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FREQUENCY OF COPRIMALITY OF THE VALUES OF A POLYNOMIAL AND A PRIME-INDEPENDENT MULTIPLICATIVE FUNCTION*

CLAUDIA SPIRO

1. Introduction. Let H, J, and n denote positive integers, take P to be a polynomial with integer coefficients, and assume that M is a nonzero integer-valued multiplicative function such that

$$M(p) = H, \qquad M(p^2) = J, \tag{1}$$

for every prime p. In 1976, E. J. Scourfield **[4]** obtained the estimate

$$\#\{n \leqq x: (M(n), n) = 1\} \sim c_M x, \tag{2}$$

as x tends to ∞. She obtained equally precise results for elements of the class of "polynomial-like" arithmetic functions—a class which includes ϕ and σ. Five years later, in the Ph.D. dissertation of the author **[6**, §3.4], we obtained an estimate for the left side of (2) which is more precise, if a certain convergence condition is satisfied. For example, we showed that

$$\#\{n \leqq x: (d(n), n) = 1\} = c_d x + 0(\sqrt{x}(\log x)^3), \tag{3}$$

where $d(n)$ is the number of positive integers dividing n, and c_d is a computable constant with $0 < c_d < 1$. In this paper, we derive the following estimate for $\#\{n \leqq x: (M(n), P(n)) = 1\}$.

THEOREM 1. *$\#\{n \leqq x: (P(n), M(n)) = 1\} = C_{M,P}x + 0(\sqrt{x}(\log x)^{2J} E(x, M))$, where*

$$C_{M,P} = \frac{6}{\pi^2 H} \sum_{t=1}^{\infty} \frac{1}{tU} \prod_{p \mid tU} (1 - p^{-2})^{-1} \sum_{\substack{b \bmod U \\ (P(bt), U) = (b, U, t) = 1 \\ \mu((q, U)) \neq 0}} \prod_{\substack{p \mid t(b, U) \\ p \nmid U/(b, U)}} (1 - p^{-1}),$$

where $U = HM(t)$, and

$$E(x, M) = \sum_{\substack{t \leq x \\ t \text{ cubefull}}} 2^{\omega(t)} |M(t)| t^{-1/2}.$$

If $E(\infty, M)$ converges, then $E(x, M)$ can be omitted from the error term. As a special case of this result, we show that

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$$\#\{n \leqq x: (d(n), P(n)) = 1\} = C_{d,P}x + 0(\sqrt{x}(\log x)^6).$$

We also give improved results for the case when $P(n) = n$, which enable us to replace $(\log x)^3$ by $(\log x)^2$ in (3) (see Corollary 2, below, with $k = 2$).

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2. Preliminary results. Throughout this paper, we will let m, n, t denote positive integers, take p to represent a prime, and restrict x to be a positive real number. Thus, sums of the form $\sum_{n\leqq x}$ will run over positive integers not exceeding x, and products of the shape $\prod_{p|n}$ will extend over the prime divisors of n. Set $d(n)$ equal to the number of positive integers dividing n, put $\omega(n)$ for the number of distinct prime divisors of n, and let μ denote the Möbius function.

LEMMA 1. *Let h, k, and q be integers with k and q positive, and $(h, k) = 1$. Then we have*

$$\#\{n \leqq x: \mu(n) \neq 0, (n, q) = 1, n \equiv h(\mathrm{mod}\ k)\} = C_{k,q} + 0(2^{\omega(q)}\sqrt{x}),$$

where the implied constant is absolute, and where

$$C_{k,q} = \frac{6}{\pi^2}\frac{1}{k}\prod_{\substack{p|q\\ p\nmid k}}(1 - \mathrm{p}^{-1}) \cdot \prod_{p|qk}(1 - \mathrm{p}^{-2})^{-1}.$$

