}(z) \end{pmatrix} = \delta_n \begin{pmatrix} P_{2n}(z) \\ Q_{2n}(z) \end{pmatrix} + (1-|\delta_n|^2)z \begin{pmatrix} P_{2n-1}(z) \\ Q_{2n-1}(z) \end{pmatrix},$$

n=1,2,3, $\ldots$. It follows from these that, for $n \geq 1$, $P_{2n}(2), Q_{2n}(z)$, $P_{2n+1}(z)$ and $Q_{2n+1}(z)$ are polynomials in z of degree at most n, with $Q_{2n}(0) = 1$ and $Q_{2n+1}(z) = z^n + \cdots + \delta_n$. The proof of the following theorem given in [**4**, Theorems 2.1, 2.2, and 2.3] is based on constructive linear algebraic methods. The symbol $0(z^r)$ is used to denote a *fps* in increasing powers of z starting with a power not less than r. If $\mathbf{R}$ is a rational function, then the symbols $\Lambda_0(\mathbf{R})$ and $\Lambda_\infty(\mathbf{R})$ denote the Taylor and Laurent series expansions of $\mathbf{R}$ about 0 and ∞, respectively.

THEOREM 1.1. (A) *Let* (1.1) *be a given* PPC*-fraction. Then there exists a unique pair* (L_0, L_∞) *of fps* (1.2) *such that, for* $n = 0, 1, 2, \ldots,$

$$(1.5a) \qquad L_0 - \Lambda_0\left(\tfrac{P_{2n}}{Q_{2n}}\right) = -2\delta_0\overline{\delta}_{n+1}\prod_{j=1}^{n}(1-|\delta_j|^2)z^{n+1} + O(z^{n+2}),$$

(1.5b)
$$L_\infty - \Lambda_\infty\left(\tfrac{P_{2n+1}}{Q_{2n+1}}\right) = 2\delta_0\delta_{n+1}\prod_{j=1}^{n}(1-|\delta_j|^2)z^{-n-1} + O\Big(\Big(\frac{1}{z}\Big)^{n+2}\Big),$$

and

$$(1.6a) \qquad Q_{2n}L_0 - P_{2n} = -2\delta_0\overline{\delta}_{n+1}\prod_{j=1}^{n}(1-|\delta_j|^2)z^{n+1} + O(z^{n+2}),$$

$$(1.6b) \qquad Q_{2n}L_\infty - P_{2n} = -2\delta_0\prod_{j=1}^{n}(1-|\delta_j|^2) + O\Big(\frac{1}{z}\Big),$$

$$(1.6c) \qquad Q_{2n+1}L_0 - P_{2n+1} = 2\delta_0\prod_{j=1}^{n}(1-|\delta_j|^2)z^n + O(z^{n+1}),$$

$$(1.6d)\quad Q_{2n+1}L_\infty - P_{2n+1} = 2\delta_0\delta_{n+1}\prod_{j=1}^{n}(1-|\delta_j|^2)z^{-1} + O\Big(\Big(\frac{1}{z}\Big)^2\Big).$$

Moreover, for $n = 1, 2, 3, \ldots,$

$$(1.7)\qquad \Delta_n > 0,\ c_0^{(0)} = c_0^{(\infty)} > 0 \text{ and } c_n = \bar{c}_{-n},$$

$$(1.8)\qquad \delta_0 = c_0^{(0)} = \frac{1}{2}\Delta_0,\ \delta_n = \frac{(-1)^n\Theta_n}{\Delta_{n-1}},$$

$$(1.9)\qquad 1 - |\delta_n|^2 = \frac{\Delta_n\Delta_{n-2}}{\Delta_{n-1}^2} > 0.$$

(B) *Conversely, let* (L_0, L_∞) *be a given pair of fps* (1.2) *such that* (1.7) *holds. Let* $\{\delta_n\}_0^\infty$ *be defined by* (1.8). *Then* (1.9) *holds* (*by the Jacobi identity* (1.3), *since* $\Phi_n = \overline{\Theta}_n$). *Thus* (1.1b) *holds, and so* (1.1a) *is a PPC-fraction. Moreover,* (1.1) *corresponds to* (L_0, L_∞) *in the sense that* (1.5) *and* (1.6) *hold.*

