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# THE Q-ANALOGUE OF STIRLING'S FORMULA

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ABSTRACT. F.H. Jackson defined a $q$-analogue of the factorial $n! = 1 \cdot 2 \cdot 3 \cdots n$ as $(n!)_q = 1 \cdot (1 + q) \cdot (1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^{n-1})$, which becomes the ordinary factorial as $q \to 1$. He also defined the $q$-gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0 < q < 1,$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x} q^{\binom{x}{2}},\ q > 1,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

It is known that if $q \to 1$, $\Gamma_q(x) \to \Gamma(x)$, where $\Gamma(x)$ is the ordinary gamma function. Clearly $\Gamma_q(n+1) = (n!)_q$, so that the $q$-gamma function does extend the $q$ factorial to non integer values of $n$. We will obtain an asymptotic expansion of $\Gamma_q(z)$ as $|z| \to \infty$ in the right halfplane, which is uniform as $q \to 1$, and when $q \to 1$, the asymptotic expansion becomes Stirling's formula.

**1. Introduction.** In recent years many of the classical facts about the ordinary gamma function have been extended to the $q$-gamma function. See Askey [2], and [5], [6]. Using an identity of Euler,

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n}, \tag{1.1}$$

$\Gamma_q(x)$ can be written as,

$$\Gamma_q(x) = (q;q)_\infty (1-q)^{1-x} \sum_{n=0}^{\infty} \frac{q^{nx}}{(q;q)_n},\ 0 < q < 1, \tag{1.2}$$

and

$$\Gamma_q(x) = (q^{-1};q^{-1})_\infty\, q^{\binom{x}{2}} (q-1)^{1-x} \sum_{n=0}^{\infty} \frac{q^{-nx}}{(q^{-1};q^{-1})_n},\ q > 1. \tag{1.3}$$

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(1.1) is a consequence of the $q$-binomial theorem

$$(1.4) \qquad \frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n, \; 0 < q < 1,$$

where $a = 0$. As easy proof of (1.4) can be found in [**1**, p. 66].

Unfortunately, (1.2) or (1.3) does not become Stirling's formula when $q \to 1$. Neither the infinite series nor the infinite product converge uniformly as $q \to 1$, although they are asymptotic series for large $x$. There is, however, another asymptotic expansion which becomes Stirling's formula when $q \to 1$ and is uniform for $q$ near 1.

**2. The main result.** The basic tool we will use is the following Lemma.

LEMMA A. *If $f \in C^{2m}[a, b]$, $a$ and $b$ integers, then*

$$(2.1) \qquad \begin{aligned} \sum_{n=a}^{b} f(n) &= \int_a^b f(t)dt + \frac{1}{2} f(a) + \frac{1}{2} f(b) \\ &+ \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \{f^{(2k-1)}(b) - f^{(2k-1)}(a)\} - \int_a^b \frac{B_{2m}(x - [x])}{(2m)!} f^{(2m)}(x)dx \end{aligned}$$

*where $B_n(x)$ is the nth Bernouilli polynomial, $B_n = B_n(0)$, and $m$ is any positive integer.*

If $\{(-1)^m f^{(m)}(x)\}_{m=2}^{\infty}$ all have one sign on $[a, b]$, the error in terminating the series on the right at $k = m$ is less in absolute value than that of the first term neglected and has the same sign.

For a proof see [**9**, p. 128]. To simplify the following discussion, we will use the following definition.

DEFINITION. An asymptotic series $f(z) \sim \sum_{k=1}^{\infty} a_k \beta_k(z)$ is *proper* if the $a_i$ are all real and there exists a sequence of positive integers $n_1 < n_2 < \cdots$ such that

(i) $f(z)$ and $\beta_k(z)$ are analytic for $-\pi/2 < \arg z < \pi/2$.

(ii) $\beta_k(z) = O(z^{-n_k})$ as $z \to \infty$ uniformly in every sector $-\pi/2 + \varepsilon < \arg(z) < \pi/2 - \varepsilon$, $0 < \varepsilon < \pi/2$.

(iii) For every $m$, $(f(z) - \sum_{k=1}^{m} a_k \beta_k(z)) \cdot z^{n_m} \to 0$ as $z \to \infty$ uniformly in each sector $-\pi/2 + \varepsilon < \arg(z) < \pi/2 - \varepsilon$, $0 < \varepsilon < \pi/2$.