PROOF. We observe that

$$\begin{aligned}\#\{n \leqq x: \mu(n) \neq 0, (n, q) = 1, n \equiv h(\mathrm{mod}\ k)\} &= \sum_{\substack{n\leqq x\\ n\equiv h(\mathrm{mod} k)\\ (n,q)=1}} \sum_{\ell^2|n}\mu(\ell) \\ &= \sum_{\substack{\ell\leqq\sqrt{x}\\ (\ell,q)=1}}\mu(\ell)\sum_{\substack{m\leqq x/\ell^2\\ m\ell^2\equiv h(\mathrm{mod} k)}}\sum_{\substack{d|m\\ d|q}}\mu(d),\end{aligned} \tag{4}$$

since the sum on d is 1 if m and q are coprime, and 0 otherwise. When we interchange the two inner sums, we find that

$$\begin{aligned}\sum_{\substack{m\leqq x/\ell^2\\ m\ell^2\equiv h(\mathrm{mod} k)}}\sum_{\substack{d|m\\ d|q}}\mu(d) &= \sum_{d|q}\mu(d)\sum_{\substack{m\leqq x/\ell^2\\ m\ell^2\equiv h(\mathrm{mod} k)\\ d|m}} 1 \\ &= \sum_{d|q}\mu(d)\sum_{\substack{r\leqq x/(\ell^2 d)\\ r\ell^2 d\equiv h(\mathrm{mod} k)}} 1\end{aligned} \tag{5}$$

Now, h and k are coprime, so that the congruence condition on the last sum has one solution r modulo k if $(\ell^2 d, k) = 1$, and no solutions otherwise. Hence (4) and (5) imply that

$$\begin{aligned}&\#\{n \leqq x: \mu(n) \neq 0, (n, q) = 1, n \equiv h(\mathrm{mod}\ k)\} \\ &= \sum_{\substack{\ell\leqq\sqrt{x}\\ (\ell,qk)=1}}\mu(\ell)\sum_{\substack{d|q\\ (d,k)=1}}\mu(d)\left(\frac{x}{\ell^2 dk} + 0(1)\right)\end{aligned} \tag{6}$$

$$= \frac{x}{k} (\prod_{\substack{p|q \\ p \nmid k}} (1 - p^{-1})) \sum_{\substack{\ell \leq \sqrt{x} \\ (\ell, qk)=1}} \frac{\mu(\ell)}{\ell^2} + 0(\sum_{\ell \leq \sqrt{x}} \sum_{d|q} |\mu(d)|).$$

We can conclude from the definitions of ω and μ that the remainder term in (6) is $0(\sqrt{x}\, 2^{\omega(q)})$. Finally, we note that

$$\sum_{\substack{\ell \leq \sqrt{x} \\ (\ell, qk)=1}} \frac{\mu(\ell)}{\ell^2} = \sum_{\substack{\ell=1 \\ (\ell, qk)=1}}^{\infty} \frac{\mu(\ell)}{\ell^2} + 0(\sum_{\ell > \sqrt{x}} \ell^{-2})$$

$$= \frac{6}{\pi^2} \prod_{p|qk} (1 - p^{-2})^{-1} + 0(x^{-1/2}),$$

to complete the proof. We remark that our Lemma 1 is a generalization of Theorem 1 of [**1**].

DEFINITION. For each positive integer k, we define the multiplicative functions $d_k(n)$ by

$$(\sum_{n=1}^{\infty} n^{-s})^k = \sum_{n=1}^{\infty} d_k(n) n^{-s}.$$

Thus [**3**, Theorem 299, p. 255], we have

$$d_k(p^a) = \frac{k(k+1) \cdots (k+a-1)}{a!} \tag{7}$$

for every p and each nonnegative integer a.

LEMMA 2. (i) *For fixed* k, $d_k(n) = n^{o(1)}$.
(ii) *The multiplicative function* $2^{\omega(n)}/\sqrt{n}$ *is bounded.*

PROOF. The lemma follows immediately from Theorem 3.6 on p. 260 of [**3**]. For i), we apply it to the multiplicative function $d_k(n)n^{-\varepsilon}$, where $\varepsilon > 0$ is fixed but arbitrary, and to obtain ii), we apply it to $2^{\omega(n)}/\sqrt{n}$.

For convenience, we put

$$\mathscr{S} = \{n\colon \text{if } p|n \text{ then } p^2|n\} = \{\text{squarefull numbers}\},$$
$$\mathscr{C} = \{n\colon \text{if } p|n \text{ then } p^3|n\} = \{\text{cubefull numbers}\}.$$

LEMMA 3. *In the notation of* §1, *we have*

$$\sum_{\substack{n \leq x \\ n \in \mathscr{S}}} 2^{\omega(n)} |M(n)|^{-1/2} n^{-1/2} \ll (\log x)^{2J}\, E(x, M).$$

Designate the expression on the left by $T(x, M)$.