We note that (1.6a,b) and (1.6c,d) imply that P_{2n}/Q_{2n} and P_{2n+1}/Q_{2n+1} are weak (n,n) two-point Padé approximants for (L_0, L_∞) of order $(n+1, n)$ and $(n, n+1)$, respectively. For $n = 1, 2, 3, \ldots,$ we define the linear fractional transformations (l.f.t)

$$\begin{aligned}
s_0(z,w) &:= \delta_0 + w, \quad s_{2n}(z,w) := \frac{1}{\bar{\delta}_n z + w},\\
s_1(z,w) &:= \frac{-2\delta_0}{1+w}, \quad s_{2n+1}(z,w) := \frac{(1-|\delta_n|^2)z}{\delta_n + w},\\
S_0(z,w) &:= s_0(z,w), \quad S_n(z,w) := S_{n-1}(z, s_n(z,w)),\\
r_0(z,w) &:= s_0 \circ s_1(z,w), \quad r_n(z,w) := \frac{1}{s_{2n}\circ s_{2n+1}(z,w^{-1})},\\
R_0(z,w) &:= r_0(z,w), \quad R_n(z,w) := R_{n-1}(z, r_n(z,w)),
\end{aligned}$$

where composition $\circ$ is taken with respect to w. Then, from continued fraction theory [**8**, (2.1.7)], for $n = 1, 2, 3, \ldots,$

$$S_n(z,w) = \frac{P_n(z) + wP_{n-1}(z)}{Q_n(z) + wQ_{n-1}(z)},$$

and hence

$$R_n(z,w) = S_{2n+1}(z,w^{-1}) = \frac{P_{2n+1}(z)w + P_{2n}(z)}{Q_{2n+1}(z)w + Q_{2n}(z)} \tag{1.10}$$

and

(1.11)

$$\frac{P_{2n}(z)}{Q_{2n}(z)} = S_{2n}(z,0) = R_n(z,0), \ \frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = S_{2n+1}(z,0) = R_n(z,\infty).$$

It follows from the mapping properties of l.f.t.'s that, for $n = 1,2,3,\dots,$

$$\left|\frac{P_{2n}(z)}{Q_{2n}(z)} - \delta_0\frac{1+\rho^2}{1-\rho^2}\right| \le \frac{2\rho\delta_0}{1-\rho^2}, \quad |z| \le \rho < 1, \tag{1.12a}$$

and

$$\left|\frac{P_{2n+1}(z)}{Q_{2n+1}(z)} + \delta_0\frac{\rho^2+1}{\rho^2-1}\right| \le \frac{2\rho\delta_0}{\rho^2-1}, \quad |z| \le \rho > 1. \tag{1.12b}$$

2. Deonominators of PPC-Fractions. In this section we derive some properties of the polynomial denominators $Q_n(z)$ of a PPC-fraction (1.1) that will be used in §3. We begin with the following Christoffel-Darboux formula

$$\sum_{j=0}^{n} K_j^2 Q_{2j-1}(x)\overline{Q_{2j-1}(y)} = \frac{K_n^2(Q_{2n}(x)\overline{Q_{2n}(y)} - x\overline{y}Q_{2n+1}(x)\overline{Q_{2n+1}(y)})}{1-x\overline{y}}, \tag{2.1}$$

where

$$K_n := \sqrt{\Delta_{n-1}/\Delta_n} > 0, \ n = 0,1,2,\dots. \tag{2.2}$$

To prove (2.1) we note that, from the difference equations (1.4), it follows that

$$Q_0(z) = Q_1(z) = 1 \tag{2.3a}$$

and for $n = 1, 2, 3, \ldots$,

(2.3b)
$$Q_{2n+1}(z) = zQ_{2n-1}(z) + \delta_n Q_{2n-2}(z), \; Q_{2n}(z) = \overline{\delta}_n z Q_{2n-1}(z) + Q_{2n-2}(z).$$

From this one easily obtains, for $n = 1, 2, 3, \ldots$,

$$(2.4) \qquad \begin{aligned} zQ_{2n-1}(z) &= \frac{Q_{2n+1}(z) - \delta_n Q_{2n}(z)}{1 - |\delta_n|^2}, \\ Q_{2n-2}(z) &= \frac{Q_{2n}(z) - \overline{\delta}_n Q_{2n+1}(z)}{1 - |\delta_n|^2}. \end{aligned}$$

From (1.9) and (2.2) it follows that

$$(2.5) \qquad K_n^2 = \frac{K_{n-1}^2}{1 - |\delta_n|^2} = \frac{K_0^2}{\prod_{j=1}^n (1 - |\delta_j|^2)}, \; n = 1, 2, 3, \ldots .$$

