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CONVERGENCE IN PROBABILISTIC SEMIMETRIC SPACES

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ABSTRACT . Let (S, F) denote a probabilistic semimetric space. An induced Cauchy structure on S is shown to be a natural means of studying convergence and completion theory. A convergent sequence is also required to be Cauchy. This is a fundamental difference between our convergence notion and that studied previously. However, if (S, F, τ) is a probabilistic metric space with τ continuous, then the definitions coincide because of the triangle inequality. Moreover, the requirement that F be Cauchy-continuous, along with the restriction that a convergent sequence be Cauchy, helps to capture the geometry of the triangle inequality.

It is shown that every probabilistic semimetric space (S, F) is the image of a simple space under a Cauchy-quotient map whenever F is Cauchy-continuous. Sufficient conditions are given to ensure that a probabilistic semimetric space has a completion (compactification).

1. Introduction. The concept of a probabilistic metric space is a generalization of that of a metric space. Associated with each pair of elements of an abstract set is a probability distribution function. The assignment function is required to satisfy axioms resembling those of a metric and is called a probabilistic metric. The origin of the theory dates back to a paper published by K. Menger in 1942 [**16**]. A foundational paper on the subject was written by E. Schweizer and A. Sklar in 1960 [**18**] and numerous articles followed thereafter. The latter two authors gave an excellent updated treatment of the subject in their book published in 1983 [**19**].

The purpose of this study is to investigate a natural convergence setting for these spaces. This is the concept of a Cauchy space which was introduced by H. Kowalsky in 1954 [**12**] and refined to the present form

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by H.H. Keller in 1968 [**6**]. Cauchy spaces are presently being used in various areas. For example, R.N. Ball [**1, 2**] has made extensive use of Cauchy spaces on completions of lattices and lattice ordered groups, and K. McKennon [**15**] has applied this notion in the study of C*-algebras. There is little doubt that additional applications of Cauchy spaces will be forthcoming as familiarity with the subject becomes more widespread. A survey article on the subject was written in 1983 [**10**].

2. Preliminaries. Basic definitions and terminology needed here are given below. The reader interested in further details is referred to references [**19**] and [**10**]. The symbol Δ^+ denotes the set of all nondecreasing functions F from $\mathbf{R}^+ = [0, \infty]$ into $[0,1]$ which are left continuous on $(0, \infty)$ and such that $F(0) = 0$ and $F(\infty) = 1$. The set Δ^+ is equipped with the modified Levy metric d_L. This still implies that a sequence $\{F_n\}$ converges to F in (Δ^+, d_L) if and only if $F_n(x) \to F(x)$ on $\mathbf{R}$ whenever x is a point of continuity of F. More generally, a filter ϕ on Δ^+ converges to F if and only if $\phi(x) \to F(x)$ on $\mathbf{R}$ whenever x is a point of continuity of F. It is well-known that (Δ^+, d_L) is a compact metric space.

Define the element ε_0 in Δ^+ by

$$\varepsilon_0(x) = \begin{cases} 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

A binary operation τ on Δ^+ is called a *triangle function* provided τ is commutative, associative, nondecreasing in each place, and has ε_0 as its identity. Let S denote a nonempty set, F a function from $S \times S$ into Δ^+. Consider the following conditions for all $p, q, r \in S$: (i)$F(p,p) = \varepsilon_0$, (ii)$F(p,q) \neq \varepsilon_0$ whenever $p \neq q$, (iii) $F(p,q) = F(q,p)$, and (iv) there is a triangle function τ on Δ^+ such that $F(p,q) \geq \tau[F(p,r), F(r,q)]$. If conditions (i) and (iii) are satisfied, then (S, F) is called a *probabilistic premetric space* (abb., *pre*-PM *space*). If conditions (i) - (iii) hold, then (S, F) is called a *probabilistic semimetric space* (abb., PSM *space*). A *probabilistic pseudometric space* (abb., PPM *space*) (S, F, τ) satisfies conditions (i), (iii), and (iv). Finally, a *probabilistic metric space* (abb., PM *space*) (S, F, τ) satisfies conditions (i) - (iv). Note that the four conditions given above resemble those possessed by a metric on S.

The notion of a uniform convergence space was introduced by C.H. Cook and H.R. Fischer in 1967 [**4**]. Shortly thereafter, H.H. Keller [**6**] gave the present definition of a Cauchy space by characterizing the allowable set of Cauchy filters for a uniform convergence space. Cauchy spaces provide a more natural foundation upon which to base a general completion theory. Given a nonempty set S, let $\dot{p}$ denote the filter on S consisting on all subsets of S which contain p as an element. Let ϕ, ψ denote two filters on S; then ϕ is *finer* (*coarser*) than ψ, written $\phi \geq \psi$ ($\phi \leq \psi$), provided $\psi \subseteq \phi (\phi \subseteq \psi)$. A maximal filter in this ordering is called an *ultrafilter*. If A and B are not disjoint for each $A \in \phi$ and $B \in \psi$, then $\phi \vee \psi$ is said to *exist*. Let $\phi \cap \psi$ denote the filter obtained by set theoretic intersection of ϕ and ψ.

Suppose that S is a nonempty set. A *Cauchy structure* C on S is a collection of filters on S satisfying the following conditions: (i) $\dot{p} \in C$ whenever $p \in S$, (ii) if $\phi \in C$ and $\psi \geq \phi$, then $\psi \in C$, and (iii) if $\phi, \psi \in C$ and $\phi \vee \psi$ exists, then $\phi \cap \psi \in C$. The pair (S, C) is called a *Cauchy space* and each element of C is called a *Cauchy filter*. A filter $\phi \in C$ is said to *converge to* p provided $\phi \cap \dot{p} \in C$. Note that a convergent filter is necessarily Cauchy. If S is either a metric or uniform space, then the set of all Cauchy filters forms a Cauchy structure and convergence agrees with that given above.

