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BOUNDARY VALUE PROBLEMS FOR PAIRS OF SECOND ORDER EQUATIONS CONTAINING A SMALL PARAMETER

WALTER G. KELLEY

Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. In this paper we study weakly coupled second order systems

$$\varepsilon x'' = F(t, x, x', y), \quad y'' = G(t, x, y, y'), \tag{1}$$

where ε is a small positive parameter, with boundary conditions

$$x(0) = A, \ x(1) = B, \ y(0) = C, \ y(1) = D. \tag{2}$$

We associate with boundary value problems of type (1), (2) certain "reduced" problems

$$0 = F(t, x, x', y), \quad y'' = G(t, x, y, y'), \tag{3}$$

with a proper subset of the boundary conditions (2). Such problems have been previously examined by Hoppensteadt [4], Vasil'eva and Butusov [11], Fife [1] and others. These authors have demonstrated that solutions to (1), (2) exist under certain conditions and exhibit boundary layer behavior. Also, the occurence of internal layers for certain autonomous problems has been studied by Fife [1] for boundary conditions of type (2) and by Mimura, Tabata and Hosono [10] for boundary conditions of Neumann type. However, for a given problem, the conditions imposed by the above authors are difficult to check.

The purpose of this paper is to present explicit conditions on (1) so that for each "stable" solution of the reduced problem (3), there is a solution of (1), (2) which is approximated by the solution of the reduced problem for small $\varepsilon > 0$ outside of boundary layers at one or both endpoints of [0, 1]. Since our main technique is comparison with linear

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problems using differential inequalities, we present three simple linear examples.

EXAMPLE 1.

$$\varepsilon x'' = x, \qquad x(0) = x(1) = 1,$$
$$y'' = x - \pi^2 y, \quad y(0) = y(1) = 0.$$

From the first equation and boundary conditions,

$$x(t) = (1 - e^{-2/\sqrt{\varepsilon}})^{-1}(e^{-t/\sqrt{\varepsilon}} + e^{-(1-t)/\sqrt{\varepsilon}} - e^{(t-2)/\sqrt{\varepsilon}} - e^{-(1+t)/\sqrt{\varepsilon}}).$$

Substituting $x(t)$ into the second equation, we see that this problem has no solution. The reduced problem is

$$y'' = -\pi^2 y, \quad y(0) = y(1) = 0,$$

which has the family of solutions $y(t) = C \sin \pi t$.

EXAMPLE 2.

$$\varepsilon x'' = x + \pi^2 y, \quad x(0) = x(1) = 1,$$
$$y'' = x, \qquad y(0) = y(1) = 0.$$

This system is equivalent to $\varepsilon y^{(4)} = y'' + \pi^2 y$. Again, the reduced problem has a family of solutions. However, in this case the original problem has a unique solution, which is approximated by one of the reduced solutions outside of the boundary layers at $t = 0$ and $t = 1$.

EXAMPLE 3.

$$\varepsilon x'' = -kx' + \ell y, \quad x(0) = 1, \; x(1) = 0,$$
$$y'' = \mu x, \qquad y(0) = y(1) = 0,$$

where k, ℓ, μ are positive constants. The appropriate reduced problem is

$$y''' - \frac{\ell\mu}{k} y = 0, \; y(0) = y(1) = y''(1) = 0.$$

For $\ell\mu/k$ sufficiently small, the reduced problem has only the zero solution and the original problem has a unique solution which is close to zero for small $\varepsilon > 0$, except that $x(t)$ exhibits boundary layer behavior near $t = 0$. However, when $\ell\mu/k$ equals the first eigenvalue of the reduced problem (about 27.46), we have again the situation encountered in example 2.

For nonlinear problems, we avoid these difficulties by imposing bounds on various derivatives of F and G (see also Fife [1]) together with disconjugacy conditions on associated linear problems.

2. Boundary layers at both endpoints. We consider first the case that F and G are independent of the derivatives x' and y'. Then the first equation in (3) is an algebraic equation, and in general we cannot expect a solution of (3) to satisfy $x(0) = A$ or $x(1) = B$. Consequently, we select the reduced problem to be (3) together with the boundary conditions $y(0) = C$, $y(1) = D$. Suppose the reduced problem has a C^2 solution pair $x = u(t)$, $y = v(t)$, $0 \leqq t \leqq 1$. Making a translation of dependent variables $x - u(t)$, $y - v(t)$, we obtain an equivalent system with reduced solution pair $x = 0$, $y = 0$. Then it is sufficient to consider problems of the form

$$\begin{aligned} \varepsilon x'' &= F(t, x, y) + O(\varepsilon), & x(0) &= A, x(1) = B, \\ y'' &= G(t, x, y), & y(0) &= 0, y(1) = 0, \end{aligned} \tag{4}$$

where $F(t, 0, 0) = G(t, 0, 0) = 0$ for $0 \leqq t \leqq 1$.