(iv) If $z$ is real and positive, the $\beta_i(z)$ are real and the error in truncating the series is less in absolute value than that of the first term neglected and has the same sign.

Stirling's formula

$$(2.2) \qquad \Gamma(z) \sim (z - 1/2)\ln z - z + (1/2)\ln(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k-1}$$

is one example of a proper asymptotic expansion. Many other classical functions of mathematical physics have proper asymptotic expansions, and as we shall see, $\Gamma_q(z)$ has one too.

THEOREM 1. *Let* $\Psi_q(z) = (d/dz)\Gamma_q(z)/\Gamma_q(z)$, *then*

$$\begin{aligned}\Psi_q(z) &\sim \ln\left(\frac{1-q^z}{1-q}\right)+\frac{\ln q}{2(q^{-z}-1)}-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left\{\left(\frac{d}{dt}\right)^{2k-1}\left(\frac{q^{z+t}\ln q}{1-q^{z+t}}\right)\right\}\Bigg|_{t=0} \\ &= \ln\left(\frac{1-q^z}{1-q}\right)+\frac{\ln q}{2(q^{-z}-1)}-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\ln q}{(1-q)^z}\right)^{2k} q^z P_{2k-2}(q^z)\end{aligned} \tag{2.3}$$

*where* $P_n(z)$ *is a polynomial of degree n satisfying*

$$P_n(z) = (z - z^2)P'_{n-1}(z) + (nz + 1)P_{n-1}(z),\ P_0 = 1,\ n \geqq 1. \tag{2.4}$$

*Here it is assumed that the principal branch of the logarithm is taken. Moreover* (2.3) *is a proper asymptotic expansion with* $n_k = 2k$.

PROOF. We first show by induction that

$$\left(\frac{d}{dt}\right)^n\left(\frac{q^t\ln q}{1-q^t}\right)=\left(\frac{\ln q}{1-q^t}\right)^{n+1} q^t P_{n-1}(q^t),\ n = 1, 2, 3,\ldots \tag{2.5}$$

It is easy to verify that (2.5) holds for $n = 1$. Assuming that (2.5) holds for $n = k$, we have

$$\begin{aligned}\left(\frac{d}{dt}\right)^{k+1}\left(\frac{q^t\ln q}{1-q^t}\right) &= \frac{d}{dt}\left(\left(\frac{\ln q}{1-q^t}\right)^{k+1} q^t P_{k-1}(q^t)\right) \\ &= (k+1)(\ln q)^{k+2}(1-q^t)^{-k-2}q^{2t}P_{k-1}(q^t) \\ &\quad + (\ln q)^{k+2}(1-q^t)^{-k-1}q^t P_{k-1}(q^t) \\ &\quad + (\ln q)^{k+2}(1-q^t)^{-k-1}q^{2t}P'_{k-1}(q^t) \\ &= \left(\frac{\ln q}{1-q^t}\right)^{k+2} q^t[(k+1)q^t P_{k-1}(q^t) \\ &\quad + (1-q^t)P_{k-1}(q^t) + q^t(1-q^t)P'_{k-1}(q^t)] \\ &= \left(\frac{\ln q}{1-q^t}\right)^{k+2} q^t P_k(q^t),\end{aligned}$$

by (2.4). This proves (2.5).

There are some properties of $P_n(x)$ which will be needed and they are summarized in the following Lemma.

LEMMA 1.
(i) *The coefficients of* $P_n(z)$ *are all positive.*
(ii) $P_n(1) = (n + 1)!$

PROOF OF (i). We will prove (i) by induction on $n$. For $n = 0$, (i) is clear. Assuming that (i) holds for $n = k$, we have by (2.4), $P_{k+1}(z) = (z - z^2)\cdot P'_k(z)+((k+1)z+1)P_k(z) = (zP'_k(z) + P_k(z)) + ((k+1)zP_k(z) - z^2P'_k(z))$.

