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A NOTE ON MONOTONICITY PROPERTIES OF A FREE BOUNDARY PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

Introduction. A number of results have appeared in which the method of lines (also known as Rothe's method) is used to construct approximate solutions to two dimensional parabolic free boundary problems. This method replaces the partial differential equation with a sequence of ordinary differential equations by substituting a difference term for the time derivatives. For example, in a paper by Sackett [5], the solution to a problem for the heat equation is reduced to a sequence of problems for $u_n(x) \cong u(x, nh)$ of the form

$$\begin{gathered} u'' = h^{-1}(u - k_n(x)),\ x \in (0, s_n); \\ u(0) = f_{1n}, \\ u(s_n) = f_{2n},\ u'(s_n) = g_n, \\ u_n(x) := u(x). \end{gathered}$$

where f_{1n}, f_{2n}, and g_n are given from the boundary conditions of the two dimensional problem and $k_n(x)$ is a suitably extended solution $u_{n-1}(x)$ of the previous problem in the sequence. In [4], Meyer considers a problem which produces a similar sequence of ordinary differential equations, but with the boundary conditions

$$\begin{gathered} u'(0) = \alpha_n, \\ u(s_n) = u'(s_n) = 0. \end{gathered}$$

In both of these results, monotonicity properties of the boundary value problems with respect to the boundary conditions and the functions $k_n(x)$ are used to obtain bounds on the solutions and on the location of the free boundary.

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Monotonicity results which apply to free boundary value problems for nonlinear ordinary differential equations of the form

$$u'' = k(1 + u'^2),\ x \in (0, s);$$
$$u(0) = \alpha,$$
$$u(s) = u'(s) = 0,$$

have also appeared in papers by L. Collatz [1] and K. Glashoff and B. Werner [2].

In this note, the theory of differential inequalities will be used to establish some monotonicity properties of the free boundary problem

$$\begin{aligned} -u'' + f(x, u, u') &= 0,\ x \in (0, s); \\ u(0) &= \alpha, \\ u(s) = u'(s) &= 0. \end{aligned} \tag{1}$$

The main theorem. The main result in this note is the following theorem on differential inequalities for problem (1).

THEOREM 1. *Let α satisfy the inequality $\alpha > 0$ and suppose that $f(x, u, u')$ satisfies the following conditions:*

(i) $f(x, 0, 0) > 0$;

(ii) $f(x, u, u')$ *is nondecreasing in u for fixed u', i.e., the inequalities $u \leqq v$ and $u' = v'$ imply $f(x, u, u') \leqq f(x, v, v')$; and*

(iii) $f(x, u, u')$ *satisfies a Lipschitz condition in u' on bounded subsets of its domain in* $\mathbf{R}^3$.

Further, let there exist numbers s and s^* and functions $u \in C^2[0, s]$ and $v \in C^2[0, s^*]$ such that the following inequalities hold:

$$\begin{aligned} -u'' + f(x, u, u') &\leqq 0,\ x \in (0, s), \\ u(0) &\leqq \alpha, \\ u(s) = u'(s) &= 0; \end{aligned} \tag{2}$$

and

$$\begin{aligned} -v'' + f(x, v, v') &\geqq 0,\ x \in (0, s^*), \\ v(0) &\geqq \alpha, \\ v(s^*) = v'(s^*) &= 0; \end{aligned} \tag{3}$$

and in addition $v(x) \geqq 0$ for $x \in (0, s^*)$. Then $s \leqq s^*$ and $u(x) \leqq v(x)$ on the interval (0, s).

PROOF. Let the number T be chosen so that the inequality $T \geqq \max\{s, s^*\}$ is satisfied and extend $u(x)$ and $v(x)$ to the interval $[0, T]$ by defining them to be zero on the intervals $(s, T]$ and $(s^*, T]$, respectively.

Before proceeding with the proof, we will establish two lemmas on solutions to differential inequalities.

LEMMA 1. *Let* (ii) *and* (iii) *be satisfied and suppose there exist functions* $u(x)$ *and* $v(x)$ *in* $C^2[x_1, x_2]$ *satisfying the inequalities*

$$-u'' + f(x, u, u') \leqq -v'' + f(x, v, v'),\ x \in (x_1, x_2),$$
$$u(x_1) \leqq v(x_1),\ u(x_2) \leqq v(x_2).$$

Then $u(x) \leqq v(x)$ *on the interval* $[x_1, x_2]$.

PROOF. This is a well known result in the theory of ordinary differential inequalities (see, e.g., [3]).

LEMMA 2. *Let* $u(x)$ *satisfy* (2) *and suppose* (i) *and* (ii) *are satisfied. Then* $u(x) > 0$ *on the interval* $(0, s)$.