A homogeneous, linear second order equation is disconjugate on an interval J if every nontrivial solution has at most one zero on J. If J is open or closed and bounded, then the equation is disconjugate if and only if it has a positive solution on J. Various tests for disconjugacy can be found in Hartman [3], Willett [12] and Fink [2].

The following theorem extends a theorem of Howes [5].

THEOREM 1. *Let n be a nonnegative integer and $\delta > 0$. Define $\mathscr{R} = \{(t, x, y) : 0 \leqq t \leqq 1, |x| \leqq d(t), |y| \leqq \delta\}$, where $d(t) \leqq \delta$ for $\delta \leqq t \leqq 1 - \delta$, $d(t) \geqq |A|$ for $0 \leqq t \leqq \delta/2$ and $d(t) \geqq |B|$ for $1 - \delta/2 \leqq t \leqq 1$. Assume*

(a) *$F, G : \mathscr{R} \to R^1$, F is of class C^{2n+2}, G is of class C^2, and $F(t, 0, 0) = G(t, 0, 0) = 0$ for $0 \leqq t \leqq 1$;*

(b) *$F_x^{(i)}(t, 0, 0) = F_y^{(i)}(t, 0, 0) = 0 \quad (i = 1, \ldots, 2n)$, $F_x^{(2n+1)}(t, 0, 0) = q(t) > 0$ for $0 \leqq t \leqq 1$;*

(c) *$\int_0^\alpha F(0, x, 0)dx > 0$ if $0 < \alpha \leqq A$ or $A \leqq \alpha < 0$, and $\int_0^\beta F(1, x, 0)dx > 0$ if $0 < \beta \leqq B$ or $B \leqq \beta < 0$; and*

(d) *the equation*

$$\theta'' + \left[\left(\frac{\ell(t)}{q(t)}\right)^{1/(2n+1)} m(t) - G_y(t, 0, 0)\right]\theta = 0$$

is disconjugate on $[0, 1]$, where $\ell(t) = |F_y^{(2n+1)}(t, 0, 0)|$ and $m(t) \geqq |G_x(t, x, y)|$ for small $|x|$ and $|y|$, $0 \leqq t \leqq 1$.

Then (4) has a solution pair $x(t, \varepsilon)$, $y(t, \varepsilon)$ for small $\varepsilon > 0$, and $y(t, \varepsilon) = O(\varepsilon^{1/(2n+1)})$ for $0 \leqq t \leqq 1$, $x(t, \varepsilon) = O(\varepsilon^{1/(2n+1)})$ for $O(\sqrt{\varepsilon}) \leqq t \leqq 1 - O(\sqrt{\varepsilon})$.

PROOF. We use the method of upper and lower solutions (see Jackson [7]). We will define a pair $\phi_1(t, \varepsilon)$, $\phi_2(t, \varepsilon)$, which will serve as upper solutions for the system (4). Lower solutions can be constructed in a similar manner. Furthermore, we consider only the case $A > 0$ and $B > 0$, in which boundary layer corrections must be included in the upper solution ϕ_1 at both endpoints.

Let $w_1(t, \varepsilon)$ and $w_2(t, \varepsilon)$ be certain positive solutions of

$$\begin{aligned} &\varepsilon w_1'' < F(0, w_1, 0),\ w_1(0) = A, \\ &\varepsilon w_2'' < F(1, w_2, 0),\ w_2(1) = B. \end{aligned} \tag{5}$$

Define $\phi_1(t, \varepsilon) = w_1(t, \varepsilon) + w_2(t, \varepsilon) + \lambda(t, \varepsilon)$, where $\lambda(t, \varepsilon) > 0$, $\lambda(t, \varepsilon) = O(\varepsilon^{1/(2n+1)})$, is to be chosen.

We use Taylor's theorem to expand

$$\begin{aligned} F(t, w_1 + w_2 + \lambda, y) &= F(t, w_1 + w_2, 0) + \sum_{i=1}^{2n} F_y^{(i)}(t, w_1 + w_2, 0)\frac{y^i}{i!} \\ &+ F_y^{(2n+1)}(t, w_1 + w_2, *)\frac{y^{2n+1}}{(2n+1)!} + \sum_{i=1}^{2n} F_x^{(i)}(t, w_1 + w_2, y)\frac{\lambda^i}{i!} \\ &+ F_x^{(2n+1)}(t, **, y)\frac{\lambda^{2n+1}}{(2n+1)!}, \end{aligned}$$

where $*$ is between 0 and y, and $**$ is between $w_1 + w_2$ and $w_1 + w_2 + \lambda$. Define

$$\begin{aligned} \chi(t, \varepsilon) = &- \varepsilon w_1'' - \varepsilon w_2'' + F(t, w_1 + w_2, 0) + \sum_{i=1}^{2n} F_y^{(i)}(t, w_1 + w_2, 0)\frac{y^i}{i!} \\ &+ \sum_{i=1}^{2n} F_x^{(i)}(t, w_1 + w_2, y)\frac{\lambda^i}{i!}. \end{aligned}$$