PROOF. We can write any squarefull number n uniquely as

$$n = m^2 \ell : \ell \in \mathscr{C}, \qquad \mu(m) \neq 0,\ (m, \ell) = 1.$$

Partitioning the set of positive integers n according to the value of ℓ yields

$$T(x, M) = \sum_{\substack{\ell \leq x \\ \ell \in \mathscr{C}}} 2^{\omega(\ell)} |M(\ell)| \ell^{-1/2} \sum_{\substack{m \leq \sqrt{x/\ell} \\ \mu(m) \neq 0 \\ (m, \ell)=1}} 2^{\omega(n)} |M(m^2)| m^{-1} \tag{8}$$

since the functions $M(n)$ and $2^{\omega(m)}$ are multiplicative. Now, (1) implies that $M(m^2) = J^{\omega(m)}$ in (8). Therefore, the inner sum in (8) is

$$U(x, M) = \sum_{\substack{m \leq \sqrt{x/\ell} \\ \mu(m) \neq 0 \\ (m, \ell)=1}} (2J)^{\omega(m)} m^{-1} \tag{9}$$

Since $(2J)^{\omega(m)} m^{-1}$ is multiplicative, and the sum is over squarefree m only, we have

$$U(x, m) \leq \prod_{p \leq \sqrt{x}} \left(1 + \frac{2J}{p}\right) \leq \prod_{p \leq \sqrt{x}} \left(1 + \frac{1}{p}\right)^{2J}. \tag{10}$$

Now, the estimate $\prod_{p \leq z} (1 + 1/p) \ll \log z$ as $z \to \infty$ implies that the right side of (9) is $0(\log x)^{2J})$. Combining this result with (8) gives

$$T(x, M) \ll (\log x)^{2J} \sum_{\substack{\ell \leq x \\ \ell \in \mathscr{C}}} 2^{\omega(\ell)} |M(\ell)| \ell^{-1/2} = (\log x)^{2J} E(x, M).$$

LEMMA 4. $\sum_{\substack{\ell \leq x \\ \ell \in \mathscr{S}}} 2^{\omega(tM(t))} t^{-1/2} \ll (\log x)^2 E(x, M)$.

PROOF. By analogy with the derivation of (7), we have

$$\sum_{\substack{t \leq x \\ t \in \mathscr{C}}} 2^{\omega(tN(t))} t^{-1/2} = \sum_{\substack{\ell \leq x \\ \ell \in \mathscr{C}}} \frac{1}{\sqrt{\ell}} \sum_{\substack{m \leq \sqrt{x/\ell} \\ \mu(m) \neq 0 \\ (m, \ell)=1}} 2^{\omega(m^2 \ell M(m^2) M(\ell))} m^{-1}.$$

Since $\omega(mn) \leq \omega(m) + \omega(n)$ for all integers m and n, and since $M(m^2) = J^{\omega(m)}$ for all squarefree m by (1), we can conclude that

$$\sum_{\substack{t \leq x \\ t \in \mathscr{S}}} 2^{\omega(tM(t))} t^{-1/2} \leq \sum_{\substack{\ell \leq x \\ \ell \in \mathscr{C}}} \frac{2^{\omega(\ell M(\ell))}}{\sqrt{\ell}} \sum_{\substack{m \leq \sqrt{x/\ell} \\ \mu(m) \neq 0 \\ (m, \ell)=1}} 2^{\omega(m)} 2^{\omega(J)} m^{-1}. \tag{11}$$

But from the estimate $0((\log x)^{2J})$ of the quantity in (9) with that J replaced by 1, we can deduce that the inner sum is $0((\log x)^2)$. The lemma immediately follows.

LEMMA 5. (i) $\sum_{\substack{t \leq x \\ t \in \mathscr{S}}} 1 \ll \sqrt{x}$.

(ii) $\sum_{\substack{t \leq x \\ t \in \mathscr{C}}} 1 \ll x^{1/3}$.

PROOF. This lemma is immediate from results of Erdös and Szekeres **[2]**. However, for completeness, we present a proof. For part i), we observe that any squarefull number can be uniquely written as $t = m^2 k^3$, with $\mu(k) \neq 0$. Hence,

$$\sum_{\substack{t\leq x\\ t\in\mathscr{S}}} 1 = \sum_{\substack{k\leq x^{1/3}\\ \mu(k)\neq 0}} \sum_{m\leq x^{1/2}k^{-3/2}} 1 \leq \sum_{k\leq x^{1/3}} x^{1/2}k^{-3/2} \ll x^{1/2}.$$

