ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 1, Winter 1985

THE SCHUR MULTIPLIER OF THE AUTOMORPHISM GROUP OF THE MATHIEU GROUP M_{22}

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In [1], Gagola and Garrison show that the Sylow 2-subgroups of the Schur multiplier of the automorphism group of M_{22} are cyclic of order two. In this note, we make use of this result to prove the following.

THEOREM. *The Schur multiplier of the automorphism group of the Mathieu group M_{22} is cyclic of order two.*

PROOF. Let α be a cocycle of $G = \text{Aut}\, M_{22}$ of odd prime order p. On restriction to M_{22}, α gives rise to a cocycle β say. Since $|G: M_{22}| = 2$, it follows by Satz IX of [5], that β also has order p. However, Mazat [4], has shown that the Schur multiplier of M_{22} is cyclic of order 12, so we deduce that p must be 3. Thus there is a group 3.G with a cyclic central subgroup A whose quotient is isomorphic to G and $3.G$ has a subgroup $3.M_{22}$ of index 2. Fixing a nontrivial irreducible character λ of A, the irreducible representations of $3.M_{22}$ which restrict to A as a multiple of λ may be regarded as projective representations of M_{22} and as such their characters are well-known (see [3]). The degrees of these irreducible representations are 21, 45, 45, 99, 105, 105, 210, 231, 231, 330 and 384. Since the irreducible Brauer characters modulo 3 of $3.M_{22}$ are precisely those of M_{22}, we may use the results of [3] to see that the restrictions of the characters 21 and 210 to 3-regular conjugacy classes give irreducible Brauer characters. We also note that the product character 21.21 has the decomposition

$$21.21 = 21 + 105_1 + 105_2 + 210 \tag{1}$$

into irreducible characters.

The two classes of elements of order 11 in M_{22} fuse into one class in G. This means that the two characters of degree 105 (being exceptional for 11) give rise to an irreducible character of $3.G$ of degree 210. For each other irreducible representation D of $3.M_{22}$ whose restriction to A is a multiple of λ, there is a pair D^+, D^- of representations of $3.G$. If θ is the character of D, the characters θ^+ and θ^- agree on $3.M_{22}$ and $\theta^+(g) = -\theta^-(g)$ for all $g \in 3.G\backslash 3.M_{22}$. Thus (1) gives

Received by the editors on January 2, 1983.

(2) $$21^+21^+ = 21^{\pm} + 210 + 210^{\pm}$$

where the $\pm$ signs are to be determined. The restrictions of 21^+ and 21^- to 3-regular conjugacy classes will clearly give rise to irreducible Brauer characters of $3.G$ which may be regarded as irreducible Brauer characters of G (and similarly for 210^+ and 210^-). The character table of G is well-known and is given, for example, as Table IV of **[6]**. Considering a class of involutions in $G\backslash M_{22}$ corresponding to elements of Sym (22) of cycle type 2^8, we see that $21^{\pm}$ take the values ± 7 on this class while $210^{\pm}$ take the values ± 14 on this class. Since 210 vanishes on $G\backslash M_{22}$, equation (2) cannot hold and so G has no cocycle of order 3. The result now follows by the result of Gagola and Garrison [**1**, Theorem 5.4].

REMARK. Apart from M_{22}, the only Mathieu group with nontrivial outer automorphism group is M_{12}. It was shown in Theorem 2.4 of **[2]** that the Schur multiplier of $\mathrm{Aut}(M_{12})$ is also cyclic of order 2.

REFERENCES

1. S.M. Gagola, and S.C. Garrison, *Real characters, double covers and the multiplier*, J. Algebra **74** (1982), 20–51.

2. J.F. Humphreys, *The projective characters of the Mathieu group M_{12} and of its automorphism group*, Math. Proc. Cambridge Phil. Soc. **87** (1980), 401–412.

3. G.D. James, *The modular characters of the Mathieu groups*, J. Algebra **27** (1973), 57–111.

4. P. Mazat, *Sur le multiplicateur de Schur du groupe de Mathieu M_{22}*, C.R. Acad. Sci. Paris **289** (1979), 659–661.

5. I. Schur, *Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. reine angew. Math. **127** (1904), 20–50.

6. J.A. Todd, *A representation of the Mathieu group M_{22} as a collineation group*, Ann. Mat. Pura Appl. (4) **71** (1966), 199–238.

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