A Cauchy space (S, C) is called *totally bounded* whenever each ultrafilter belongs to C. It is said to be T_2 provided no filter converges to two distinct elements of S. Let A be any subset of S; then $p \in \mathrm{cl}A$ whenever there exists a filter containing A and converging to p. Denote by $\mathrm{cl}\phi$ the filter whose base is $\{\mathrm{cl}A : A \in \phi\}$. The Cauchy space (S, C) is called *regular* if $\mathrm{cl}\phi \in C$ whenever $\phi \in C$. A regular T_2 Cauchy space is said to be T_2.

A function $f : (S, C) \to (T, D)$ between two Cauchy spaces is said to be *Cauchy-continuous* if $f\phi \in D$ whenever $\phi \in C$; that is, Cauchy filters are preserved under f. A Cauchy structure D_1 on T is said to be *finer* than D, written $D_1 \geq D$, whenever $D_1 \subseteq D$. The function f above is called a *Cauchy-quotient map* provided f is onto and D is the finest Cauchy structure on T such that f is Cauchy-continuous.

3. Associated Cauchy Space. This section is devoted to showing that there is a natural Cauchy space associated with each probabilistic semimetric space (or probabilistic premetric space). This is shown to be a satisfactory generalization to these less restrictive spaces.

Let (S, F) denote a probabilistic semimetric space and let $\{p_n\}$ denote a sequence in S. Then $\{p_n\}$ is usually defined to be Cauchy whenever $F(p_n, p_m) \to \varepsilon_0$ in Δ^+ as $n, m \to \infty$. The natural extension is to call a filter ϕ on S Cauchy whenever $F(\phi \times \phi) \to \varepsilon_0$ in Δ^+. The collection $\underline{A}$ of all such filters may not satisfy axiom (iii) of a Cauchy structure defined in §2; however, it does generate a Cauchy structure on S. More precisely, C is the finest Cauchy structure on S containing $\underline{A}$. It is straightforward to verify that C is the collection of all filters on S of the form $\phi \geq \cap_1^n \psi_i$, for some finite collection $\psi_i \in A$ such that $\psi_i \vee \psi_{i+1}$ exists, $1 \leq i \leq n-1$. The space (S, C) is called the *Cauchy space associated with* (S, F).

The convergence definition most often studied in general probabilistic spaces is the following: $p_n \to p$ if and only if $F(p_n, p) \to \varepsilon_0$ in Δ^+. A thorough treatment of this concept is given in papers authored by E.O. Thorp [**23**], D.H. Muštari and A.N. Šerstnev [**17**], R. Fritsche [**5**], and R.M. Tardiff [**22**]. The latter two authors include the possibility of using a profile function other than ε_0.

An example is given by D.H. Muštari and A.N. Šerstnev [**17**] of a PM space (S, F, τ) such that τ is not continuous at $(\varepsilon_0, \varepsilon_0)$. A sequence $\{p_n\}$ is constructed in this example having the property that $F(p_n, p) \to \varepsilon_0$ in Δ^+ but $F(p_n, p_m)$ fails to converge in Δ^+. This implies that the filter generated by $\{p_n\}$ does not belong to C and hence fails to converge in our setting. This is desirable; otherwise, a convergent filter need not be Cauchy. However, if τ is continuous at $(\varepsilon_0, \varepsilon_0)$, then $F(p_n, p) \to \varepsilon_0$ in Δ^+ implies that $F(p_n, p_m) \to \varepsilon_0$ in Δ^+ due to the triangle inequality.

A PSM space (S, F) is said to be *first-countable* if, for each $\phi \in C$, there exists a $\psi \in C$ such that $\phi \geq \psi$ and ψ has a countable base. The following result shows, in particular, that points of closure of subsets of (S, C) as well as Cauchy continuity of $F : (S, C) \times (S, C) \to \Delta^+$ are determined by sequences.

PROPOSITION 3.1. *Every PSM space is first-countable.*

PROOF. Let (S, F) denote a PSM space and let $\phi \in A$. Denote by

$N_{\varepsilon_0}(t) = \{F \in \Delta^+ : d_L(F, \varepsilon_0) < t\}$. Then, for each $\delta_n = 1/n$, there exists an $A_n \in \phi$ such that $F(A_n \times A_n) \subseteq N_{\varepsilon_0}(\delta_n)$. The sequence $\{A_n, n \geq 1\}$ may be chosen to be decreasing and thus is a base for a filter ψ on S. Note that $\psi \in C, \phi \geq \psi$, and ψ has a countable base.

Recall that A generates the associated Cauchy space C; that is, if $\phi \in C$ then $\phi \geq \cap_1^n \psi_i$, where $\phi_i \in A$ and $\phi_i \vee \phi_{i+1}$ exists. It follows from the preceding paragraph that each ψ_i may be assumed to have a countable base and thus $\cap_i^n \psi_i$ has a countable base. Hence (S, F) is first-countable.□

Suppose that (S, d) denotes a metric space and G is an element in Δ^+ such that $G \neq \varepsilon_0$ and $\lim_{y \to \infty} G(y) = 1$. Let us adopt the convention that $G(0/0) = G(0) = 0$ and $G(x/0) = G(\infty) = 1$ whenever $x > 0$. Define $F(p, q)(x) = G(x/d(p, q))$ whenever $p, q \in S$ and $x \geq 0$. Then (S, F) is called a *simple space*, denoted by (S, F, G), and in fact is a PM space under an appropriate triangle function.

