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MORE q-BETA INTEGRALS

RICHARD ASKEY[1] AND RANJAN ROY

ABSTRACT. Two new q-extensions of Barnes' beta integral are found. A third one found by Watson is reproved, and in the course of doing this another q-beta integral is discovered.

1. Introduction. Barnes [7] evaluated an integral of the product of four gamma functions which usually goes under the name of Barnes' lemma. This integral is an extension of Euler's classical beta function, so we will call it Barnes' beta integral. When Re$(a, b, c, d) > 0$ this integral is

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} &\Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)dt \\ &= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.\end{aligned} \tag{1.1}$$

To see that this extends Euler's beta integral replace b by $b - iw$, d by $d + iw$ and set $t = wx$. Then Stirling's formula can be used to obtain

$$\int_{-\infty}^{\infty} x_+^{a+c-1}(1-x)_+^{b+d-1}\,dx = \frac{\Gamma(a+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \tag{1.2}$$

where $x_+ = x$ if $x \geqq 0$, $x_+ = 0$ if $x < 0$.

Watson [12] found a q-extension of (1.1). This will be given in §3. There are others, and the easiest way to find them was given by Titchmarsh [11, pp.193–194]. Here is his proof of (1.1). Take Euler's beta integral on $[0, \infty)$

$$\int_0^{\infty} \frac{t^{\alpha-1}\,dt}{(1+t)^{\alpha+\beta}} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.3}$$

and consider it as a Mellin transform. Then use the L^2 theory of Mellin transforms. One of these results is the following. If

$$F_j(x) = \int_0^{\infty} t^{x-1} f_j(t)dt$$

then

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$$\int_0^\infty f_1(t)f_2(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty F_1(ix)F_2(1-ix)dx \tag{1.4}$$

under suitable assumptions on $f_1(t)$ and $f_2(t)$. See **[11**, (2.1.12)]

To obtain (1.1) take

$$f_1(t) = t^\alpha(1+t)^{-\alpha-\gamma}, f_2(t) = t^{\delta-1}(1+t)^{-\beta-\delta}.$$

To find q-extensions of (1.1) it is natural to look at q-extensions of (1.3) and use an appropriate L^2 theory.

We would like to thank G. Gasper for drawing our attention to Watson's paper and for a couple of useful comments.

2. Two q-extensions of Barnes' integral. Ramanujan found two identities which extend (1.3). Before stating them the standard notation will be recalled. Let $|q| < 1$ be given and define

$$\begin{aligned}(a;q)_\infty &= \prod_{n=0}^\infty (1-aq^n)\\ (a;q)_n &= (a;q)_\infty/(aq^n;q)_\infty.\end{aligned} \tag{2.1}$$

For $0 < q < 1$ the q-gamma function is defined by

$$\Gamma_q(x) = (q;q)_\infty(1-q)^{1-x}/(q^x;q)_\infty. \tag{2.2}$$

One of Ramanujan's extensions of (1.3) is

$$\int_0^\infty t^{x-1}\frac{(-ctq^{x+y};q)_\infty\,dt}{(-ct;q)_\infty} = \frac{\Gamma(x)\Gamma(1-x)\Gamma_q(y)}{c^x\Gamma_q(1-x)\Gamma_q(x+y)}. \tag{2.3}$$

Take

$$f_1(t) = \frac{t^\alpha(-tq^{\alpha+\gamma};q)_\infty}{(-t;q)_\infty}$$

$$f_2(t) = \frac{t^{\delta-1}(-tq^{\alpha+\beta+\gamma+\delta};q)_\infty}{(-tq^{\alpha+\gamma};q)_\infty}.$$