It clearly suffices to show that $(k+1)zP_k(z) - z^2P_k(z)$ has positive coefficients. This will be true if $(k+1)z^{j+1} - z^2(d/dz)z^j$ has positive coefficients for $0 \leqq j \leqq k$. It clearly does,

PROOF OF (ii). Let $\sigma_n = P_n(1)$, then by (2.5),

$$\lim_{q\to 1}\left(\frac{d}{dt}\right)^{n+1}\left(\frac{q^t \ln q}{1-q^t}\right) = \lim_{q\to 1}\left(\frac{\ln q}{1-q^t}\right)^{n+2} q^t P_n(q^t) = \sigma_n(-t)^{-n-2}. \tag{2.6}$$

Now $q^t \ln q/(1-q^t)$ is an analytic function of $t$ on any punctured disk of the form $\{z: 0 < |z| < R\}$ when $q$ is close enough to one. On any compact subset of such a punctured disk, $q^t \ln q/(1-q^t)$, coverges uniformly to $-1/t$ as $q \to 1$. We can therefore justify interchanging $\lim_{q\to 1}$ and $(d/dt)^{n+1}$ in (2.6), yielding $(n+1)!\,(-t)^{-n-2} = \sigma_n(-t)^{-n-2}$, or $\sigma_n = (n+1)!$ proving (ii).

Now let $n_k = 2k$, $k = 1, 2, \ldots$, and $\beta_k(z) = (\ln q/(1-q^z))^{2k} q^z P_{2k-2}(q^z)$. $z^n q^z \to 0$ as $z \to \infty$ uniformly in any closed subsector of $\{z: -\pi/2 < \arg(z) < \pi/2\}$. So (i) and (ii) hold. By a direct calculation; $\Psi_q(z) = -\ln(1-q) + (\ln q)\sum_{n=0}^{\infty} 1/(q^{-n-z}-1)$. Now let $f(x) = \ln q/(q^{-x-z}-1)$, $a = 0$, and $b \to \infty$ in (2.1). Then

$$\begin{aligned}\Psi_q(z) = {}& -\ln(1-q) + (\ln q)\int_0^\infty \frac{dt}{q^{-t-z}-1} + \frac{\ln q}{2(q^{-z}-1)} \\ & - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left(\frac{d}{dt}\right)^{2k-1}\left(\frac{\ln q}{q^{1-z-t}-1}\right)\Bigg|_{t=0} + R_{2m}(z;q),\end{aligned} \tag{2.7}$$

where

$$R_{2m}(z;q) = -\int_0^\infty \frac{B_{2m}(x-[x])}{(2m)!}\left(\frac{d}{dt}\right)^{2m}\left(\frac{\ln q}{q^{-z-t}-1}\right)dt.$$

Using (2.5) and lemma 1, one can easily justify letting $b \to \infty$ in (2.1). The first integral in (2.7) can be evaluated explicitly:

$$(\ln q)\int_0^\infty \frac{dt}{q^{-t-z}-1} = \int_0^\infty \frac{q^{t+z}\ln q}{1-q^{t+z}}dt = -\ln(1-q^{t+z})\Big|_{t=0}^{t=\infty} = \ln(1-q^z).$$

Substituting this into (2.7) we obtain (2.3). It remains to show that the remainder term has the needed properties to make the asymptotic expansion proper with $n_k = 2k$. To do that we need another lemma.

LEMMA 2. *If* $0 < q \leqq 1$ *and* $\mathrm{Re}(z) > 0$, *then*

$$\left|\left(\frac{d}{dt}\right)^n\left(\frac{\ln q}{q^{-z-t}-1}\right)\right| \leqq \left(\frac{d}{dt}\right)^n\left(\frac{-\ln q}{q^{-\mathrm{Re}(z)-t}-1}\right),\ t > 0. \tag{2.8}$$

PROOF. If $0 < q < 1$, $\ln q/(q^{-z-t}-1) = (\ln q)\sum_{k=1}^{\infty} q^{kz+kt}$, so that

$$\left(\frac{d}{dt}\right)^n\left(\frac{\ln q}{q^{-z-t}-1}\right) = (\ln q)^{n+1}\sum_{k=1}^{\infty} k^n q^{kz+kt}.$$

Hence

$$\left|\left(\frac{d}{dt}\right)^n\left(\frac{\ln q}{q^{-z-t}-1}\right)\right| < |\ln q|^{n+1}\sum_{k=1}^{\infty} k^n\ |q^{kz+kt}|$$
$$= (-\ln q)^{n+1}\sum_{k=1}^{\infty} k^n q^{k\mathrm{Re}(z)+kt} = \left(\frac{d}{dt}\right)^n\left(\frac{-\ln q}{q^{-\mathrm{Re}(z)-t}-1}\right).$$

At $q = 1$, $\ln q/(q^{-z-t} - 1)$ has a removable singularity, in fact $\lim_{q\to 1}(\ln q/(q^{-z-t} - 1)) = -1/(z + t)$. Then $|(d/dt)^n(-1/(z + t))| = |n!/(z+t)^{n+1}| = n!/|z+t|^{n+1} \leqq n!/(\mathrm{Re}(z) + t)^{n+1}$ So the lemma also holds when $q = 1$ also.