By using (2.4) and (2.5) one can show that, for $j = 0, 1, 2, \ldots$,

$$(2.6) \qquad \begin{aligned} &\frac{K_{j-1}^2 (Q_{2j-2}(x)\overline{Q_{2j-2}(y)} - x\overline{y} Q_{2j-1}(x)\overline{Q_{2j-1}(y)})}{1 - x\overline{y}} \\ &\quad + K_j^2 Q_{2j+1}(x)\overline{Q_{2j+1}(y)} \\ &= \frac{K_j^2 (Q_{2j}(x)\overline{Q_{2j}(y)} - x\overline{y} Q_{2j+1}(x)\overline{Q_{2j+1}(y)})}{1 - x\overline{y}}, \end{aligned}$$

where we set $K_{-1} = 0$ and $Q_{-1}(x) := Q_{-2}(x) := 0$. Summing both sides of (2.6) and cancelling like terms yields (2.1). By setting $x = y = z$ in (2.1) we obtain the inequality

$$K_0^2 \le \sum_{j=0}^n K_j^2 |Q_{2j+1}(z)|^2 = \frac{K_n^2}{1 - |z|^2} (|Q_{2n}(z)|^2 - |zQ_{2n+1}(z)|^2).$$

Applying (2.5) to this yields, for $n = 1, 2, 3, \ldots$,

$$(2.7) \qquad |Q_{2n}(z)|^2 - |zQ_{2n+1}(z)|^2 \ge (1 - |z|^2) \prod_{j=1}^n (1 - |\delta_j|^2).$$

3. Convergence of the even approximants. The main result of this paper is

THEOREM 3.1. *Let* (1.1) *be a given PPC-fraction and let* $f_n(z) := P_{2n}(z)/Q_{2n}(z)$ *denote its* 2n^{th} *approximant. Then:* (A) $\{f_n(z)\}$ *converges to a (normalized Carathéodory) function* $f(z)$, *analytic for* $|z| < 1$ *and satisfying*

$$f(0) = \delta_0 > 0, \ \operatorname{Re} f(z) \geq 0, \ |z| < 1. \tag{3.1}$$

The convergence is uniform on every compact subset of $|z| < 1$.

(B) *For* $n = 1, 2, 3, \ldots,$

$$|f(z) - f_n(z)| \leq \frac{4\delta_0 \prod_{j=1}^{n}(1 - |\delta_j|^2)|z|^{n+1}}{|Q_{2n}(z)|^2 - |zQ_{2n+1}(z)|^2} \quad (a \textit{ posteriori} \text{ bound}). \tag{3.2}$$

(C) *For* $n = 1, 2, 3, \ldots,$

$$|f(z) - f_n(z)| \leq \frac{4\delta_0 |z|^{n+1}}{(1 - |z|^2)} \quad (a \textit{ priori} \text{ bound}). \tag{3.3}$$

(D) *The a posteriori bound* (3.2) *is best.*

Before proving Theorem 3.1 we give some basic lemmas and definitions that are used. We denote by $\Gamma^+(z)$ the class of all PPC-fractions with z fixed, $|z| < 1$. We say that a sequence of non-empty subsets $\{V_n\}_{n=1}^{\infty}$ of $\hat{\mathbb{C}}$ is a *sequence of value regions for* $\{P_{2n}(z)/Q_{2n}(z)\}$ *with respect to* $\Gamma^+(z)$ if

$$r_n(z, 0) \in V_n, \ n \geq 0 \text{ and } r_n(z, V_n) \subseteq V_{n-1}, \ n \geq 1. \tag{3.4}$$

We denote by $\mathcal{V}(z)$ the family of all sequences of value regions for $\{P_{2n}(z)/Q_{2n}(z)\}$ with respect to $\Gamma^+(z)$. It follows that

$$[0 \in V_{n-1} \text{ and } r_n(z, V_n) \subseteq V_{n-1}, n \geq 1] \Rightarrow \{V_n\} \in \mathcal{V}(z). \tag{3.5}$$

It is readily seen that if (3.5) holds, then

$$f_n(z) := \frac{P_{2n}(z)}{Q_{2n}(z)} = R_n(z,0) \in R_n(z,V_n) \subseteq R_{n-1}(z,V_{n-1}), \tag{3.6}$$

$$n = 1,2,3,\dots .$$

Hence $\{R_n(z,V_n)\}$ is a nested sequence of subsets of $\hat{\mathrm{C}}$ satisfying

$$f_{n+m}(z) \in R_n(z,V_n),\ n = 1,2,3,\dots, m = 0,1,2,\dots\ s. \tag{3.7}$$

It follows that

$$|f_{n+m}(z) - f_n(z)| \le \operatorname{diam} R_n(z,V_n),\quad \begin{matrix} n = 1,2,3,\dots, \\ m = 0,1,2,\dots, \end{matrix} \tag{3.8}$$