Then

$$\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\Gamma(\alpha+is)\Gamma(1-\alpha-is)\Gamma_q(\gamma-is)\Gamma(\delta-is)\Gamma(1-\delta+is)\Gamma_q(\beta+is)ds}{\Gamma_q(1-\alpha-is)\Gamma_q(\alpha+\gamma)\Gamma_q(1-\delta+is)\Gamma_q(\beta+\delta)q^{(\alpha+\gamma)(\delta-is)}}$$

$$= \int_0^\infty t^{\alpha+\delta-1}\frac{(-tq^{\alpha+\beta+\gamma+\delta};q)_\infty\,dt}{(-t;q)_\infty} = \frac{\Gamma(\alpha+\delta)\Gamma(1-\alpha-\delta)\Gamma_q(\beta+\gamma)}{\Gamma_q(1-\alpha-\delta)\Gamma_q(\alpha+\beta+\gamma+\delta)}$$

or

$$\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{\Gamma_q(\beta+is)\Gamma_q(\gamma-is)q^{(\alpha+\gamma)is}\,ds}{\Gamma_q(1-\alpha-is)\Gamma_q(1-\delta+is)\sin\pi(\alpha+is)\sin\pi(\delta-is)}$$

$$(2.4)\qquad =q^{(\alpha+\gamma)\delta}\frac{\Gamma_q\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma_q(\beta+\gamma)\Gamma_q(\beta+\delta)\Gamma(1-\alpha-\delta)}{\Gamma_q(\alpha+\beta+\gamma+\delta)\Gamma_q(1-\alpha-\delta)}=$$

$$\frac{1}{2\pi}\int_{\infty}^{-\infty}\frac{\Gamma(\alpha+is)\Gamma_q(\beta+is)\Gamma_q(\gamma-is)\Gamma(\delta-is)\Gamma(1-\alpha-\mathrm{is})\Gamma(1-\delta+is)q^{(\alpha+\gamma)is}ds}{\Gamma_q(1-\alpha-is)\Gamma_q(1-\delta+is)}$$

Ramanujan found a second extension of (1.3). It is

$$(2.5)\qquad \sum_{-\infty}^{\infty}\frac{(bq^n;q)_\infty}{(aq^n;q)_\infty}x^n=\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty(q;q)_\infty\left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty\left(\frac{b}{ax};q\right)_\infty(a;q)_\infty\left(\frac{q}{a};q\right)_\infty}$$

or

$$(2.6)\qquad \sum_{-\infty}^{\infty}\frac{(a;q)_n}{/(b;q)_n}x^n=\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty(q;q)_\infty\left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty\left(\frac{b}{ax};q\right)_\infty(b;q)_\infty\left(\frac{q}{a};q\right)_\infty}$$

when $|q|<1$, $|x|<1$, $|b/ax|<1$.

See **[5]** or **[8]** for simple proofs of (2.6) and **[4]** for the connection with (1.3).

Take $x=re^{i\theta}$ in (2.6) and then $x=se^{-i\theta}$ with (a,b) replaced by (b,c) in the second case. Multiply the series and integrate on $[-\pi,\pi]$. The result is

$$(2.7)\qquad \begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(are^{i\theta};q)_\infty(qe^{-i\theta}/ar;q)_\infty(bse^{-i\theta};q)_\infty(qe^{i\theta}/bs;q)_\infty\,d\theta}{(re^{i\theta};q)_\infty(be^{-i\theta}/ar;q)_\infty(se^{-i\theta};q)_\infty(ce^{i\theta}/bc;q)_\infty}\\&=\frac{(b;q)_\infty(q/b;q)_\infty(ars;q)_\infty(q/ars;q)_\infty(c/a;q)_\infty}{(q;q)_\infty(rs;q)_\infty(c/ars;q)_\infty(b/a;q)_\infty(c/b;q)_\infty}.\end{aligned}$$