We now use the lemma to obtain an estimate of the magnitude of $R_{2m}(z)$. We can write $R_{2m}(z)$ as

$$\begin{aligned} R_{2m}(z) = & -\frac{B_{2m+2}}{(2m+2)!}\left(\frac{d}{dt}\right)^{2m+1}\left(\frac{\ln q}{q^{-z-t}-1}\right)\bigg|_{t=0} \\ & -\int_0^{\infty}\frac{B_{2m+2}(t-[t])}{(2m+2)!}\left(\frac{d}{dt}\right)^{2m+2}\left(\frac{\ln q}{q^{-z-t}-1}\right)dt. \end{aligned} \tag{2.9}$$

Then using (2.5), lemma 1, and lemma 2

$$\begin{aligned} |R_{2m}(z)| \leqq & \frac{|B_{2m+2}|}{(2m+2)!}\left(\frac{(\ln q)q^{\mathrm{Re}(z)}}{1-q^{\mathrm{Re}(z)}}\right)^{2m+2} q^{\mathrm{Re}(z)}\, P_{2m}(q^{\mathrm{Re}(z)}) \\ & + \int_0^{\infty}\frac{|B_{2m+2}(t-[t])|}{(2m+2)!}\left(\frac{d}{dt}\right)^{2m+2}\left(\frac{-\ln q}{q^{-\mathrm{Re}(z)-t}-1}\right)dt. \end{aligned}$$

It is known, [7, p. 533–538] that $|B_{2m}(t)| \leqq |B_{2m}|$, $t \in [0, 1]$, $m = 1, 2, 3, \ldots$ Then

$$|R_{2m}(z)| \leqq \frac{2|B_{2m+2}|}{(2m+2)!}\left(\frac{(\ln q)q^{\mathrm{Re}(z)}}{1-q^{\mathrm{Re}(z)}}\right)^{2m+2} q^{\mathrm{Re}(z)}\, P_{2m}(q^{\mathrm{Re}(z)}). \tag{2.10}$$

For $|\arg(z)| < \pi/2 - \varepsilon$, we have $\mathrm{Re}(z) > |z| \sin\varepsilon$, hence

$$\begin{aligned} & |z^{2m+2}R_{2m}(z)| \\ & \leqq \frac{2|B_{2m+2}|}{(\sin\varepsilon)^{2m+2}(2m+2)!}\left(\frac{(\ln q)\mathrm{Re}(z)q^{\mathrm{Re}(z)}}{1-q^{\mathrm{Re}(z)}}\right)^{2m+2} q^{\mathrm{Re}(z)}P_{2m}\,(q^{\mathrm{Re}(z)}). \end{aligned} \tag{2.11}$$

It suffices to show that the expression on the right is uniformly bounded for $q \leqq 1$ and $|\arg(z)| < \pi/2 - \varepsilon$. Let $u = \mathrm{Re}(z) \ln q$. Then $u \leqq 0$, and $(\ln q)\ \mathrm{Re}(z)q^{\mathrm{Re}(z)}/(1 - q^{\mathrm{Re}(z)}) = ue^u/(1 - e^u)$, which is bounded for $u \leqq 0$. It follows that the right side of (2.11) is bounded for $\mathrm{Re}(z) \geqq 0$ and $q \leqq 1$. Consequently $\lim_{z\to\infty}|z^{2m}R_{2m}(z)| = 0$ uniformly for $0 < q \leqq 1$ and $|\arg(z)| < \pi/2 - \varepsilon$. So property (iii) holds. To prove (iv), note that lemma 1 and (2.5) imply that $(-1)^{n-1}(d/dt)^n(q^t \ln q/(1-q^t)) > 0$. By lemma A, (iv) follows.