PROOF. Suppose to the contrary that $u(x) \leqq 0$ for some $\hat{x} \in (0, s)$. Since $u''(s-0) = f(s-0, 0, 0) > 0$, it follows that $u(x)$ must have a positive maximum on $[\hat{x}, s]$ at some point $\bar{x}$. However, at $\bar{x}$ we have

$$u''(\bar{x}) \geqq f(\bar{x}, u(\bar{x}), u'(\bar{x})) \geqq f(\bar{x}, 0, 0) > 0,$$

since $u'(\bar{x}) = 0$. This contradicts the assumption that $u(x)$ has a positive relative maximum at $\bar{x}$. It follows that $u(x)$ must be positive on $(0, s)$.

Returning to the proof of the theorem, we will first show that the inequality $s^* < s$ is not possible. Suppose to the contrary that $s^* < s$. Let us define a function $\phi(x) := u(x) - v(x)$ on $[0, t]$. $\phi(x)$ has the following properties: $\phi(x)$ is differentiable, $\phi(0) \leqq 0$, $\phi(T) = 0$, and $\phi(s^*) > 0$. It follows that $\phi(x)$ has a positive maximum at a point $\hat{x} \in (0, T)$. That this leads to a contradiction will be shown by considering the following cases.

Case I. $0 < \hat{x} < s^* < s \leqq T$. Define $w(x) := v(x) + u(s^*)$. Since ϕ has a maximum at $\hat{x}$, it follows that

$$u(\hat{x}) - w(\hat{x}) = u(\hat{x}) - v(\hat{x}) - (u(s^*) - v(s^*)) = \phi(x) - \phi(s^*) > 0. \tag{4}$$

However, $u(0) - w(0) = u(0) - v(0) - u(s^*) < 0$, and $u(s^*) - w(s^*) = 0$. Furthermore,

$$\begin{aligned} -w'' + f(x, w, w') &= -v'' + f(x, v + u(s^*), v') \\ &\geqq -v'' + f(x, v, v') \\ &\geqq 0 \\ &\geqq -u'' + f(x, u, u'). \end{aligned}$$

Thus it is seen that the conditions of Lemma 1 are satisfied by $u(x)$ and $w(x)$. Therefore $u(x) \leqq w(x)$ and this is a contradiction to (4).

Case II. $0 < s^* \leqq \hat{x} < s \leqq T$. At $\hat{x}$, $v(\hat{x}) = 0$, therefore $\phi(\hat{x}) = u(\hat{x})$ and $\phi'(\hat{x}) = u'(\hat{x}) = 0$. To the right of x,

$$\begin{aligned}\phi''(x+0) &= u''(x+0)\\ &\geqq f(x+0, u(x+0), u'(x+0))\\ &\geqq f(x+0,0,0)\\ &> 0.\end{aligned}$$

Thus $\phi(x)$ is strictly concave up to the right of $\hat{x}$. It follows that there exists a point $\bar{x}$ to the right of x at which $\phi(\bar{x}) > \phi(\hat{x})$ this contradicts the maximality of $\phi(x)$ at $\hat{x}$.

From the two cases just considered, it follows that $s^* \nless s$. Therefore $s \leqq s^*$. The conclusion of the theorem now follows immediately from Lemma 1. Indeed, we have $u(0) \leqq v(0)$, $u(s) \leqq v(s)$, and from (2) and (3),

$$-u'' + f(x, u, u') \leqq 0 \leqq -v'' + f(x, v, v').$$

Lemma 1 implies that $u(x) \leqq v(x)$ for $x \in [0, s]$.

Corollaries and remarks. Two corollaries follow immediately from Theorem 1.

COROLLARY 1. *Under the hypotheses of Theorem* 1, *solutions to problem* (1) *are unique when they exist.*

COROLLARY 2. *Let* $0 < \alpha_1 \leqq \alpha_2$ *and* $f_1(x, u, u') \leqq f_2(x, u, u')$ *for* (x, u, u') *in* $[0, T] \times \mathbf{R}^2$ *and suppose conditions* (i), (ii), *and* (iii) *of Theorem* 1 *are satisfied. Let* $u_i(x)$ *be the solution of problem* (1) *with* $\alpha = \alpha_i$, $s = s_i$, *and* $f(x, u, u') = f_i(x, u, u')$, $i = 1, 2$. *Then* $s_1 \leqq s_2$ *and* $u_1(x) \leqq u_2(x)$ *on the interval* $[0, s_1]$.

REMARK. We note that the added condition on $v(x)$, $v(x) \geqq 0$, in Theorem 1 imposes less of a restriction on application of this result than it appears to at first glance. In the first place, if $v(x)$ is itself a solution, then it satisfies (2) and hence from Lemma 2 will satisfy $v(x) \geqq 0$. In the second place, suppose that $v(x)$ is function satisfying conditions (3), but that $v(x) < 0$ for some $x \in (0, s^*)$. Because $v(0) \geqq 0$ and $v(s^*) = 0$, there is an $\bar{x} \in (0, s^*)$ at which $v(x)$ has a negative minimum and $v(\bar{x}) \leqq v(x)$ for all $x \in (0, s^*)$. But then $\bar{v}(x) := v(x) - v(\bar{x})$ satisfies all of the conditions of Theorem 1 for $v(x)$ with $s^* = \bar{x}$.

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