Replace a, c, r, s using $r\to a$, $c/bs\to b$, $s\to c$, $b/ar\to d$ and relabel the original b as f. Formula (2.7) becomes

$$(2.8)\qquad \begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(fe^{i\theta}/d;q)_\infty(qde^{-i\theta}/f;q)_\infty(cfe^{-i\theta};q)_\infty(qe^{i\theta}/cf;q)_\infty\,d\theta}{(ae^{i\theta};q)_\infty(be^{i\theta};q)_\infty(ce^{-i\theta};q)_\infty(de^{-i\theta};q)_\infty}\\&=\frac{(abcd;q)_\infty(f;q)_\infty(q/f;q)_\infty(cf/d;q)_\infty(qd/cf;q)_\infty}{(ac;q)_\infty(ad;q)_\infty(bc;q)_\infty(bd;q)_\infty(q;q)_\infty}\end{aligned}$$

when $\max(|q|,|a|,|b|,|c|,|d|)<1$ and $cdf\neq 0$.

Take $f=q^{1/2}e$ and let $q=0$. The result is the elementary integral

$$\begin{aligned}(2.9)\qquad &\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{(1-ae^{i\theta})(1-be^{i\theta})(1-ce^{-i\theta})(1-de^{-i\theta})}\\ &=\frac{1-abcd}{(1-ac)(1-ad)(1-bc)(1-bd)}\end{aligned}$$

when $\max(|a|, |b|, |c|, |d|) < 1$.

This integral was stated by Li and Soto-Andrade [10], and was the reason we decided to look for a q-extension of Barnes' beta integral.

Since

$$\lim_{q\to 1}\Gamma_q(x) = \Gamma(x)$$

[4], it is clear that (2.4) generalizes (1.1). To see that (2.8) also does take $0 < q < 1$ and set

$$e^{i\theta} = q^{it},\ a = q^{\alpha},\ b = q^{\beta},\ c = q^{\gamma},\ d = q^{\delta}, f = q^{\varepsilon}.$$

Then the integral can be rewritten as

$$\begin{aligned}(2.10)\qquad &\frac{1}{2\pi}\int_{\pi/\log q}^{-\pi/\log q}\Gamma_q(\alpha+it)\Gamma_q(\beta+it)\Gamma_q(\gamma-it)\Gamma_q(\delta-it)w_q(t)dt\\ &=\frac{\Gamma_q(\alpha+\gamma)\Gamma_q(\alpha+\delta)\Gamma_q(\beta+\gamma)\Gamma_q(\beta+\delta)(1-q)}{\Gamma_q(\alpha+\beta+\gamma+\delta)\,[-\log q]}\end{aligned}$$

where

$$w_q(t)=\frac{(q^{\varepsilon-\delta+it};q)_\infty(q^{\delta+1-\varepsilon-it};q)_\infty(q^{\gamma+\varepsilon-it};q)_\infty(q^{1-\gamma-\varepsilon+it};q)_\infty}{(q^{\varepsilon};q)_\infty(q^{1-\varepsilon};q)_\infty(q^{\gamma+\varepsilon-\delta};q)_\infty(q^{\delta+1-\gamma-\varepsilon};q)_\infty}.$$

Using the triple product for the theta function

$$(2.11)\qquad (a;q)_\infty(q/a;q)_\infty(q;q)_\infty = \sum_{-\infty}^{\infty}(-1)^n q^{n(n-1)/2}a^n$$

(see [2] or [4] for simple proofs) and the modular transformation

$$\sum_{-\infty}^{\infty} e^{-\pi n^2 t+2\pi i n z} = t^{-1/2}\sum_{-\infty}^{\infty} e^{-\pi(n+z)^2/t}$$

(see [9] for a proof) it is easy to show that

$$\lim_{q\to 1} w_q(t) = 1.$$

Since $\lim_{q\to 1} -\log q/(1-q) = 1$ it is clear that (2.10) becomes (1.1) when $q \to 1$.