COROLLARY. *Let* $q \leqq 1$, *then*

$$\ln \Gamma_q(z) \sim (z-1/2)\ln\left(\frac{q^z-1}{q-1}\right) + \frac{1}{\ln q}\int_{-\ln q}^{-z\ln q} \frac{u\,du}{e^u-1} + C_q - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\frac{d}{dz}\right)^{2k-2}\left(\frac{q^z \ln q}{1-q^z}\right), \tag{2.12}$$

*or by* (2.5)

$$\ln \Gamma_q(z) \sim (z-1/2)\ln\left(\frac{q^z-1}{q-1}\right) + \frac{1}{\ln q}\int_{-\ln q}^{-z\ln q} \frac{u\,du}{e^u-1} + C_q + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\frac{\ln q}{q^z-1}\right)^{2k-1} q^z P_{2k-3}(q^z), \tag{2.13}$$

*where* $C_q$ *is a constant depending on q. Moreover, these asymptotic expansions are proper with* $n_k = 2k-1$, *and uniform for* $0 < q \leqq 1$ *and* $|\arg(z)| < \pi/2 - \varepsilon$, $\varepsilon > 0$.

PROOF. We integrate each term of (2.3) from 1 to $z$ inside the sector $|\arg(z)| < \pi/2$. The resulting asymptotic expansion is still proper with $n_k = 2k-1$ and $\beta_k(z) = (\ln q/(1-q^z))^{2k-1} q^z P_{2k-3}(q^z)$. We still need to show that when (2.3) is integrated term by term from 1 to $z$ we obtain (2.12) and (2.13). This is clearly true for the terms of the series. For the terms preceding the series we have

$$\int_1^z \ln\left(\frac{1-q^t}{1-q}\right) + \frac{\ln q}{2(q^{-t}-1)}\,dt$$

$$= \int_1^z \ln(1-q^t)\,dt - (z-1)\ln(1-q) + \frac{1}{2}\int_1^z \frac{\ln q}{q^{-t}-1}\,dt$$

$$= t\ln(1-q^t)\Big|_1^z + \ln q\int_1^z \frac{tq^t dt}{1-q^t} - (z-1)\ln(1-q) + \frac{1}{2}\int_1^z \frac{\ln q\,dt}{q^{-t}-1}$$

$$= z\ln\left(\frac{1-q^z}{1-q}\right) + \frac{1}{2}\int_1^z \frac{(\ln q)q^t dt}{1-q^t} + \ln q\int_1^z \frac{t\,dt}{q^{-t}-1}$$

$$= z\ln\left(\frac{1-q^z}{1-q}\right) - \frac{1}{2}\ln(1-q^t)\Big|_1^z + \frac{1}{\ln q}\int_{-\ln q}^{-z\ln q} \frac{u\,du}{e^u-1}.$$

Substituting this back into the integrated asymptotic series for $\ln(\Gamma_q(x))$ we obtain (2.12) and (2.13).

It remains to obtain an asymptotic expansion for $\Gamma_q(x)$ for $q > 1$, and to find the constant term $C_q$. We then obtain the main result.

THEOREM 2.

$$
\begin{aligned}
\ln \Gamma_q(z) \sim (z-1/2)\ln\left(\frac{1-q^z}{1-q}\right) + \frac{1}{\ln q}\int_{-\ln q}^{-z\ln q} \frac{u\,du}{e^u-1} + C_{\hat q} \\
+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\frac{\ln \hat q}{\hat q^z - 1}\right)^{2k-1} \hat q^z P_{2k-3}(\hat q^z)
\end{aligned} \tag{2.14}
$$

*where*

$$
\hat q = \begin{cases} q, & 0 < q \leqq 1 \\ q^{-1}, & q \geqq 1, \end{cases}
$$

*and* $P_{n+1}(z) = (z - z^2)P_n'(z) + ((n+1)z + 1)P_n(z)$, $n \geqq 0$, $P_0(z) = 1$, *and*

$$
C_q = \frac{1}{2}\ln(1-q) + \ln(q;q)_\infty - \frac{\pi^2}{6\ln q} + \frac{1}{\ln q}\int_0^{-\ln q}\frac{u\,du}{e^u-1}. \tag{2.15}
$$

$$
\begin{aligned}
C_q = \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln\left(\frac{q-1}{\ln q}\right) - \frac{1}{24}\ln q + \frac{1}{\ln q}\int_0^{-\ln q}\frac{u\,du}{e^u-1} \\
+ \ln \sum_{m=-\infty}^{\infty} r^{m(6m+1)} - r^{(3m+1)(2m+1)}.
\end{aligned} \tag{2.16}
$$

*where* $r = \exp(4\pi^2/\ln q)$.