so that $\{f_n(z)\}$ is convergent if

$$\lim_{n\to\infty} \operatorname{diam} R_n(z,V_n) = 0. \tag{3.9}$$

For a fixed z with $|z| < 1$ we define

$$\begin{aligned} W_n(z) := \{r_{n+1} \circ r_{n+2} \circ \cdots \circ r_{n+m}(z,0) : |\delta_k| < 1, \\ n+1 \le k \le n+m, m = 1,2,3,\dots\}, \end{aligned} \tag{3.10}$$

where composition $\circ$ is with respect to w in $r_k(z,w)$. By a well known argument (see, for example, [**5**, Theorem 4.1]) it can be shown that

$$\{W_n\} \in \mathcal{V}(z) \text{ and } [\{V_n\} \in \mathcal{V}(z) \Rightarrow W_n \subseteq V_n,\ n \ge 0]. \tag{3.11}$$

In view of property (3.11), we call $\{W_n\}$, given by (3.10), the *best* sequence of value regions for $\{P_{2n}(z)/Q_{2n}(z)\}$ with respect to $\Gamma_n^+(z)$.

LEMMA 3.2. *The best sequence $\{W_n\}$ of value regions for $\{P_{2n}(z)/Q_{2n}(z)\}$ with respect to $\Lambda^+(z)$ is given by*

$$W_n = U_{|z|} := [u \in \mathrm{C} : |u| < |z|],\ n = 0,1,2,\dots . \tag{3.12}$$

PROOF. Let $R := |z| < 1$. By elementary properties of l.f.t.'s it can be shown that

$$r_n(z, U_R) = \{u : |u - \Gamma_n| \leq \rho_n\}, \ n = 1, 2, 3, \ldots, \tag{3.13a}$$

where

$$\Gamma_n := \frac{z\bar{\delta}_n(1 - R^2)}{1 - R^2|\delta_n|^2}, \ \rho_n := \frac{R^2(1 - |\delta_n|^2)}{1 - R^2|\delta_n|^2}. \tag{3.13b}$$

An immediate consequence of this is that, for $n \geq 1$,

$$r_n(z, U_R) \subseteq U_R, \tag{3.14}$$

if and only if

$$|\Gamma_n| + \rho_n \leq R := |z|. \tag{3.15}$$

To verify (3.15) we substitute (3.13b), divide by R, multiply by $(1 - R^2|\delta_n|^2)$, and rearrange terms to obtain the equivalent inequality

$$R(1 - |\delta_n|^2) \leq (1 - |\delta_n|) + R^2|\delta_n|(1 - |\delta_n|).$$

Now, dividing this by $(1 - |\delta_n|)$ and rearranging terms, yields the equivalent inequality

$$(1 - R)(1 - R|\delta_n|) \geq 0$$

which clearly holds. This proves (3.15) and hence also (3.14). Since $|z| < 1$ we have, for $n \geq 0$,

$$[r_{n+1}(z, 0) : |\delta_{n+1}| < 1] = [\bar{\delta}_{n+1}z : |\delta_{n+1}| < 1] = U_R. \tag{3.16}$$

Hence by (3.10), $U_R \subseteq W_n$, $n \geq 0$. It suffices to show that

$$W_n \subseteq U_R, \ n = 0, 1, 2, \ldots. \tag{3.17}$$

By (3.16) and (3.14), for $n \geq 0$, $m \geq 1$,

$$\begin{aligned} r_{n+1} \circ \cdots \circ r_{n+m}(z, 0) &\in r_{n+1} \circ \ldots r_{n+m}(z, U_R) \\ \subseteq r_{n+1} \circ \cdots \circ r_{n+m-1}(z, U_R) &\subseteq \cdots \subseteq r_{n+1}(z, U_R) \subseteq U_R. \end{aligned} \tag{3.18}$$

Therefore (3.17) follows from this and (3.10). □

We now denote by PPC $(z, \{\delta_n\})$ a given (fixed) PPC-fraction (1.1). It can be seen from (3.10) and (3.12) that, for $n \geq 1$, the set $R_n(z, U_{|z|})$ consists of all $(2n+2m)^{\text{th}}$ approximants (with $m \geq 0$) of PPC-fractions for which the coefficients $\delta_0, \delta - 1, \dots, \delta_n$ are given (and fixed) and the other coefficients $\hat{\delta}_m$, $m > n$, satisfy $|\hat{\delta}_n| < 1$. Therefore following standard continued fraction terminology (see, for example, [**2**] and [**5**, p. 297]) we call the closed set

$$c(R_n(z, U_{|z|}))$$

the *best* n^{th} *inclusion region for* PPC $(z, \{\delta_n\})$. Here $c(A)$ is used to denote the closure of a set A. It can be seen from (3.8) that the diameter of this set, $\text{diam} R_n(z, U_{|z|})$, is the best truncation errors bound for $f_n(z) := P_{2n}(z)/Q_{2n}(z)$. It follows from (3.14) and mapping properties of l.f.t.'s, that $R_n(z, U_{|z|})$ is an open circular (bounded) disk. The following lemma gives the radius of this disk.