Then

$$\begin{aligned} &F(t, w_1 + w_2 + \lambda, y) + O(\varepsilon) - \varepsilon\phi_1''(t, \varepsilon) \\ &= O(\varepsilon) - \varepsilon\lambda'' + \chi + F_y^{(2n+1)}(t, w_1 + w_2, *)\frac{y^{2n+1}}{(2n+1)!} \\ &\quad + F_x^{(2n+1)}(t, **, y)\frac{\lambda^{2n+1}}{(2n+1)!}. \end{aligned} \tag{6}$$

We can select a function $\rho(\varepsilon) = O(\sqrt{\varepsilon})$ and a number t_0 so that $\chi(t, \varepsilon)$ exceeds a given multiple of ε for $0 \leqq t \leqq \rho(\varepsilon)$ and $1 - \rho(\varepsilon) \leqq t \leqq 1$ and so that $\chi(t, \varepsilon) > 0$ for $\rho(\varepsilon) \leqq t \leqq t_0$ and $1 - t_0 \leqq t \leqq 1 - \rho(\varepsilon)$. This is possible because of (5) and assumptions (b) and (c).

Write $F_y^{(2+1)}(t, w_1 + w_2, *) = F_y^{(2n+1)}(t, 0, 0) + O(\chi) + O(y)$ and $F_x^{(2n+1)}(t, **, y) = F_x^{(2n+1)}(t, 0, 0) + O(\chi) + O(y)$. Then (6) is equal to

$$\begin{aligned} &O(\varepsilon) + O(\varepsilon^2) + \chi + F_y^{(2n+1)}(t, 0, 0)\frac{y^{2n+1}}{(2n+1)!} + O(\chi)y^{2n+1} \\ &+ O(y)y^{2n+1} + F_x^{(2n+1)}(t, 0, 0)\frac{\lambda^{2n+1}}{(2n+1)!} + O(\chi)\lambda^{2n+1} + O(y)\lambda^{2n+1}. \end{aligned}$$

By the basic existence theorem involving upper and lower solutions, we need (6) to be non-negative for y between its upper and lower solutions. By the symmetry of the construction, it is sufficient to prove (6) is non-negative for $|y| \leqq \phi_2(t, \varepsilon)$. By the properties of χ, (6) is certainly nonegative for $0 \leqq t \leqq \rho(\varepsilon)$ and $1 - \rho(\varepsilon) \leqq t \leqq 1$ and small $\varepsilon > 0$. For other values of t, $\chi + O(\chi)y^{2n+1} + O(\chi)\lambda^{2n+1}$ is either positive or transcendentally small, so (6) is positive for small $\varepsilon > 0$ provided

$$(7) \qquad O(\varepsilon) - \frac{\ell(t)}{(2n+1)!}\phi_2^{2n+1}(r, \varepsilon) + \frac{q(t)}{(2n+1)!}\lambda^{2n+1}(t, \varepsilon) > 0$$

for $0 \leqq t \leqq 1$.

By hypothesis (d), we can choose a function $\theta(t) > 0, 0 \leqq t \leqq 1$, so that

$$\left[G_y(t, 0, 0) - \left(\frac{\ell(t)}{q(t)}\right)^{1/(2n+1)} m(t)\right]\theta - \theta'' = \delta\theta,$$

for $0 \leqq t \leqq 1$ and some choice of $\delta > 0$. Define

$$\lambda(t, \varepsilon) = C\varepsilon^{1/(2n+1)}\left[\left(\frac{\ell(t)}{q(t)}\right)^{1/(2n+1)} + \frac{\delta}{2m(t)}\right]\theta(t).$$

Choose $N = \sup\{|G_x(t, x, y)| : (t, x, y) \in \mathscr{R}\}$. We wish to select $\phi_2(t, \varepsilon) > 0$ so that

$$\phi_2'' = -(N+1)(w_1 + w_2) + C\varepsilon^{1/(2n+1)}\,\theta''.$$

Estimates on the functions w_1 and w_2 yield that $\phi_2 = O(\varepsilon^{1/(2n+1)}) + C\varepsilon^{1/(2n+1)}\,\theta > 0$ for C sufficiently large.