*Moreover this asymptotic expansion is proper with* $n_k = 2k - 1$ *and uniform for* $\delta < q < 1/\delta$ *and* $|\arg(z)| < \pi/2 - \varepsilon$, $\varepsilon > 0$, $\delta > 0$.

PROOF. First we obtain the constant term for $0 < q < 1$. Comparing (1.2) with (2.12), we have

$$
\begin{aligned}
&\ln(q;q)_\infty + (1-z)\ln(1-q) + \ln\left(\sum_{n=0}^{\infty}\frac{q^{nz}}{(q;q)_n}\right) \\
&= (z-1/2)\ln\left(\frac{q^z-1}{q-1}\right) + \frac{1}{\ln q}\int_{-\ln q}^{-z\ln q}\frac{u\,du}{e^u-1} + C_q + O(1/z)
\end{aligned}
$$

as $z \to \infty$, $|\arg z| < \pi/2 - \varepsilon$, $q$ fixed, $0 < \varepsilon < \pi/2$. Letting $z \to \infty$ on the real axis, we obtain

$$
C_q = -\frac{1}{\ln q}\int_{-\ln q}^{\infty}\frac{u\,du}{e^u-1} + \ln(q;q)_\infty + \frac{1}{2}\ln(1-q) - \lim_{z\to+\infty}(z-1/2)\ln(1-q^z).
$$

Thus

$$
C_q = -\frac{1}{\ln q}\int_{-\ln q}^{\infty}\frac{u\,du}{e^u-1} + \ln(q;q)_\infty + \frac{1}{2}\ln(1-q).
$$

It is known [8, p. 265–169] that $\int_0^\infty u\,du/(e^u-1) = \zeta(2) = \pi^2/6$. We now have

$$
C_q = \frac{1}{\ln q}\int_0^{-\ln q}\frac{u\,du}{e^u-1} + \ln(q;q)_\infty - \frac{\pi^2}{6\ln q} + \frac{1}{2}\ln(1-q).
$$

This proves (2.15). The difficulty with (2.15) is that it is hard to evaluate $(q;q)_\infty$ for $q$ close to one. We need to have an alternate way of computing $(q;q)_\infty$. We start with Ramanujan's sum [3, (3.15)]

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n \tag{2.17}$$
$$= \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty \left(\frac{b}{a};q\right)_\infty (q;q)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty \left(\frac{q}{a};q\right)_\infty (b;q)_\infty}, \quad |b/a| < |x| < 1, \ |q| < 1.$$

Now let $b = 0$ and replace $x$ by $x/a$ and let $a \to \infty$. We then obtain Jacobi's triple product

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (x;q)_\infty \left(\frac{q}{x};q\right)_\infty (q;q)_\infty. \tag{2.18}$$

Now replace $q$ by $q^3$ and let $x = q^2$. Then we have Euler's series

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2/2+n/2} &= (q;q^3)_\infty (q^2;q^3)_\infty (q^3;q^3)_\infty \\ &= \prod_{n=0}^{\infty} (1-q^{1+3n})(1-q^{2+3n})(1-q^{3+3n}) \\ &= \prod_{n=0}^{\infty} (1-q^n).\end{aligned}$$

Thus

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2/2+n/2} = (q;q)_\infty. \tag{2.19}$$

We still have problems computing $(q;q)_\infty$ when $q$ is close to one, but at least we have quadratic rather than linear convergence. There is, however, an important formula of Poisson which completely solves all these convergence problems.