Barnes showed that the restrictions $\mathrm{Re}(a, b, c, d) > 0$ made for (1.1) can be relaxed if the path of integration is deformed to keep the poles of $\Gamma(a+it)$ and $\Gamma(b+it)$ and those of $\Gamma(c-it)$ and $\Gamma(d-it)$ on opposite sides of the contour. The same can be done for (2.8). Let C be a contour that is a deformation of the unit circle so the zeros of

$$\prod_{n=0}^{\infty}(1-cq^n/z)(1-dq^n/z)$$

are inside the contour and the zeros of

$$\prod_{n=0}^{\infty}(1-aq^nz)(1-bq^nz)$$

are outside the contour. Assume that there are no poles of second order inside the contour and for the time being assume that $|ad|<1$ and $|bc|<1$. Let C_N be the contour obtained by shrinking C so that N zeros of $\prod_{n=0}^{\infty}(1-cq^n/z)$ and N zeros of $\prod_{n=0}^{\infty}(1-dq^n/z)$ are outside of C_N and the rest are inside C_N. Then

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_C \frac{(fz/d;q)_\infty(qd/fz;q)_\infty(qz/cf;q)_\infty(cf/z;q)_\infty}{(az;q)_\infty(bz;q)_\infty(c/z;q)_\infty(d/z;q)_\infty}\frac{dz}{z} \\ &= \sum_{k=0}^{N-1}\frac{(fcq^k/d;q)_\infty(qd/cfq^k;q)_\infty(q^{k+1}/f;q)_\infty(f/q^k;q)_\infty}{(acq^k;q)_\infty(bcq^k;q)_\infty(q^{-k};q)_k(q;q)_\infty(d/cq^k;q)_\infty} \\ &\quad+\sum_{k=0}^{N-1}\frac{(fq^k;q)_\infty(q^{1-k}/f;q)_\infty(q^{k+1}d/cf;q)_\infty(cf/dq^k;q)_\infty}{(adq^k;q)_\infty(bdq^k;q)_\infty(c/dq^k;q)_\infty(q^{-k};q)_k(q;q)_\infty} \\ &\quad+\frac{1}{2\pi i}\int_{C_N}. \end{aligned} \tag{2.12}$$

When $N\to\infty$ the series become

$$\begin{aligned} &\frac{(cf/d;q)_\infty(qd/cf;q)_\infty(q/f;q)_\infty(f;q)_\infty}{(ac;q)_\infty(bd;q)_\infty(q;q)_\infty(d/c;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}ac,\ bc\\ cq/d\end{matrix};q,q\right) \\ &+\frac{(f;q)_\infty(q/f;q)_\infty(cf/d;q)_\infty(dq/cf;q)_\infty}{(ad;q)_\infty(bd;q)_\infty(q;q)_\infty(c/d;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}ad,\ bd\\ dq/c\end{matrix};q,q\right) \end{aligned} \tag{2.13}$$

where

$${}_2\varphi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}x^n.$$

When the contour C_N is taken to be bounded away from the N^{th} and $(N+1)^{st}$ zeros in each sequence, the integrand is bounded and the length of the path of integration is of the order of q^N, so the integral over C_N vanishes as $N\to\infty$.

Neither of the series in (2.13) can be summed but they can be combined into a single series that can be evaluated. To do this use Heine's transformation, [**1**, Corollary 2.3]

$${}_2\varphi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};q,x\right)=\frac{(ax;q)_\infty(b;q)_\infty}{(x;q)_\infty(c;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}c/b,\ x\\ ax\end{matrix};q,b\right)$$

on each series, using ac as a in the first series and bd as a in second. The result is

$$\frac{(f;q)_\infty(q/f;q)_\infty(cf/d;q)_\infty(dq/cf;q)_\infty(1-d/c)}{(q;q)_\infty^2(d/c;q)_\infty(c/d;q)_\infty(1-bd)}$$
$$\left[\sum_{n=1}^{\infty}\frac{((bd)^{-1};q)_n(bc)^n}{(ac;q)_n}+\sum_{n=0}^{\infty}\frac{(q/ac;q)_n(ad)^n}{(bdq;q)_n}\right].$$