Next, write

$$\begin{aligned} G(t, x, \phi_2) &= G(t, 0, \phi_2) + G_x(t, \Delta, \phi_2)x \\ &= G_y(t, 0, 0)\phi_2 + O(\phi_2^2) + G_x(t, \Delta, \phi_2)x, \end{aligned}$$

where Δ is between 0 and x. Then

$$(8) \qquad \begin{aligned} G(t, x, \phi_2) - \phi_2'' \geqq\ & G_y(t, 0, 0)\phi_2 + O(\phi_2^2) - |G_x(t, \Delta, \phi_2)|\,|x| \\ & + (N+1)(w_1 + w_2) - C\varepsilon^{1/(2n+1)}\,\theta''. \end{aligned}$$

We require (8) $\geqq 0$ for x between $\phi_1(t, \varepsilon)$ and the corresponding lower solution. Letting $|x| = w_1 + w_2 + \lambda$ and noting the definition of N, we see that it is sufficient to require

$$(9) \quad O(\varepsilon^{1/(2n+1)}) + G_y(t, 0, 0)C\varepsilon^{1/(2n+1)}\,\theta - m(t)\lambda - C\varepsilon^{1/(2n+1)}\,\theta'' > 0$$

for small $\varepsilon > 0$ and $0 \leqq t \leqq 1$. Using the definitions of θ and λ, (9) is precisely

$$O(\varepsilon^{1/(2n+1)}) + C\varepsilon^{1/(2n+1)}\,\frac{\delta}{2}\,\theta > 0,$$

which is true for sufficiently large C.

Finally, we return to (7) and substitute the expressions for ϕ_2 and λ, obtaining

$$
\begin{aligned}
O(\varepsilon) - \frac{\varepsilon \ell}{(2n+1)!}(O(1) + C\theta)^{2n+1} &+ \frac{C^{2n+1}\varepsilon q}{(2n+1)!}\left[\left(\frac{\ell}{q}\right)^{1/(2n+1)} + \frac{\delta}{2m}\right]^{2n+1} \theta^{2n+1} \\
&> - \frac{\varepsilon \ell}{(2n+1)!} C^{2n+1}\theta^{2n+1} + \frac{C^{2n+1}\varepsilon q}{(2n+1)!}\left(\frac{\ell}{q}\right)\theta^{2n+1} = 0
\end{aligned}
$$

for sufficiently large C.

For simplicity, we have assumed that G is independent of y' in the above discussion. However, Theorem 1 can easily be extended to include this case. We omit the details. Also, if $F_x^{(2n+1)} \geqq q > 0$ in R, then

$$
x(t, \varepsilon) = \begin{cases} O(|A|e^{-t\sqrt{q/\varepsilon}}) + O(|B|e^{-(1-t)\sqrt{q/\varepsilon}}) + O(\varepsilon), \text{ if } n = 0 \\ O(|A|(1 + \sigma_1 t/\sqrt{\varepsilon})^{-1/n}) + O(|B|(1 + \sigma_2(1-t)/\sqrt{\varepsilon})^{-1/n}) \\ \qquad + O(\varepsilon^{1/(2n+1)}), \text{ if } n \geqq 1, \end{cases}
$$

where σ_1, σ_2 are positive constants.

EXAMPLE 4. Consider the system

$$
\begin{aligned}
\varepsilon x'' &= \frac{1}{6}(x^3 + y^3), & x(0) = A,\ x(1) = B, \\
y'' &= x + y - (4\pi \sin 2\pi t)(e^y - 1), & y(0) = y(1) = 0.
\end{aligned}
$$

Here hypotheses (a), (b) and (c) of Theorem 1 are satisfied with $n = 1$ and $q(t) = 1$. In (d), $\ell(t) = 1$ and $m(t) = 1$. Since $G_y(t, 0, 0) = 1 - 4\pi \sin 2\pi t$, the equation for θ is

$$\theta'' + (4\pi \sin 2\pi t)\theta = 0.$$

By Liapunov's inequality (see [9]), this equation is disconjugate on $[0, 1]$ if $\int_0^1 (4\pi \sin 2\pi t)^+ dt \leqq 4$. In fact, $\int_0^1 (4\pi \sin 2\pi t)^+ dt = \int_0^{1/2} 4\pi \sin 2\pi\, t dt = 4$, so Theorem 1 can be applied, and there is a solution pair $x(t, \varepsilon)$, $y(t, \varepsilon)$ satisfying the estimates of Theorem 1. Note that if A and B are not zero, then $x(t, \varepsilon)$ exhibits boundary layer behavior at both endpoints.

Theorem 1 can easily be extended to vector systems by comparison techniques (see also Howes [6]). Consider

$$
\begin{aligned}
\varepsilon x'' &= \mathscr{F}(t, x, y) + O(\varepsilon), & x(0) = A,\ x(1) = B, \\
y'' &= \mathscr{G}(t, x, y), & y(0) = 0,\ y(1) = 0,
\end{aligned}
\tag{10}
$$

where x and y are vectors.