Another important space is the following. Let (S, F, τ) denote a PM space. It is shown in Theorem 12.1.5 [**19**] that, whenever τ is continuous at $(\varepsilon_0, \varepsilon_0)$, sets of the form $V(t) = \{(p, q) | d_L(F(p, q), \varepsilon_0) < t\}, t > 0$, are a base for a uniformity on S. This uniformity is called the *strong uniformity* on S and the topology induced is called the *strong topology.* The next result shows that the associated Cauchy space is as expected for the two important spaces mentioned above.

PROPOSITION 3.2. (i) *Let (S, d) be a metric space and (S, F, G) the corresponding simple space. Then $\phi \in C$ if and only if ϕ is a Cauchy filter on (S, d).*

(ii) *Suppose that (S, F, τ) is a PM space such that τ is continuous at $(\varepsilon_0, \varepsilon_0)$. Then $\phi \in C$ if and only if ϕ is a Cauchy filter with respect to the strong uniformity.*

(iii) *$F : (S, C) \times (S, C) \to \Delta^+$ is Cauchy-continuous in each of the two spaces mentioned above.*

PROOF. (i) This follows by applying Theorem 12.3.2 [**19**] to the metric space completion of (S, d).

(ii) The proof follows from the definition of the Cauchy filter in each setting.

(iii) This follows directly from Theorem 12.2.2 [**19**]. □

Let (S, F) be a PSM space. Then F is said to be Cauchy-continuous whenever condition (iii) of Proposition 3.2 is satisfied. The following example of E.O. Thorp [**23**] shows that F may fail to be Cauchy-continuous.

EXAMPLE 3.1. [**23**]. Let (S, d) denote the two dimensional Euclidean space and defined for $x > 0$,

$$F(p,q)(x) = \begin{cases} \varepsilon_0(x-1), & \text{if } d(p,q) \geq 1 \text{ or slope between p and q} \\ & \text{is irrational,} \\ 1 - d(p,q), & \text{if } d(p,q) < 1 \text{ and slope between p and q} \\ & \text{is rational, } 0 < x \leq 1, \\ 1, & \text{if } d(p,q) < 1 \text{ and slope between p and q} \\ & \text{is rational, } x > 1. \end{cases}$$

Then (S, F) determines a PSM space. Let L_1 and L_2 denote two distinct line segments in the first quadrant through $p = (0,0)$, each having rational slope. Select sequences $\{p_n\}$ on L_1 and $\{q_n\}$ on L_2 such that $p_n \to p$, $q_n \to q$ on (S, d) and the slope of the line segment from p_n to q_n is irrational, $n \geq 1$. Denote by ϕ and ψ the filters generated by $\{p_n\}$ and $\{q_n\}$, respectively; then $\phi \cap \dot{p} \in C$ and $\psi \cap \dot{p} \in C$. However, $F(p_n, q_n)(x) = 0$ whenever $0 \leq x \leq 1$ and thus it follows that $F(\phi \times \psi)$ fails to converge to ε_0 in Δ^+ and hence F is not Cauchy-continuous.

4. Quotients. Recall that an onto function $f : (S, C) \to (T, D)$ between two Cauchy spaces is a Cauchy quotient map provided D is the finest Cauchy structure such that f is Cauchy-continuous. It is straightforward to verify that $\psi \in D$ if and only if $\psi \geq \cap_1^n f\phi_i$ for some finite collection $\phi_i \in C$ such that $f\phi_i \vee f\phi_{i+1}$ exists, $1 \leq i \leq n-1$. Given a PPM space (S, F, τ), it is shown in Lemma 8.1.3 [**19**] that (S, F, τ) has an associated quotient PM space. The latter space is obtained by introducing the equivalence relation $p \sim q$ if and only if $F(p,q) = \varepsilon_0$. The triangle function τ is needed to verify this. A similar result is given below whenever the triangle function is replaced by the requirement that F is Cauchy-continuous.

PROPOSITION 4.1. *Let (S, F) denote a pre-PM space and assume that F is Cauchy-continuous. There is an associated PSM space (S^*, F^*) such that F^* is also Cauchy-continuous. Moreover, the natural injection from (S, C) onto (S^*, C^*) is a Cauchy-quotient map.*

PROOF. Let C denote the Cauchy structure associated with (S, F). Define $p \sim q$ if and only if $F(p, q) = \varepsilon_0$. Note that if $p \sim q$ and $q \sim r$, then $F[(\dot{p} \cap \dot{q}) \times (\dot{p} \cap \dot{q})] \to \varepsilon_0$ in Δ^+ and thus $\dot{p} \cap \dot{q} \in C$. Similarly, $\dot{q} \cap \dot{r} \in C$ and since $(\dot{p} \cap \dot{q}) \vee (\dot{q} \cap \dot{r})$ exists, then $\dot{p} \cap \dot{q} \cap \dot{r} \in C$ and thus $\dot{p} \cap \dot{r} \in C$. The Cauchy-continuity of F implies that $F(p, r) = \varepsilon_0$ and thus the relation is an equivalence relation. Let $[p]$ denote the equivalence class containing p, $S^* = \{[p] | \in S\}$, $j(p) = [p]$, and $F^*([p], [q]) = F(p, q)$. It is straightforward,using the Cauchy-continuity of F, to verify that F^* is well-defined. It then follows that (S^*, F^*) is a PSM space.

Let C^* denote the Cauchy structure associated with (S^*, F^*) and let us show first that $j : (S, C) \to (S^*, C^*)$ is Cauchy-continuous. If $\phi \in A$, then $F^*(j\phi \times j\phi) = F(\phi \times \phi) \to \varepsilon_0$ in Δ^+ and thus $j\phi \in C^*$. Suppose that $\psi \in C$; then $\psi \geq \cap_1^n \psi_i$ for some finite collection $\psi_i \in A$, such that $\psi_i \vee \psi_{i+1}$ exists. It follows that $j\psi \geq \cap_i^n j\psi_i \in C^*$ since each $j\psi \in C^*$ and $j\psi_i \vee j\psi_{i+1}$ exists. Thus j is Cauchy-continuous.