LEMMA 3.3. *For z fixed with $|z| < 1$, and $n = 1, 2, 3, \dots$,*

$$\kappa_n := \text{rad} R_n(z, U_{|z|}) = \frac{2\delta_0 \prod_{j=1}^{n} (1 - |\delta_j|^2)|z|^{n+1}}{|Q_{2n}(z)|^2 - |zQ_{2n+1}(z)|^2} \tag{3.19}$$

PROOF. Let w_n denote the point inside $U_{|z|}$ such that $R_n(z, w_n)$ is the center of the disk $R_n(z, U_{|z|})$. Let $g_n := Q_{2n}(z)/Q_{2n+1}(z)$. Then $R_n(z, -g_n) = \infty$. Since $R_n(z, w_n)$ and $R_n(z, -g_n) = \infty$ are inverses of each other with respect to the circle $\partial R_n(z, U_{|z|})$, and since inverses are preserved under the l.f.t. $R_n^{-1}(z, w)$, it follows that w_n and $-g_n$ are inverses of each other with respect to the circle $\partial U_{|z|}$. Hence

$$\tau_n := \text{Arg}(w_n) = \text{Arg}(-g_n) \text{ and } |w_n| \cdot |g_n| = |z|^2. \tag{3.20}$$

Therefore we can write

$$v_n := |z| e^{i\tau_n} \text{ and } w_n = |w_n| e^{i\tau_n}. \tag{3.21}$$

It follows from (1.10) and the determinant formula [**5**, (2.1.9)] that
(3.22)

$$\kappa_n = |R_n(z,w_n) - R_n(z,v_n)| = \left|\frac{w_nP_{2n+1}+P_{2n}}{w_nQ_{2n+1}+Q_{2n}} - \frac{v_nP_{2n+1}+P_{2n}}{v_nQ_{2n+1}+Q_{2n}}\right|$$
$$= \frac{2\delta_0\prod_{j=1}^n(1-|\delta_j|^2)|z|^n\cdot|w_n-v_n|}{|Q_{2n+1}|^2\cdot|w_n+g_n|\cdot|v_n+g_n|}.$$

From (3.20) and (3.21) one has

$$(3.23) \qquad |w_n+g_n| = |g_n| - |\omega_n| \text{ and } |v_n+g_n| = |g_n| - |z|.$$

Using $|w_n - v_n| = |z| - |w_n| = |z| - |z^2|/|g_n|$, we conclude that (3.19) follows from (3.22) and (3.23). □0

PROOF OF THEOREM 3.1. It follows from (3.8) and Lemma 3.3 that, for $n \geq 1,\ m \geq 0$,

$$(3.24) \qquad |f_{n+m}(z) - f_n(z)| \leq 2\kappa_n = \frac{4\delta_0\prod_{j=1}^n(1-|\delta_j|)^2|z|^{n+1}}{|Q_{2n}(z)|^2 - |zQ_{2n+1}(z)|^2}.$$

From (2.7) and (3.24) we obtain
(3.25)

$$|f_{n+m}(z) - f_n(z)| \leq \frac{4\delta_0|z|^{n+1}}{1-|z|^2},\ n = 1,2,3,\dots,\ m = 0,1,2,\dots .$$

It follows that $\{f_n(z)\}$ converges uniformly on every compact subset of $|z| < 1$ to a function $f(z)$ analytic for $|z| < 1$. That $\text{Re}f(z) \geq 0$ for $|z| < 1$ follows from (1.12a). The normalization $f(0) = \delta_0 > 0$ follows from (1.4a,b), since $f_n(0) = f_{n-1}(0) = f_0(0) = \delta_0,\ n \geq 1$. This proves (A). (B) follows from (3.24) and (C) from (3.25). (D): The best *a posteriori* truncation error bound for $f_n(z)$ is given by (3.2) since $c(R_n(z, U_{|z|}))$ is the best n$^{\text{th}}$ inclusion region for PPC $(z, \{\delta_n\})$. □

REMARK. It can be shown that the corresponding *fps* L_0 in (1.2) is the Taylor series for $f(z)$ at $z = 0$ [**4**, Theorem 3.2(A)]. However, we do not yet have a constructive proof of this.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, CO 80309-0426