Similarly, for part ii), we notice that any cubefull number can be written as $t = m^3n^4k^5$, where $\mu(n) \neq 0$, $\mu(k) \neq 0$, and $(n, k) = 1$. Therefore,

$$\sum_{\substack{t\leq x\\ t\in\mathscr{C}}} 1 \leqq \sum_{n\leq x^{1/4}} \sum_{k\leq x^{1/5}n^{-4/5}} \sum_{m\leq x^{1/3}n^{-4/3}k^{-5/3}} 1 \leq \sum_{n\leq x^{1/4}} \sum_{k\leq x^{1/5}n^{-4/5}} x^{1/3}n^{-4/3}k^{-5/3} \ll x^{1/3}.$$

3. The proof of Theorem 1. For ease and clarity of exposition, place

$F(x) = F(x, M, P) = \#\{n \leqq x: (M(n), P(n)) = 1\}$;
$Q(x, t) = \{n \leqq x/t: \mu(n) \neq 0, (n, t) = 1\}$;
$A(t) = \{b \bmod HM(t): (HM(t), P(bt)) = 1\}$; and
$B(t) = \{b \bmod HM(t): (b, HM(t), t) = 1, \mu(b, HM(t)) \neq 0\}$.

Any positive integer n can be factored uniquely as

$$n = mt: \mu(m) \neq 0,\ t \in \mathscr{S},\ (m, t) = 1.$$

If we partition the $n \leqq x$ according to the value of t, we find that

$$F(x) = \sum_{\substack{t\leq x\\ t\in\mathscr{S}}} \sum_{\substack{m\in Q(x,t)\\ (M(mt),P(mt))=1}} 1. \tag{12}$$

The definition of $Q(x, t)$, the multiplicativity of M, and (1) imply that $M(mt) = H^{\omega(m)}M(t)$, for every m contributing to the inner sum. Since $\omega(m) = 0$ if and only if $m = 1$, we can conclude that

$$F(x) = \sum_{\substack{t\leq x\\ t\in\mathscr{S}}} \Big\{ \sum_{\substack{m\in Q(x,t)\\ (HM(t),P(mt))=1}} 1 + 0(1)\Big\}.$$

By the first part of Lemma 5, we have

$$F(x) = \sum_{\substack{t\leq x\\ t\in\mathscr{S}}} \sum_{\substack{m\in Q(x,t)\\ (HM(t),P(mt))=1}} 1 + 0(\sqrt{x}).$$

Subdivide the set of m contributing to the inner sum, according to their remainder modulo $HM(t)$, i.e.,

$$F(x) = \sum_{\substack{t\leq x\\ t\in\mathscr{S}}} \sigma(x, t) + 0(\sqrt{x}), \tag{13}$$

where

$$\sigma(x, t) = \sum_{b\in A(t)} \sum_{\substack{m\in Q(x,t)\\ m\equiv b (\mathrm{mod} HM(t))}} 1 . \tag{14}$$

By the definition of $Q(x, t)$, any m contributing to the inner sum in (14) must satisfy the conditions $(b, HM(t))|m$, $(m, t) = 1$, $\mu(m) \neq 0$. Consequently, it follows from the definitions of $A(t)$ and $B(t)$ that

$$\sigma(x, t) = \sum_{b\in A(t)\cap B(t)} \sum_{\substack{m\in Q(x,t)\\ m\equiv b (\mathrm{mod} HM(t))}} 1$$

Next, we make the substitution $\ell = m/(b, HM(t))$ in the inner sum

$$\sigma(x, t) = \sum_{b \in A(t) \cap B(t)} \sum_{\substack{\ell \in Q(x, (b, HM(t))t) \\ \ell \equiv b/(b, HM(t)) \ (\mathrm{mod} HM(t)/(b, HM(t)))}} 1.$$

We now apply Lemma 1 to the inner sum, with $k = HM(t)/(b, HM(t))$ and with $q = (b, HM(t))t$. When we combine the result with (13), we obtain

$$F(x) = \frac{6x}{\pi^2 H} \sum_{\substack{t \leq x \\ t \in \mathcal{J}}} \frac{1}{tM(t)} \sum_{b \in A(t) \cap B(t)} \prod_{p | HtM(t)} (1 - p^{-2})^{-1} \prod_{\substack{p | (b, HM(t))t \\ p \nmid HM(t)/(b, HM(t))}} (1 - p^{-1})$$