Finally, let D denote the finest Cauchy structure on S^* such that $j : (S, C) \to (S^*, D)$ is Cauchy-continuous. Then $D \geq C^*$. Let $\psi \in C^*$; then $\psi \geq \cap_i^n \psi_i$ for some finite collection ψ_i such that $F^*(\psi_i \times \psi_i) \to \varepsilon_0$ in Δ^+ and $\psi_i \vee \psi i + 1$ exists. Since $F(j^{-1}\psi_i \times j^{-1}\psi_i) = F^*(\psi_i \times \psi_i) \to \varepsilon_0$ in Δ^+ and $j^{-1}\psi_i \vee j^{-1}\psi_{i+1}$ exists, then $j^{-1}\psi_i \in C$ and $\cap_1^n j^{-1}\psi_i \in C$. Hence $j^{-1}\psi \in C$ and thus $\psi = j(j^{-1}\psi) \in D$. Therefore $C^* - D$ and thus $j : (S, C) \to (S^*, C^*)$ is a Cauchy-quotient map. The Cauchy continuity of F^* is now straightforward to verify. □

The image of any complete Cauchy space under a Cauchy-quotient map is also complete [8]; Proposition 2.1. This gives the following result.

COROLLARY 4.1. *Let (S, F) be a complete pre-PM space such that F*

is Cauchy-continuous. Then the quotient space (S^*, F^*) *is a complete PSM space.*

An argument similar to that given in the latter part of the proof of Proposition 4.1 may be used to prove the following. For sake of brevity, the proof is omitted here.

PROPOSITION 4.2. Assume that (S, F) and (T, G) are two pre-PM spaces such that F and G are each Cauchy-continuous. Suppose that j and k denote the Cauchy-quotient maps into the quotient spaces (S^*, F^*) and (T^*, G^*), respectively. If f is Cauchy-continuous, then there exists a Cauchy-continuous map g such that the diagram below commutes. Moreover, if f is a Cauchy-quotient map, then g is also a Cauchy-quotient map.

$$\begin{tikzcd}
(S,F) \arrow[r, "f"] \arrow[d, "j"'] & (T,G) \arrow[d, "k"] \\
(S^*,F^*) \arrow[r, "g"] & (T^*,G^*).
\end{tikzcd}$$

Proposition 4.2 shows, in particular, that the PSM spaces form a reflective subcategory of the pre-PM spaces, where morphisms are Cauchy-continuous functions.

D.C. Kent [**7**]; Theorem 1 has shown that a convergence space is first-countable if and only if it is the convergence quotient of a pseudo-metric space. A parallel result in our setting is given below. First, suppose that S is any nonempty set and ψ is a filter on S such that ψ has a countable base and $\cap\{A|A \in \psi\}$ contains at most one element of S. Denote the diagonal of S by B; then $V = (\psi \times \psi) \cap \dot{B}$ is a uniformity on S having a countable base. It follows from Theorem 11.5.1 [**24**] that V is metrizable; that is, there exists a metric d on S such that the uniformity V is generated by sets of the form $\{(p,q)|d(p,q) < t\}, t > 0$. The Cauchy structure associated with (S, d) is $\{\phi|\phi \geq \psi \text{ or } \phi = \dot{p}, p \in S\}$.

THEOREM 4.1. *Let (S, F) be any PSM space such that F is Cauchy-continuous. Then (S, F) is the image of a simple space under a Cauchy-quotient map.*

PROOF. Let C denote the Cauchy structure associated with (S, F) and suppose that $\phi \in C$ and ϕ has a countable base. Since F is Cauchy-continuous, it follows that $\cap\{A|A \in \phi\}$ contains at most one element of S. Consider $\{S_\phi|\phi \in C$ and ϕ has a countable base$\}$, where each S_ϕ is a copy of S, S_ϕ and S_ψ are disjoint whenever $\phi \neq \psi$. It follows from the discussion preceding this proposition that there exists a metric d_ϕ on S_ϕ such that the Cauchy structure associated with (S_ϕ, d_ϕ) is $C_\phi = \{\psi|\psi \geq \phi$ or $\psi = \dot{p}, p \in S\}$. Define a metric d on $T = \cup\{S_\phi|\phi \in C$ and ϕ has a countable base $\}$ by

$$d(p,q) = \begin{cases} d_\phi(p,q), & \text{if } p,q \in S_\phi \\ 1, & \text{otherwise.} \end{cases}$$

Let $G \in \Delta^+$ such that $G \neq \varepsilon_0$ and $\lim_{y+\infty} G(y) = 1$. Define $F_T(p,q)(x) = G(x/d(p,q)), x \geq 0$. Then, by Proposition 3.2 (i), (T, F_T, G) is a simple space whose associated Cauchy structure C_T coincides with that of (T, d). However, ψ is a Cauchy filter on (T,d) if and only if there exists an $S_\phi \in \psi$ such that ψ restricted to (S_ϕ, d_ϕ) is Cauchy. It follows from this that the natural map from (T, C_T) onto (S, C) is a Cauchy-quotient map.□

5. Properties of $(\mathbf{S}, \mathbf{C})$. Let (S, F) denote a PSM space and let C denote the associated Cauchy structure. Assume that F is Cauchy-continuous and note that $\phi \sim \psi$ if and only if $\phi \cap \psi \in C$ defines an equivalence relation in C. Let $[\phi]$ denote the equivalence class containing $\phi \in C, T = \{[\phi]|\phi \in C\}$, and $j(p) = [\dot{p}]$. Define $G([\phi], [\psi]) = \lim F(\phi \times \psi); G$ is well-defined since F is Cauchy-continuous and, moreover, (T, G) is a PSM space. The space (T, G) is a possible candidate for a completion of (S, F). The PSM space (S, F) is said to be *Cauchy-separated* if $[\phi] \neq [\psi]$ implies that there exists a real-valued Cauchy-continuous function f defined on (S, C) such that $\lim f\phi \neq \lim f\psi$ on the real line.