THEOREM 2. *Let* $\mathscr{R}' = \{(t, x, y)\colon 0 \leqq t \leqq 1, 0 \leqq x \leqq d(t), 0 \leqq y \leqq \delta\}$, *where* $d(t)$ *and* δ *are given in Theorem* 1. *Assume*

(a) $\mathscr{F}$ and $\mathscr{G}$ *are continuous, vector-valued functions and* $\mathscr{F}(t, 0, 0) = \mathscr{G}(t, 0, 0) = 0$; *and*

(b) *there are scalar-valued functions* F *and* G *having the properties of Theorem* 1 *with* $\mathscr{R}$ *replaced by* $\mathscr{R}'$ *and* A, B *replaced by* $\|A\|$, $\|B\|$, *respectively, so that*

$$\frac{x}{\|x\|} \cdot \mathscr{F}(t, x, y) \geqq F(t, \|x\|, \|y\|), \tag{11}$$

$$\frac{y}{\|y\|} \cdot \mathscr{G}(t, x, y) \geqq G(t, \|x\|, \|y\|), \tag{12}$$

for $0 \leqq t \leqq 1$, $\|x\| \leqq d(t)$, $\|y\| \leqq \delta$.

Then for small $\varepsilon > 0$, (10) *has a solution pair* $x(t, \varepsilon)$, $y(t, \varepsilon)$ *with* $y(t, \varepsilon) = O(\varepsilon^{1/(2n+1)})$ *for* $0 \leqq t \leqq 1$ *and* $x(t, \varepsilon) = O(\varepsilon^{1/(2n+1)})$ *for* $O(\sqrt{\varepsilon}) \leqq t \leqq 1 - O(\sqrt{\varepsilon})$.

PROOF. The proof of Theorem 1 yields a pair of positive functions $\phi_1(t, \varepsilon)$, $\phi_2(t, \varepsilon)$ so that

$$\varepsilon\phi_1'' \leqq F(t, \phi_1, v) + O(\varepsilon), \quad 0 \leqq v \leqq \phi_2,$$
$$\phi_2'' \leqq G(t, u, \phi_2), \qquad 0 \leqq u \leqq \phi_1,$$

$\phi_1(0, \varepsilon) \geqq \|A\|$, $\phi_1(1, \varepsilon) \geqq \|B\|$, $\phi_2(t, \varepsilon) = O(\varepsilon^{1/(2n+1)})$ for $0 \leqq t \leqq 1$ and $\phi_1(t, \varepsilon) = O(\varepsilon^{1/(2n+1)})$ for $O(\sqrt{\varepsilon}) \leqq t \leqq 1 - O(\sqrt{\varepsilon})$ and small $\varepsilon > 0$.

Let $r_1(t, x, \varepsilon) = \|x\| - \phi_1(t, \varepsilon)$ and $r_2(t, y, \varepsilon) = \|y\| - \phi_2(t, \varepsilon)$. Then by (11)

$$\frac{\partial^2 r_1}{\partial t^2} + \frac{1}{\varepsilon}\frac{x}{\|x\|} \cdot [\mathscr{F}(t, x, y) + O(\varepsilon)]$$
$$\geqq -\phi_1'' + \frac{1}{\varepsilon}[F(t, \phi_1, \|y\|) + O(\varepsilon)] \geqq 0,$$

whenever $r_1 = 0$ and $r_2 \leqq 0$. Similarly, by (12)

$$\frac{\partial^2 r_2}{\partial t^2} + \frac{y}{\|y\|} \cdot [\mathscr{G}(t, x, y)]$$
$$\geqq -\phi_2'' + G(t, \|x\|, \phi_2) \geqq 0,$$

whenever $r_1 \leqq 0$ and $r_2 = 0$. By Theorem 5 of [**8**], problem (10) has a solution pair $x(t, \varepsilon)$, $y(t, \varepsilon)$ for small $\varepsilon > 0$ and $r_1(t, x(t, \varepsilon)) \leqq 0$, $r_2(t, y(t, \varepsilon)) \leqq 0$ for $0 \leqq t \leqq 1$.

EXAMPLE 5. Consider the problem

$$\begin{aligned}
\varepsilon x_1'' &= 2x_1 - x_1^2 - x_2 + y_2, & x(0) &= A,\ x(1) = B,\\
\varepsilon x_2'' &= 3x_1 + 2x_2 - x_2^2 + y_1, & y(0) &= y(1) = 0,\\
y_1'' &= ax_1 - by_1e^{y_2}, & b &\geqq 0,\\
y_2'' &= cx_2 - dy_2e^{y_1}, & d &\geqq 0.
\end{aligned}$$

First, we compute

$$(13)\qquad \frac{x}{\|x\|}\cdot \mathscr{F}(t,x,y)=\frac{2x_1^2+2x_1x_2+2x_2^2-(x_1^3+x_2^3)+x_1y_2+x_2y_1}{(x_1^2+x_2^2)^{1/2}}.$$