(15)

$$+ 0\Big(\sqrt{x} \sum_{\substack{t \leq x \\ t \in \mathcal{J}}} \frac{2^{\omega(t)}}{\sqrt{t}} \sum_{b \bmod HM(t)} \frac{2^{\omega((b, HM(t)))}}{\sqrt{(b, HM(t))}}\Big).$$

Since $\pi_p(1 - p^{-2})^{-1}$ converges, and $1 - p^{-1} \leqq 1$, both products over primes occuring in (15) are bounded, Hence, the main term in (15) is

$$\frac{6x}{\pi^2 H} \sum_{\substack{t=1 \\ t \in \mathcal{J}}} \frac{1}{tM(t)} \sum_{b \in A(t) \cap B(t)} \Pi\ \Pi' + 0\Big(x \sum_{\substack{t > x \\ t \in \mathcal{S}}} \frac{1}{t|M(t)|} \sum_{b \bmod HM(t)} 1\Big),$$

where the symbols Π Π' stand for the same products over primes that occurred in (15). Now, partial summation of the first part of Lemma 5 yields

(16) $$\sum_{\substack{t \leq x \\ t \in \mathcal{J}}} \frac{1}{t} \ll \frac{1}{\sqrt{x}}.$$

Accordingly, the main term in (15) equals

$$\frac{6x}{\pi^2 H} \sum_{\substack{t=1 \\ t \in \mathcal{J}}}^{\infty} \frac{1}{tM(t)} \sum_{b \in A(t) \cap B(t)} \Pi\,\Pi' + 0(\sqrt{x}).$$

Furthermore, it follows from the second part of Lemma 2 that the error term in (15) is

(17) $$0\Big(\sqrt{x} \sum_{\substack{t \leq x \\ t \in \mathcal{J}}} \frac{2^{\omega(t)}|M(t)|}{\sqrt{t}}\Big).$$

By Lemma 3, this remainder is $0(\sqrt{x}(\log x)^{2J} E(x, M))$. Combining this fact with our estimate of the main term in (15) yields the theorem.

4. The result for P(n) = n.

THEOREM 2. *Under the hypotheses of Theorem* 1, *we have*

$$\#\{n \leqq x \colon (n, M(n)) = 1\} = \mathscr{C}_M x + 0(\sqrt{x}(\log x)^2 \mathscr{E}(x, M)),$$

where

$$\mathscr{E}(x, M) = \sum_{\substack{t \leq x \\ t \in \mathscr{S}}} 2^{\omega(tM(t))} t^{-1/2}.$$

If $\mathscr{E}(\infty, M)$ converges, then $\mathscr{E}(x, M)$ can be omitted from the error term.

REMARK. From the fact that $\omega(mn) \leqq \omega(m) + \omega(n)$, for all m, n, and the inequality $2^{\omega(n)} \leqq n$, we deduce that

$$\mathscr{E}(x, M) \leqq \sum_{\substack{t \leq x \\ t \in \mathscr{C}}} 2^{\omega(t)} 2^{\omega(M(t))} t^{-1/2} \leqq \sum_{\substack{t \leq x \\ t \in \mathscr{C}}} 2^{\omega(t)} |M(t)| t^{-1/2} = E(x, M). \tag{18}$$

Consequently, the estimate of Theorem 2 is at least as sharp as, and in general sharper than, the approximation given in Theorem 1.

PROOF. Set $F(x) = \#\{n \leqq x: (M(n), n) = 1\}$. By (12),

$$F(x) = \sum_{\substack{t \leq x \\ t \in \mathscr{S}}} \sum_{\substack{m \leq x/t \\ \mu(m) \neq 0,\ (m, t) = 1 \\ (HM(t), mt) = 1}} 1 + 0(\sqrt{x}).$$

Consequently,

$$F(x) = \sum_{\substack{t \leq x \\ t \in \mathscr{S} \\ (t, HM(t)) = 1}} \sum_{\substack{m \leq x/t \\ \mu(m) \neq 0 \\ (m, HtM(t)) = 1}} 1 + 0(\sqrt{x}).$$

Applying Lemma 1 with $q = Ht|M(t)|$ and $k = 1$ to the inner sum yields

$$F(x) = \frac{6x}{\pi^2} \sum_{\substack{t \leq x \\ t \in \mathscr{S} \\ (t, HM(t)) = 1}} t^{-1} \prod_{p | HtM(t)} (1 + p^{-1})^{-1} + 0\left(\sqrt{x} \sum_{\substack{t \leq x \\ t \in \mathscr{S}}} \frac{2^{\omega(tM(t))}}{\sqrt{t}}\right). \tag{19}$$