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+z)^2} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t + 2\pi n z i}, \quad \mathrm{Re}(t) > 0. \tag{2.20}$$

See [4, p. 40]. Let $q^{3/2} = e^{-\pi t}$, $z = 1/6 + \pi/(3i \ln q)$ in (2.20). We obtain

$$\begin{aligned}(q;q)_\infty &= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2/2+n\backslash 2} \\ &= (r/q)^{1/24} \sqrt{2\pi/(-\ln q)} \sum_{n=-\infty}^{\infty} r^{n(6n+1)} - r^{(3n+1)(2n+1)},\end{aligned} \tag{2.21}$$

where $r = \exp(4\pi^2/\ln q)$. The series on the right converges extraordinarily rapidly for $q$ near one. For example, if we let $q = .8$, then $r \doteq 1.46 \times 10^{-77}$! In fact it is always true that either $q$ or $r$ is less than $2 \times 10^{-3}$.

(2.21) can be substituted into (2.15) to obtain (2.16), This proves the theorem for $0 < q < 1$. When (2.16) is used for the constant term in (2.14) and when $q \to 1$, the ordinary Stirling's formula for the gamma function emerges.

To extend these results for $q > 1$, we use the definition of $\Gamma_q(x)$ to obtain

$$\Gamma_q(z) = q^{z^2/2-3z/2+1}\Gamma_{q^{-1}}(z). \tag{2.22}$$

For $q \geqq 1$, we can use the asymptotic expansion on $\Gamma_{q^{-1}}(z)$. The result is

$$\begin{aligned}\ln \Gamma_q(z) \sim &\left(\frac{z^2}{2} - \frac{3z}{2} + 1\right)\ln q + (z - 1/2)\ln\left(\frac{1-q^{-z}}{1-q^{-1}}\right) \\ &- \frac{1}{\ln q}\int_{\ln q}^{z\ln q} \frac{u\,du}{e^u - 1} + C_{q^{-1}} \\ &+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\frac{\ln q^{-1}}{q^{-z}-1}\right)^{2k-1} q^{-z}P_{2k-3}(q^{-z}).\end{aligned} \tag{2.23}$$

Now looking at the terms preceding the constant term, we see that

$$\begin{aligned}&\left(\frac{z^2}{2} - \frac{3}{2}z + 1\right)\ln q + (z - 1/2)\ln\left(\frac{1-q^{-z}}{1-q^{-1}}\right) - \frac{1}{\ln q}\int_{\ln q}^{z\ln q}\frac{u\,du}{e^u-1} \\ &= \left(\frac{z^2}{2} - \frac{3z}{2} + 1\right)\ln q + (z - 1/2)\ln(q^{1-z}) \\ &\quad + (z - 1/2)\ln\left(\frac{1-q^z}{1-q}\right) - \frac{1}{\ln q}\int_{\ln q}^{z\ln q}\frac{u\,du}{e^u-1} \\ &= \left(-\frac{z^2}{2} + \frac{1}{2}\right)\ln q + \frac{1}{\ln q}\int_{-\ln q}^{-z\ln q}\frac{1-e^{-u}+e^{-u}}{1-e^{-u}}u\,du + (z - 1/2)\ln\left(\frac{1-q^z}{1-q}\right) \\ &= \left(-\frac{z^2}{2} + \frac{1}{2}\right)\ln q + \frac{1}{2(\ln q)}u^2\Big|_{-\ln q}^{-z\ln q} \\ &\quad + \frac{1}{\ln q}\int_{-\ln q}^{-z\ln q}\frac{e^{-u}u\,du}{1-e^{-u}} + (z - 1/2)\ln\left(\frac{1-q^z}{1-q}\right) \\ &= (z - 1/2)\ln\left(\frac{1-q^z}{1-q}\right) + \frac{1}{\ln q}\int_{-\ln q}^{-z\ln q}\frac{u\,du}{e^u-1}.\end{aligned}$$

These are the same two terms in the asymptotic expansion for $\Gamma_q(z)$ for $0 < q \leqq 1$. Theorem 2 now follows for all $q > 0$.

**3. Some remarks on the asymptotic expansion.** Despite the apparent complexity of (2.14) it does provide a practical way of computing $\Gamma_q(z)$ for $q$ arbitrarily close to one. One must use Taylor expansions for expressions like $u/(e^u - 1)$ for $u$ near zero, otherwise numerical cancellation problems will occur. Once these precautions are taken, (2.14) will yield good results; for $x \geqq 30$, five terms in the sum will yield at least 14 decimal place accuracy and often more, regardless of the value of $q$. To evaluate the constant term $C_{\hat{q}}$, (2.16) is used if $\hat{q}$ is close to one, and (2.15) is used for small values of $\hat{q}$, where the Euler series

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2/2-n/2}$$

is used to compute $(q;q)_\infty$.