Now $(x_1-x_2)^2\geqq 0$ implies $x_1^2+x_2^2\geqq 2x_1x_2$, so $x_1^4x_2^2+x_1^2x_2^4\geqq 2x_1^3x_2^3$, and finally $(x_1^2+x_2^2)^3\geqq(x_1^3+x_2^3)^2$. Also, $(x_1+x_2)^2\geqq 0$ implies $2x_1^2+2x_1x_2+2x_2^2\geqq x_1^2+x_2^2$, and

$$|x_1y_2+x_2y_1|\leqq (x_1^2+x_2^2)^{1/2}(y_1^2+y_2^2)^{1/2}.$$

From (13)

$$\begin{aligned}\frac{x}{\|x\|}\cdot\mathscr{F}(t,x,y)&\geqq\frac{\|x\|^2-\|x\|^3-\|x\|\,\|y\|}{\|x\|}\\ &=\|x\|-\|x\|^2-\|y\|\equiv F(\|x\|,\|y\|).\end{aligned}$$

Next, we have

$$\begin{aligned}\frac{y}{\|y\|}\cdot\mathscr{G}(t,x,y)&=\frac{ax_1y_1+cx_2y_2-by_1^2e^{y_2}-dy_2^2e^{y_1}}{\|y\|}\\ &\geqq\frac{-\|x\|(a^2y_1^2+c^2y_2^2)^{1/2}-by_1^2e^{\delta}-dy_2^2e^{\delta}}{\|y\|}\\ &\geqq-\max\{|a|,|c|\}\,\|x\|-\max\{b,d\}e^{\delta}\|y\|\equiv G(\|x\|,\|y\|),\end{aligned}$$

where $\delta>0$ is small. Checking the hypotheses of Theorem 1, (b) is satisfied with $n=0$ and $q=1$. For (c), we need

$$\int_0^{\alpha}(x-x^2)dx=\frac{\alpha^2}{2}-\frac{\alpha^3}{3}>0$$

for $0<\alpha\leqq\|A\|$ and $0<\alpha\leqq\|B\|$, so we require $\|A\|<3/2$ and $\|B\|<3/2$. Finally, in (d), $\ell=1$, $m\geqq\max\{|a|,|c|\}$, $G_y(t,0,0)=-\max\{b,d\}e^{\delta}$, so we require

$$\theta''+[\max\{|a|,|c|\}+\max\{b,d\}e^{\delta}]\theta=0$$

to be disconjugate on $[0,1]$, or $\max\{|a|,|c|\}+\max\{b,d\}<\pi^2$.

3. A boundary layer at one endpoint. We now consider system (1) in the case that F depends on x'. Since the first equation in (3) is now a first order differential equation, we require a solution pair for (3) to satisfy one of the conditions on x in (2) as well as the y conditions. As in §2, we assume a change of variables has been made so that this reduced problem has the zero solution. Then we consider the problem

$$(14)\qquad\begin{aligned}\varepsilon x''&=F(t,x,x',y)+O(\varepsilon), & x(0)&=A,\ x(1)=0\\ y''&=G(t,x,y,y'), & y(0)&=0,\ y(1)=0\end{aligned}$$

where $F(t, 0, 0, 0) = G(t, 0, 0, 0) = 0$ for $0 \leqq t \leqq 1$. Note that we have assumed the solution of (3) satisfies the boundary condition on x at $t = 1$, so we expect the solution of (14) to exhibit boundary layer behavior at $t = 0$. It is then necessary to restrict $F_{x'}$ to be negative in an appropriate domain. For a boundary layer at $t = 1$, one would take $F_{x'}$ to be positive.

THEOREM 3. *Define* $\mathscr{D}_1 = \{(t, x, x', y)\colon 0 \leqq t \leqq 1, |x| \leqq d_1(t), |x'| \leqq d_2(t), |y| \leqq \delta\}$ *and* $\mathscr{D}_2 = \{(t, x, y, y')\colon 0 \leqq t \leqq 1, |x| \leqq d_1(t), |y| \leqq \delta, y' \in R^1\}$, *where* $\delta > 0$, $d_1(t) \leqq \delta$ *for* $\delta \leqq t \leqq 1$ *and* $d_1(t) \geqq |A|$ *for* $0 \leqq t \leqq \delta$, *and* $d_2(t) \leqq \delta$ *for* $\delta \leqq t \leqq 1$ *and* $d_2(t) \to \infty$ *as* $t \downarrow 0$. *Assume*

(a) *F and G are real-valued and of class C^2 on $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively, and* $F(t, 0, 0, 0) = G(t, 0, 0, 0) = 0$ *for* $0 \leqq t \leqq 1$;

(b) $F_{x'}(t, x, x', y) \leqq -k < 0$ *for* $(t, x, x', y) \in \mathscr{D}_1$; *and*