Replace n by $-n$ in the second sum and use

$$\frac{(a;q)_{-n}}{(b;q)_{-n}}=\frac{(a;q)_\infty(bq^{-n};q)_\infty}{(aq^{-n};q)_\infty(b;q)_\infty}=\frac{(bq^{-n};q)_n}{(aq^{-n};q)_n}=\frac{(q/b;q)_n}{(q/a;q)_n}\left(\frac{b}{a}\right)^n$$

to obtain

$$\frac{(f;q)_\infty(q/f;q)_\infty(cf/d;q)_\infty(dq/cf;q)_\infty}{(q;q)_\infty^2(dq/c;q)_\infty(c/d;q)_\infty(1-bd)}\sum_{-\infty}^{\infty}\frac{((bd)^{-1};q)_n}{(ac;q)_n}(bc)^n.$$

This series is just Ramanujan's (2.6), and using the sum of this series the result is the right hand side of (2.8) as it should be. The condition that there is no double pole inside the unit circle is just $c \neq dq^k$, $k = 0, \pm 1, \cdots$. This condition insures that $(c/d;q)_\infty(d/c;q)_\infty \neq 0$, as was needed in the proof. However in the final identity this condition is not needed, so it can be removed by continuity. The conditions $|bc| < 1$ and $|ad| < 1$ were used in the convergence of Ramanujan's ${}_1\psi_1$ sum. Also we needed the conditions $ac \neq q^{-k}$ and $bd \neq q^{-k}$, $k = 0, 1, \cdots$, so there were no infinite terms in this sum. Analytic continuation can be used to replace $|ad| < 1$ and $|bc| < 1$ by $ad \neq q^{-k}$ and $bc \neq q^{-k}$ for $k = 0, 1, \cdots$.

It is interesting to compare this use of Cauchy's theorem with Barnes' use of it. See [**13**, §14. 52] for his argument. Barnes needed to assume $\mathrm{Re}(a + b + c + d) < 1$ to have convergence of the series he obtained. That is not necessary in the q case. Barnes could sum both of the series he obtained, while that cannot be done in the q-case. After summing his series Barnes needed to use some trigonometry to combine the two sums into a single one. In the q case that had to be done before summing the series. Watson had the same problem in evaluating his extension of Barnes' beta integral, and he solved it differently than we did since he was unaware of Ramanujan's sum (2.6) in 1910. The same problem arose when evaluating another q-extension of the beta integral [**3**], and the argument we gave above was first given there. It was repeated here since it is a nice argument and should be better known.

3. Watson's q-extension of Barnes' beta integral and an extension of one of Ramanujan's q-beta integrals. The beta integral on $[0, \infty)$ can be written as

$$\int_0^\infty \frac{t^{c-1}\,dt}{(1+t)^{a+c}(1+t^{-1})^{b-c}}=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{3.1}$$

This suggests consideration of

(3.2) $$f(a, b, c) = \int_0^\infty t^{c-1} \frac{(-q^{a+c}t; q)_\infty(-q^{b+1-c}t^{-1}; q)_\infty\, dt}{(-t; q)_\infty(-qt^{-1}; q)_\infty}.$$

The reason for using the theta function product $(-t; q)_\infty(-qt^{-1}; q)_\infty$ (recall (2.11)) is that it works, but there are a number of reasons for suspecting it is the right product to use. A pair of theta products are used in the second q-Barnes beta integral above, and also in the integral that is responsible for the orthogonal polynomials in [6].