It follows from (16) that the main term in (19) is

$$\frac{6x}{\pi^2} \sum_{\substack{t = 1 \\ t \in \mathscr{S} \\ (t, HM(t)) = 1}}^{\infty} \frac{1}{t} \prod_{p | HtM(t)} (1 + p^{-1})^{-1} + 0\left(x \sum_{\substack{t > x \\ t \in \mathscr{S}}} \frac{1}{t}\right)$$

$$= \frac{6x}{\pi^2} \sum_{\substack{t = 1 \\ t \in \mathscr{S} \\ (t, HM(t)) = 1}}^{\infty} \frac{1}{t} \prod_{p | HtM(t)} (1 + p^{-1})^{-1} + 0(\sqrt{x}).$$

Finally, if we apply Lemma 4, we can deduce that this error is $0(\mathscr{E}(x, M))$, and the theorem follows.

5. Corollaries and remarks.

COROLLARY 1. *For every integer $k > 1$ and every polynomial P with integer coefficients, there is a positive constant $c(k, P) < 1$ such that*

$$\#\{n \leqq x: (d_k(n), P(n)) = 1\} = c(k, Px + 0(\sqrt{x}(\log x)^{k(k+1)}).$$

COROLLARY 2. *For every integer $k > 1$, there is a positive constant $c(k) < 1$ for which*

$$\#\{n \leqq x: (d_k(n), n) = 1\} = c(k)x + 0(\sqrt{x}(\log x)^2).$$

PROOF OF COROLLARIES: According to (7), we can apply Theorems 1 and 2 with $M = d_k$, $H = k$, and $J = 1/2k(k + 1)$. Therefore, it suffices to prove that

$$E(\infty, d_k) = \sum_{t \in \mathscr{C}} 2^{\omega(t)} t^{-1/2} d_k(t), \ \mathscr{E}(\infty, d_k) = \sum_{t \in \mathscr{C}} 2^{\omega(td_k(t))} t^{-1/2}$$

both converge. The inequality $2^{\omega(n)} \leqq d(n)$ implies that

$$E(\infty, d_k) \leqq \sum_{t \in \mathscr{C}} d_2(t) d_k(t) t^{-1/2}. \tag{20}$$

Combining (20) with (18) and the first part of Lemma 2 yields

$$0 \leqq \mathscr{E}(\infty, d_k) \leqq E(\infty, d_k) \ll \sum_{t \in \mathscr{C}} t^{-4}.$$

The convergence of $E(\infty, d_k)$ and $\mathscr{E}(\infty, d_k)$ can now be readily obtained from the second part of Lemma 5.

REMARKS. If we merely estimate the error term in (15) by the expression in (17), and record the remainder in (19) as it stands, we obtain a result which is valid for all nonzero integer-valued multiplicative functions M such that $M(p) = H$ is constant for all primes p, with no restriction on the values of $M(p^2)$. These forms of the errors are, in general, less informative than the versions we presented; indeed, we claim that

$$\sum_{\substack{t \leqq x \\ t \in \mathscr{J}}} \frac{2^{\omega(t)}|M(t)|}{\sqrt{t}} \gg (\log x)^2,$$

$$\sum_{\substack{t \leqq x \\ t \in \mathscr{J}}} \frac{2^{\omega(tM(t))}}{\sqrt{t}} \gg (\log x)^2.$$

To verify these inequalities, replace $M(t)$ by 1 in both sums, sum by parts, and apply the estimate for

$$\sum_{\substack{t \leqq x \\ t \in \mathscr{J}}} 2^{\omega(t)}$$

given in [5].

In a personal communication, R. Sita Rama Chandra Rao has shown that the error term in Theorem 2 can be replaced by $0_S(\sqrt{x} \log x \, \varepsilon_S(x, M))$, where $0 < s < 1/2$,

$$\varepsilon_s(x, M) = \sum_{\substack{t \leqq x \\ t \in \mathscr{C}}} \sigma^*_{-s}(tM(t)) t^{-1/2},$$

$\sigma^*_s(n)$ denotes the sum of the s^{th} powers of the squarefree divisors of n, and the implied constant depends at most on s. He has applied this result to $M(n) = d_k(n)$, with any s strictly between 0 and 1/2, and replaced the error term in Corollary 2 by $0(x^{1/2} \log x)$.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, NEW YORK 14214