(c) *there is a positive, non-decreasing, continuous function $n(s)$ on* $[0, \infty)$ *such that* $\int^\infty s/n(s)ds = \infty$, $|F(t, x, x', y)| \leqq n(|x'|)$ *and* $|G(t, x, y, y')| \leqq n(|y'|)$ *for* $0 \leqq t \leqq 1$, $|x| \leqq d_1(t)$, $|y| \leqq \delta$ *and all x' and y'.*

(d) *Let* $\ell(t) = |F_y(t, 0, 0, 0)|$, $m(t) \geqq |G_x(t, x, y, 0)|$ *and* $q(t) \leqq F_x(t, x, 0, y)$ *for small x and y. Assume there is a positive solution pair* $\psi(t)$ *and* $\theta(t)$, *with* $\psi'(t) \leqq 0$, *to*

$$-\psi' + \frac{q}{k}\psi > \frac{\ell}{k}\theta,$$

$$-\theta'' + G_{y'}(t, 0, 0, 0)\theta' + G_y(t, 0, 0, 0)\theta > m\psi,$$

for $0 \leqq t \leqq 1$.
Then for small $\varepsilon > 0$, (17) *has a solution pair* $x(t, \varepsilon), y(t, \varepsilon)$ *so that* $x(t, \varepsilon) = O(\varepsilon) + O(|A|e^{-kt/\varepsilon})$ *and* $y(t, \varepsilon) = O(\varepsilon)$ *for* $0 \leqq t \leqq 1$.

PROOF. Again, we use the method of upper and lower solutions. This procedure requires an assumption on the growth rates of F and G with respect to x' and y', respectively. Hypothesis (c) above, a "Nagumo condition", fulfills the requirement. We will define upper solutions $\phi_1(t, \varepsilon)$ and $\phi_2(t, \varepsilon)$, assuming $A > 0$.

Define $\phi_1(t, \varepsilon) = Ae^{\mu t} + \lambda(t, \varepsilon)$, where μ is to be a certain negative constant and λ is to be a positive function with non-positive first derivative so that λ, λ' and λ'' are $O(\varepsilon)$. Write

$$\begin{aligned}
&F(t, Ae^{\mu t} + \lambda, A\mu e^{\mu t} + \lambda', y)\\
&\quad = F_y(t, 0, 0, 0)y + O(|y|^2) + F_x(t, *, 0, y)(Ae^{\mu t} + \lambda)\\
&\qquad + F_{x'}(t, Ae^{\mu t} + \lambda, **, y)(\mu Ae^{\mu t} + \lambda')\\
&\quad \geqq -\ell|y| + O(|y|^2) + N(t)(Ae^{\mu t} + \lambda) - k(\mu Ae^{\mu t} + \lambda'),
\end{aligned}$$

where $F_x(t, x, 0, y) \geqq N(t)$ for $|x| \leqq d_1(t), 0 \leqq t \leqq 1, |y| \leqq \delta$, $*$ is between 0 and $Ae^{\mu t} + \lambda$,and $**$ is between 0 and $\mu Ae^{\mu t} + \lambda'$. Then

$$
(15) \qquad \begin{aligned} &-\varepsilon\phi_1'' + F(t, Ae^{\mu t} + \lambda, A\mu e^{\mu t} + \lambda', y) + O(\varepsilon) \\ &\quad \geqq -\varepsilon A\mu^2 e^{\mu t} - \varepsilon\lambda'' - \ell|y| + O(|y|^2) \\ &\qquad + N(t)(Ae^{\mu t} + \lambda) - k(\mu A e^{\mu t} + \lambda'). \end{aligned}
$$

Choose $\mu = O(-k/\varepsilon)$ so that $-\varepsilon\mu^2 - k\mu + N(t) \geqq \rho > 0$, for $0 \leqq t \leqq 1$. Then (15) is greater than $A\rho e^{\mu t} + O(\varepsilon) - \ell|y| + N(t)\,\lambda - k\lambda'$. Thus (15) will be greater than zero for small $\varepsilon > 0$, if

$$
(16) \qquad O(\varepsilon) - \ell|y| + q\lambda - k\lambda' > 0.
$$

Let $\psi(t)$ and $\theta(t)$ be functions satisfying hypothesis (d). Define $\lambda(t, \varepsilon) = C\varepsilon\psi(t)$, where $C > 0$ is to be chosen below. Let $Q(t) = \sup\{|G_x(t, x, y, y')| : |x| \leqq d_1(t), |y| \leqq \delta, |y'| \leqq \delta\}$. Next, $\phi_2(t, \varepsilon)$ is to be defined so that

$$
\phi_2'' = C\varepsilon\theta'' - (1 + \max\{Q(t)\colon 0 \leqq t \leqq 1\})Ae^{\mu t}.
$$

Then $\phi_2 = C\varepsilon\theta + O(\varepsilon^2)$ and $\phi_2' = C\varepsilon\theta' + O(\varepsilon)$.