To evaluate (3.2) take $b - c = N$, $N = 0, 1, \ldots,$ Then

$$\begin{aligned} f(a, c+N, c) &= \int_0^\infty \frac{t^{c-1}(-q^{a+c}t; q)_\infty\, dt}{(-t; q)_\infty(-qt^{-1}; q)_N} \\ &= q^{-N(N+1)/2} \int_0^\infty \frac{t^{c+N-1}(-q^{a+c}\, t; q)_\infty\, dt}{(-tq^{-N}; q)_\infty} \\ &= q^{cN+(N^2-N)/2} \int_0^\infty \frac{t^{c+N-1}(-q^{a+N+c}\, t; q)_\infty\, dt}{(-t; q)_\infty} \\ &= \frac{q^{cN+(N^2-N)/2}\,\Gamma(c+N)\Gamma(1-c-N)\Gamma_q(a)}{\Gamma_q(1-c-N)\Gamma_q(a+N+c)} \\ &= \frac{\Gamma(c)\Gamma(1-c)\Gamma_q(a)\Gamma_q(N+c)}{\Gamma_q(c)\Gamma_q(1-c)\Gamma_q(a+N+c)}. \end{aligned}$$

Thus

(3.3) $$f(a, b, c) = \frac{\Gamma(c)\Gamma(1-c)\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(c)\Gamma_q(1-c)\Gamma_q(a+b)}$$

when $b = c, c+1, \cdots, c+N, \cdots$. Both sides of (3.3) are analytic functions of $z = q^b$ in a neighborhood of $z = 0$ and agree for infinitely many values that have $z = 0$ as a limit point, so they are equal. Thus

(3.4) $$\int_0^\infty t^{c-1}\frac{(-q^{a+c}t; q)_\infty(-q^{b+1-c}/t; q)_\infty\, dt}{(-t; q)_\infty(-q/t; q)_\infty} = \frac{\Gamma(c)\Gamma(1-c)\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(c)\Gamma_q(1-c)\Gamma_q(a+b)}$$

when $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$ and a limit is taken when $c = 0, \pm 1, \cdots$. When $b = c$ this is Ramanujan's integral (2.3).

A second fact from the L^2 theory of Mellin transforms is the following. If $f_1(t)$ and $f_2(t)$ are used to define $F_1(x)$ and $F_2(x)$ as in the introduction, then

(3.5) $$\int_0^\infty f_1\left(\frac{x}{t}\right) f_2(t) \frac{dt}{t} = \frac{1}{2\pi}\int_{-\infty}^\infty F_1(is)F_2(is)x^{-is}\, ds.$$

See [11, (2.1.17)].

If $x = q$, $f_1(t) = t^a(-q^{a+c}t; q)_\infty/(-t; q)_\infty$, and $f_2(t) = t^b(-q^{b+d}t; q)_\infty/(-t; q)_\infty$ then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma_q(c-is)\Gamma_q(d-is)q^{-is}\,ds}{\Gamma_q(1-a-is)\Gamma_q(1-b-is)\sin\pi(a+is)\sin\pi(b+is)}$$
$$=\frac{q^a\Gamma(b-a)\Gamma(a+1-b)}{\Gamma_q(b-a)\Gamma_q(a+1-b)}\cdot\frac{\Gamma_q(a+c)\Gamma_q(a+d)\Gamma_q(b+c)\Gamma_q(b+d)}{\Gamma_q(a+b+c+d)} \tag{3.6}$$

when $\operatorname{Re}(a, b, c, d) > 0$.

This is Watson's q-extension of Barnes' beta integral. He extended it in the same way that Barnes extended his integral, and as we did for (2.8), by bending the contour to separate the appropriate zeros. This can also done for (2.4). In each case Cauchy's theorem can be applied. The advantage of the L^2 theory is that it leads one to the right identities instead of having to guess them. In fact that is how (3.4) was discovered. First the argument to prove (3.6) was tried, and it led to the right Barnes type integral, but to the integral in (3.4). Since Watson had evaluated (3.6) it had to be possible to evaluate (3.4). Once this was known, it was easy to find the argument above. A similar extension of (2.5) can be attempted, but it only leads to (2.6) again.

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University of Wisconsin-Madison, Beloit College