The following Cauchy structure on T is given in [**9**]. First, if $A \subseteq S$, then denote $\Sigma A = \{[\phi]|A \in \psi$, for some $\psi \in [\phi]\}$ and let $\Sigma\phi$ denote the filter on T whose base is $\{\Sigma A | A \in \phi\}$. Define E to be the Cauchy structure $\{\Gamma | \Gamma \geq \Sigma\phi, \phi \in C\}$. The property that $G(\Sigma\phi \times \Sigma\phi) \geq \mathrm{cl}F(\phi \times \psi)$ is used below. A Cauchy space is said to induce a pretopology whenever each point has a coarsest convergent filter.

PROPOSITION 5.1. *Let (S, F) denote a PSM space such that F is Cauchy-continuous. The following properties hold:* (i) *(S, C) is regular and Cauchy-separated,* (ii) *each $[\phi]$ contains a least element, and* (iii) *(S, C) induces a pretopology on S.*

PROOF. (i). Let $\phi \in C$; then $F(\mathrm{cl}\phi \times \mathrm{cl}\phi) = F(\mathrm{cl}(\phi \times \phi)) \geq \mathrm{cl}F(\phi \times \phi) \to \varepsilon_0$ in Δ^+ and thus $\mathrm{cl}\phi \in C$. Hence (S, C) is regular. Suppose that $[\phi] \neq [\psi]$ and let $G([\phi], [\psi]) = F$ in Δ^+. Since $[\phi] \neq [\psi]$, then $F \neq \varepsilon_0$. Define $h : S \to T \times T$ by $h(p) = ([\dot{p}], [\psi])$ for each $p \in S$ and let $k : \Delta^+ \to R$ be defined by $k(G) = d_L(G, F)$ for each $G \in \Delta^+$. Finally, let $f : S \to R$ denote $f = k \circ G \circ h$. Suppose that $\Gamma \in C$; then $f(\Gamma) = kGh(\Gamma) = kG(j\Gamma \times [\psi]) \geq kG(\Sigma\Gamma \times \Sigma\psi) \geq kclF(\Gamma \times \psi) \geq \mathrm{cl}kF(\Gamma \times \psi)$ converges on R. It follows that $f : (S, C) \to R$ is Cauchy-continuous and, moreover, $f(\phi) \geq \mathrm{cl}kF(\phi \times \psi) \to k(F) = 0$ and $f(\psi) \geq \mathrm{cl}kF(\psi \times \psi) = k(\varepsilon_0)$ on R. Since $F \neq \varepsilon_0$, then $k(\varepsilon_0) > 0$ and thus (S, C) is Cauchy-separated.

(ii). Since F is Cauchy-continuous, then $F(\phi \times \psi) \to \varepsilon_0$ in Δ^+ whenever $\phi \cap \psi \in C$. It follows that

$$F\Big[\bigcap_{\phi \in [\psi]} \phi \cap \bigcap_{\alpha \in [\psi]} \alpha\Big] = \bigcap_{\phi, \alpha \in [\psi]} F(\phi \times \alpha) \to \varepsilon_0$$

in Δ^+ and thus

$$\bigcap_{\phi \in [\psi]} \phi \in C.$$

Hence $[\psi]$ contains a least element.

(iii). Let $p \in S$; recall that $\phi \to p$ if and only if $\phi \cap \dot{p} \in C$. It follows from (ii) that $\cap\{\phi | \phi \to p\}$ also converges to p and thus (S, C) induces a pretopology on S.

The PSM space (S, F) is said to be *uniformizable* whenever there exists a uniformity on S whose Cauchy filters coincide with those of (S, C). Recall that a PM space (S, F, τ) is uniformizable whenever τ is continuous at $(\varepsilon_0, \varepsilon_0)$. The strong uniformity has C as its set of Cauchy filters according to Proposition 3.2 (ii). E. Lowen-Colebunders [**14**] characterized the Cauchy spaces which are uniformizable. A sufficient condition for a T_3, totally bounded, Cauchy space to be uniformizable is that it is Cauchy-separated and that each Cauchy filter contains a minimal Cauchy filter [**10**]; Theorems 5.4. Hence, by Proposition 5.1, the following result holds.

COROLLARY 5.1. *A totally bounded PSM space (S, F) is uniformizable whenever F is Cauchy-continuous and, in particular, (S, F) induces a topology on S.*

The following example shows that, in general, a PSM space (S, F) does not induce a topology on S even whenever F is Cauchy-continuous.

EXAMPLE 5.1. Suppose that d denotes the two-dimensional Euclidean metric. Define $p_{n,k} = (\frac{1}{n}, \frac{1}{k}), p_k = (0, \frac{1}{k}), p_0 = (0,0)$, and let $S = \{p_{nk}, p_k, p_0 | n, k \geq 1\}$. For each $x > 0$, define $F : S \times S \to \Delta^+$ as follows: $F(p,q)(x) = \varepsilon_0(x-1)$ if $p = p_{kl}, q = p_{mn}$ and either $n \geq k + l$ or $l \geq m + n$; otherwise,

$$F(p,q)(x) = \begin{cases} 1 - d(p,q), & \text{if } 0 < x \leq 1 \text{ and } d(p,q) < 1 \\ 1, & \text{if } x > 1 \text{ and } d(p,q) < 1 \\ \varepsilon_0(x-1), & \text{if } d(p,q) \geq 1 \end{cases}$$

Note that $F(p,p) = \varepsilon_0$. Suppose that $F(p,q) \neq \varepsilon_0$; then $d(p,q) \neq 0$ and thus $p \neq q$. Observe that $F(p,q) = F(q,p)$ and thus (S, F) is a PSM space.