The integral $\int_0^x u\,du/(e^u-1)$ is easily computed using quadrature methods, but a better way is to let $t = e^{-u}$ and obtain

$$\int_1^{e^{-x}} \frac{\ln t\,dt}{1-t} = \mathrm{Di}(e^{-x}), \tag{3.1}$$

where $\mathrm{Di}(x)$ is the dilogarithm function. This function has been studied by a number of people including Kummer and Abel, see **[8]**. Some functional equations and series expansions have been found for it, among them being

$$\mathrm{Di}(x) = \sum_{k=1}^{\infty} (-1)^k (x-1)^k/k^2,\ 2 \geqq x \geqq 0. \tag{3.2}$$

$$\mathrm{Di}(x) + \mathrm{Di}(1-x) = -\ln x \ln(1-x) + \pi^2/6,\ 1 \geqq x \geqq 0. \tag{3.3}$$

Expanding $\mathrm{Di}(1-e^{-x})$ by (3.2) and also expanding the logarithm term in (3.3) we have $\mathrm{Di}(e^{-x}) = \pi^2/6 - \sum_{n=1}^{\infty}(nx+1)e^{-nx}/n^2$, or

$$\int_0^x u\,du/(e^u-1) = \pi^2/6 - \sum_{n=1}^{\infty} (nx+1)e^{-nx}/n^2, \tag{3.4}$$

which works well for large $x$.

**4. Calculating $(a;q)_\infty$ when q is close to one.** By the definition of the $q$-gamma function,

$$(q^z;q)_\infty = \frac{(q;q)_\infty}{\Gamma_q(z)}(1-q)^{1-z},\ 0 < q < 1. \tag{4.1}$$

Here we can let $a = q^z$ and use the asymptotic expansion (2.14) to compute $\Gamma_q(z)$ and (2.21) to compute $(q;q)_\infty$. This provides a way to compute $(a;q)_\infty$ for $q$ close to one. One result along these lines is this next theorem.

THEOREM 3. *Let* $0 < a < 1$ *and* $0 < q < 1$. *Then as* $q \to 1^-$, $\ln(a;q)_\infty \sim 1/2\ln(1-a) - B(a)/\ln q - \sum_{k=1}^{\infty} C_k(a)(\ln q)^{2k-1}$, *where*

$$\begin{aligned} B(a) &= \int_{-\ln a}^{\infty} \frac{u\,du}{e^u-1} + \ln a \ln(1-a), \\ C_k(a) &= \frac{B_{2k}}{(2k)!}\frac{a P_{2k-3}(a)}{(a-1)^{2k-1}},\ k = 1, 2, 3, \end{aligned} \tag{4.2}$$

*The $P_k(a)$ are defined in* (2.4). *Moreover the error in terminating the series is less in absolute value than that of the first term neglected and has the same sign.*

PROOF. Take logs in (4.1) and use (2.14) and (2.15) for $\Gamma_q(z)$. After some simplifying we obtain,

$$\ln(q^z;q)_\infty \sim (1/2 - z)\ln(1-q^z) - \frac{1}{\ln q}\int_0^{-z\ln q}\frac{u\,du}{e^u-1} + \frac{\pi^2}{6\ln q} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\ln q}{q^z-1}\right)^{2k-1} q^z P_{2k-3}(q^z).$$

It is known [9, p. 265–269] that $\int_0^\infty u\,du/(e^u - 1) = \zeta(2) = \pi^2/6$. We obtain

$$\ln(q^z;q)_\infty \sim (1/2 - z)\ln(1-q^z) - \frac{1}{\ln q}\int_{-z\ln q}^{\infty}\frac{u\,du}{e^u-1} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\ln q}{q^z-1}\right)^{2k-1} q^z P_{2k-3}(q^z). \tag{4.3}$$

The asymptotic expansion (4.3) is still uniform for $0 < q \leqq 1$, and proper with $n_k = 2k - 1$. We can therefore set $q^z = a$, hold $a$ fixed and let $q \to 1^-$ in (4.3) to obtain (4.2). Since the asymptotic expansion (4.3) is proper, the truncation error in (4.2) is less than that of the first term neglected and has the same sign.

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