Let $A = \{p_{kl} | k, l \geq 1\}$ and suppose $\{p_{k_n l_n}\}$ is any sequence contained in A such that $\{l_n\}$ assumes an infinite number of distinct values. If M is any positive integer, then there exists an n such that $l_n \geq k_M + l_M$. Thus $F(p_{k_n l_n} P_{k_n l_n})(x) = \varepsilon_0(x-1)$ and hence $\{p_{k_n l_n}\}$ is not a Cauchy sequence. This shows that, for any Cauchy sequence in A, there is a $y_0 > 0$ such that all values of the sequence lie above the line $y = y_0$.

For any $l \geq 1$, let ϕ_l denote the filter generated by $\{p_{kl}\}, k \geq 1$, and let ϕ_0 be the filter generated by $\{p_k\}$. The Cauchy structure associated with (S, F) is $C = \{\psi | \psi \geq \phi_k \cap \dot{p}_k \text{ or } \psi = \dot{p}, p \in S, k \geq 0\}$. It follows that $p_0 \in cl^2 A$ but $p_0 \notin \text{cl}A$ and thus the Cauchy space fails to induce a topology on S. Note that F is Cauchy-continuous.

6. Completion. More than one Cauchy structure will be considered throughout the remainder of our study. Hence, if (S, F) is a PSM space, denote $A_S = \{\phi | F(\phi\ x\ \phi) \to \varepsilon_0$ in $\Delta^+]$ and let C_S be the smallest Cauchy structure containing A_s. Recall that $\phi \in C_S$ if and only if there exists a finite collection $\phi_i \in C_S$ such that $\phi \geq \cap_i^n \phi_i$ and $\phi_i \vee \phi_{i+1}$ exists. Moreover, $C_S = A_S$ whenever F is Cauchy-continuous.

A *Cauchy space* is said to be *complete* (*compact*) whenever every Cauchy filter (ultrafilter) converges. A PSM *space* (S, F) is said to be *complete* (*compact*) whenever the associated Cauchy space (S, C_S) is complete (compact). Furthermore, (T, G, j) is called a *completion* (*compactification*) of (S, F) if (i) (T, G) is a complete (compact) PSM space, (ii) $j; (S, C_S) \to (T, C_T)$ is a dense Cauchy embedding, and (iii) $F(p, q) = G(j(p), j(q))$ for each $p, q \in S$. It should be remarked that condition (iii) implies condition (ii) whenever both F and G are Cauchy-continuous.

Let (S, F) be a PSM space and define, for $t > 0, U(t) = \{(p, q) | d_L (F(p, q), \varepsilon_0) < t\}$. Then the collection of all sets of the form $U(t), t > 0$, generates a semi-uniformity U on S. If C_S denotes the Cauchy structure associated with (S, F) and if U is a uniformity, then it is easy to verify that $\phi \in C_S$ if and only if $\phi \times \phi \geq U$. This implies that the Cauchy structure of (S, U) is C_S whenever U is a uniformity. The semi-uniformity U is a uniformity whenever (S, F, τ) is a PM space and τ is continuous at $(\varepsilon_0, \varepsilon_0)$. H. Sherwood **[20, 21]** has shown that every PM space (S, F, τ) has a completion whenever τ is continuous. The continuity of τ is crucial in the proof. The following result extends Sherwood's theorem to a larger class of spaces.

THEOREM 6.1. *Suppose that (S, F) is a PSM space. Assume that F is Cauchy-continuous and U, defined above, is a uniformity. Then (S, F) has a completion (T, G, j) and, moreover, G is Cauchy-continuous.*

PROOF. Let $\phi, \psi \in C_S$ and define $\phi \sim \psi$ if and only if $\phi \cap \psi \in C_S$. This defines an equivalence relation on C_S. Moreover, since F is Cauchy-continuous, then $\phi \sim \psi$ if and only if $F(\phi \times \psi) \to \varepsilon_0$ in Δ^+. Let

$T = \{[\phi] | \phi \in C_S\}$ denote the quotient set and define $j(p) = [\dot{p}], p \in S$. Then $G([\phi], [\psi]) = \lim F(\phi \times \psi)$ is well-defined and, furthermore, (T, G) is a PSM space.

Each equivalence class $[\phi]$ contains a least element. Given $t > 0$, define $W(t) = \{([\phi], [\psi]) | U(t) \in \phi \times \psi$, where ϕ, ψ are minimal$\}$. Sets of the form $W(t), t > 0$, are a basis for a uniformity W on T. Moreover, (T, W, j) is precisely the uniform space completion of (S, U) given in Bourbaki [**3**; Theorem 3, p. 191].

Define $N_{\varepsilon_0}(t) = \{F \in \Delta^+ | d_L(F, \varepsilon_0) < t\}$. Let us show that the set of all Cauchy filters of (T, W) is precisely C_T. Suppose that $\Gamma \in A_T$; that is, $G(\Gamma \times \Gamma) \to \varepsilon_0$ in Δ^+. It must be shown that $\Gamma \times \Gamma \geq W$. Given $t > 0$, there exists a $B \in \Gamma$ such that $G(B \times B) \subseteq N_{\varepsilon_0}(t)$. Let us show that $B \times B \subseteq W(t)$. If $([\phi], [\psi]) \in B \times B$, where ϕ and ψ are minimal Cauchy, then $G([\phi], [\psi]) = \lim F(\phi \times \psi) \in N_{\varepsilon_0}(t)$ and thus there exists an $A_1 \in \phi, A_2 \in \psi$ such that $F(A_1 \times A_2) \subseteq N_{\varepsilon_0}(t)$. This implies that $A_1 \times A_2 \subseteq U(t)$ and thus $U(t) \in \phi \times \psi$. Hence $([\phi], [\psi]) \in W(t)$ and $B \times B \subseteq U(t)$. This implies that $\Gamma \times \Gamma \geq W$. Finally, suppose that $\Gamma \geq \cap_i^n \Gamma_i$, where $\Gamma_i \vee \Gamma_{i+1}$ exists and $\Gamma_i \in A_T$. Since $\Gamma_i \times \Gamma_j = (\Gamma_i \times \Gamma_i) \circ (\Gamma_{i+1} \times \Gamma_{i+1}) \circ \cdots \circ \Gamma_j \times \Gamma_j) \geq W$, it follows that $\Gamma \times \Gamma \geq \cap_{i,j=1}^n (\Gamma_i \times \Gamma_j) \geq W$. Conversely, if $\Gamma \times \Gamma \geq W$, then an argument analogous to the above may be used to show that $\Gamma \in C_T$ and thus the Cauchy structure of (T, W) is C_T. Since (T, W, j) is a uniform space completion of (S, U), it follows that $j : (S, C_S) \to (T, C_T)$ is a dense Cauchy embedding and thus (T, G, j) is a completion of (S, F).

The Cauchy structure C_T induces a topology on T and thus if $\Gamma_i \in C_T$, then there exists a $\phi_i \in C_S$ such that $\text{cl}j\phi_i \leq \Gamma_i, i = 1, 2$. It is straightforward to verify that $G(\Gamma_1 \times \Gamma_2) \geq G(\text{cl}j\phi_1 \times \text{cl}j\phi_2)\text{cl}F(\phi_1 \times \phi_2)$. Since $\text{cl}F(\phi_1 \times \phi_2)$ is a Cauchy filter on Δ^+, it follows that G is Cauchy-continuous. □

The following result shows that the completion has an extension property for Cauchy-continuous maps.

PROPOSITION 6.1 *Assume that (S, F) is a PSM space, F is Cauchy-continuous, and U is a uniformity. Suppose that R, H is a complete PSM space and that H is Cauchy-continuous. Moreover, assume that $k : (S, C_S) \to (R, C_R)$ is any Cauchy-continuous map. There exists a*

Cauchy-continuous l such that the diagram below commutes.

$$\begin{tikzcd} (S, C_S) \arrow[rr, "j"] \arrow[dr, "k"'] & & (T, C_T) \arrow[dl, "l"] \\ & (R, C_R) & \end{tikzcd}$$

PROOF. Since H is Cauchy-continuous, then (R, C_R) is a T_3 Cauchy space. Let $\phi \in C_S$; define $l([\phi]) = \lim k\phi$ in (R, C_R). Since C_T induces a topology on T, if $\Gamma \in C_T$, there exists a $\psi \in C_S$ such that $\mathrm{cl} j\psi \leq \Gamma$. It is easily verified that $\ell\Gamma \geq l(\mathrm{cl}1j\psi) \geq \mathrm{cl}(lj\psi) = \mathrm{cl}(k\psi)$ and, since (R, C_R) is regular, $l\Gamma \in C_R$. Hence l is a Cauchy-continuous extension of k to (T, C_T).

The following result shows the uniqueness of the completion.

COROLLARY 6.1. *Suppose that (S, F) is a PSM space such that F is Cauchy-continuous and U is a uniformity. Assume that (R, H, k) is another completion of (S, F) such that H is Cauchy-continuous. Then (T, G, j) and (R, H, k) are isometric.*

PROOF. It follows from Proposition 6.1 that there exists an onto Cauchy-continuous map $l : (T, C_T) \to (R, C_R)$ such that $l \circ j = k$. Suppose that $[\phi], [\psi] \in T$. Since H is Cauchy-continuous, $H(l[\phi], l[\psi]) = H(\lim\, k\phi, \lim\, k\psi) = \lim\, H(k\phi, k\psi) = \lim\, F(\phi \times \psi) = G([\phi], [\psi])$ and thus (T, G, j) and $R, H, k)$ are isometric completions of (S, F).

7. Compactification. Let (S, F) be a PSM space and let U denote the semi-uniformity defined in the preceeding section; U, in general, is not a uniformity for S. This leads us to the following concept which was first introduced by C.H. Cook and H.R. Fisher [**4**]. A collection I of filters on $S \times S$ is called a *uniform convergence structure* if (1) $\{(p,p) | p \in S\} \in I$; (2) $\phi \in I$ implies $\phi^{-1} \in I$; (3) $\phi \geq \psi \in I$ implies $\phi \in I$; (4) $\phi, \psi, \in I$ implies $\phi \cap \psi \in I$; and (5)

$\phi, \psi \in I$ implies $\phi \circ \psi \in I$ whenever $\phi \circ \psi$ exists (i.e., $A \circ B$ is nonempty for each $A \in \phi, B \in \psi$). The pair (S, I) is called a *uniform convergence space.* A uniform convergence space (S, I) induces the Cauchy structure $C_I = \{\phi | \phi \times \phi \in I\}$ on S.

Given a PSM space (S, F) and associated semi-uniformity U, consider the uniform convergence structure $I = \{\Gamma | \Gamma \geq U^n, n \geq 1\}$, where U^n denotes the composition of U with itself $n-1$ times. Note that I is the smallest uniform convergence structure containing the element U. Let C_S denote the Cauchy structure associated with (S, F) and let C_I denote the Cauchy strucutre induced by I. Recall that if $\psi \in C_S$, then $\psi \geq \cap_i^n \phi_i$ for some $\phi_i \times \phi_i \geq U$ such that $\phi_i \vee \phi_{i+1}$ exists. Since $\phi_i \times \phi_{i+k} = (\phi_i \times \phi_i) \circ (\phi_{i+1} \times \phi_{i+1}) \circ \cdots \circ (\phi_{i+k} \times \phi_{i+k}) \geq U^{k+1}$, it follows that $\psi \times \psi \geq U^{n+1}$ and thus $\psi \in C_I$. Hence $C_S \subseteq C_I$ holds in general.

A PSM space (uniform convergence space) is totally bounded whenever its associated Cauchy space is totally bounded. Similarly, a PSM space is said to be *Cauchy-separated* whenever its associated Cauchy space is. By Proposition 5.1 (i), every PSM space (S, F), with F Cauchy-continuous, is Cauchy-separated. The following results appear as Theorem 3.4 [**9**] and Theorem 1.8 [**11**].

THEOREM 7.1. [**9, 11**]. (i) *Let (S, I) be a totally bounded uniform convergence space. Then (S, C_I) has T_3-completion if and only if* (1) $V = \cap\{\Gamma | \Gamma \in I\}$ *is a T_2 uniformity;* (2) (S, C_I) *is regular; and* (3) *if ϕ, ψ are ultrafilters on S such that $\phi \times \psi \geq V$, then $\phi \times \psi \in I$.*

(ii) *A totally bounded, T_3, Cauchy space has a T_3-completion if and only if it is Cauchy-separated.*

PROPOSITION 7.1. *Let (S, F) be a totally bounded PSM space such that F is Cauchy-continuous. Let U be the associated semi-uniformity and $V = \cap\{U^n | n \geq 1\}$. Then $C_I = C_S$ if and only if $\phi \times \psi \geq V$ implies that $\phi \cap \psi \in C_S$ whenever ϕ, ψ are ultrafilters on S. Furthermore, if $C_I = C_S$, then V is a T_2 uniformity and $C_V = C_S$.*

PROOF. It follows from Theorem 7.1 (ii) that (S, C_S) has a T_3-completion; hence, the necessity follows from Theorem 7.1 (i). Conversely, let $\psi \in C_I$ and let ϕ_1, ϕ_2 denote two ultrafilters each containing ψ. Then, for some $n, \phi_1 \times \phi_2 \geq U^n \geq V$, and, by assumption, $\phi_1 \cap \phi_2 \in C_S$. Each equivalence class $[\psi]$ of (S, C_S) contains a least element. Since $\psi = \cap\{\phi | \phi$ is an ultrafilter containing $\psi\}$, it follows that $\psi \in C_S$ and thus $C_I = C_S$. In fact, the argument given above shows that if $\psi \times \psi \geq V$, then $\psi \in C_S$. Hence the last part of the proposition follows from this fact and Theorem 7.1.

Let U_R denote the usual uniformity on the real line. A semi-uniform space (S, U) is said to be *u-separated* if for each pair ϕ, ψ of ultrafilters on S, with $\phi \times \psi \not\geq U$, there exists an $f : (S, U) \to (R, U_R)$ such that $(f \times f)(U) \geq U_R$ and $\lim f\phi \neq \lim f\psi$. Note that every totally bounded, T_2 uniform space is u-separated.

PROPOSITION 7.2. *Let (S, F) be a totally bounded PSM space such that F is Cauchy-continuous. Suppose that U denotes the associated semi-uniformity. Then U is a uniformity if and only if (S, U) is u-separated.*

PROOF. Suppose that the semi-uniform space (S, U) is u-separated. Let us show first that if ϕ, ψ are two ultrafilters on S such that $\phi \times \psi \geq V$, then $\phi \cap \psi \in C_S$. Assume that $\phi \cap \psi \notin C_S$; then $\phi \times \psi \not\geq U$ and since (S, U) is u-separated, there exists an $f : (S, U) \to (R, U_R)$ such that $(f \times f)(U) \geq U_R$ and $\lim f\phi \neq \lim f\psi$. However, $(f \times f)(V) = (f \times f)(\cap_1^\infty U^n) = \cap_1^\infty (f \times f)(U^n) \geq U_R$, and thus $f\phi \times f\psi \geq U_R$. This is contrary to the fact that $\lim f\phi \neq \lim f\psi$; hence, $\phi \cap \psi \in C_S$ and thus by Proposition 7.1, $C_V = C_S$.

It follows from results due to E. Lowen-Colebunders [**13**; Theorem 5.12] and [**14**; Theorem 5.1] that $\mu = \cap\{\phi \times \phi | \phi \in C_S\}$ is the unique uniformity having C_S as its Cauchy structure. Since $\mu \subseteq U \subseteq V$ and $C_V = C_S$, then $\mu = U = V$ is a uniformity.

A totally bounded, complete Cauchy space is compact. A necessary condition for a PSM space to have a compactification is that it be totally bounded. Let us call a PSM (S, F) u-*separated* whenever the associated semi-uniform space is u-separated. Theorem 6.1 and Proposition 7.2 combine to give the following compactification result.

THEOREM 7.2. *Let (S, F) be a totally bounded, u-separated PSM space such that F is Cauchy-continuous. Then (T, G, j) is a compactification of (S, F) and, moreover, G is Cauchy-continuous.*

Let us conclude with the remark that extension and uniqueness results for the compactification follow readily from Proposition 6.1 and Corollary 6.1 whenever (S, F) is a totally bounded, u-separated PSM with F Cauchy-continuous.

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