Using Taylor's theorem, we can expand

$$
\begin{aligned} G(t, x, \phi_2, \phi_2') &= G_x(t, \Delta, \phi_2, \phi_2')x + G_y(t, 0, 0, 0)\phi_2 \\ &\quad + G_{y'}(t, 0, 0, 0)\,\phi_2' + O(\varepsilon^2), \end{aligned}
$$

where Δ is between 0 and x, $0 \leqq t \leqq 1$. Now

$$
(17) \qquad \begin{aligned} &G(t, x, \phi_2, \phi_2') - \phi_2'' \\ &\quad \geqq -Q(t)|x| + G_y(t,0,0,0)\phi_2 + G_{y'}(t,0,0,0)\phi_2' + O(\varepsilon^2) \\ &\qquad - C\varepsilon\theta'' + (1 + \max\{Q(t)\colon 0 \leqq t \leqq 1\})Ae^{\mu t}. \end{aligned}
$$

By the basic result on differential inequalities, we need to verify (16) with $|y| \leqq \phi_2$ and show (17) is nonnegative when $|x| \leqq \phi_1$. Now (17) is greater than or equal to

$$
\begin{gathered} -Q(t)(Ae^{\mu t} + C\varepsilon\psi) + G_y(t, 0, 0, 0)(C\varepsilon\theta + O(\varepsilon^2)) \\ + G_{y'}(t, 0, 0, 0)(C\varepsilon\theta' + O(\varepsilon)) - C\varepsilon\theta'' + (1 + \max\{Q(t)\colon 0 \leqq t \leqq 1\})Ae^{\mu t}, \end{gathered}
$$

which is positive for sufficiently large C since

$$
-m\psi + G_y(t, 0, 0, 0)\theta + G_{y'}(t, 0, 0, 0)\theta' - \theta'' > 0,
$$

$0 \leqq t \leqq 1$.

Substituting $|y| = C\varepsilon\theta + O(\varepsilon^2)$ into (16), we obtain

$$
O(\varepsilon) - \ell(C\varepsilon\theta + O(\varepsilon^2)) + qC\varepsilon\psi - kC\varepsilon\psi' > 0,
$$

which is true for large C since $-\ell\theta + q\psi - \psi' > 0$.

Assumption (b) in Theorem 3 can be weakened in case F is linear or quadratic in x' (see Howes [5]). Also, note that in order for (d) to be satisfied, it is necessary, but not sufficient, for $-\theta'' + G_{y'}(t, 0, 0, 0)\theta' + G_y(t, 0, 0, 0)\theta = 0$ to be disconjugate on [0, 1].

COROLLARY 1. *If* $-\theta'' + G_y(t, 0, 0, 0)\theta' + G_y(t, 0, 0, 0)\theta = 0$ *is disconjugate on* [0, 1] *and if* $\ell m/k$ *is sufficiently small, then hypothesis* (d) *of Theorem* 3 *is satisfied.*

PROOF. Suppose $q(t) \geqq q$ and $k(t) \leqq k$, for $0 \leqq t \leqq 1$. Define $\psi(t) = Ce^{at}$, with $a = q/k - |q/k| - 1$, $C > 0$. Then

$$-\psi'(t) + \frac{q(t)}{k(t)}\psi(t) \geqq -\psi'(t) + \frac{q}{k}\psi(t)$$
$$= \left(1 + \left|\frac{q}{k}\right|\right)Ce^{at}.$$

The assumption of disconjugacy implies that there is a positive solution θ of

$$-\theta'' + G_{y'}(t, 0, 0, 0)\theta' + G_y(t, 0, 0, 0)\theta > m(t)Ce^{at}$$

on [0, 1] for small C. Furthermore,

$$m(t)\left(1 + \left|\frac{q}{k}\right|\right)Ce^{at} > \frac{\ell(t)m(t)}{k(t)}\theta(t),$$

if $\ell m/k$ is sufficiently small. Thus (d) is satisfied.

COROLLARY 2. *If* $-\theta'' + G_{y'}(t, 0, 0, 0)\theta' + G_y(t, 0, 0, 0)\theta = 0$ *is disconjugate on* [0, 1], $q(t) > 0$, $0 \leqq t \leqq 1$ *and* $\ell m/q$ *is sufficiently small, then* (d) *of Theorem* 3 *is satisfied.*

PROOF. Define $\psi(t) = C > 0$. Then proceed as in the proof of Corollary 1.

The choices of ψ made in Corollaries 1 and 2 do not give the best possible results. Consider the following example.

EXAMPLE 6.

$$\varepsilon x'' = \sin(\ell y) - kx', \qquad x(0) = A, x(1) = 0,$$
$$y'' = mx + y^2 + (y')^2, \quad y(0) = y(1) = 0,$$

where ℓ, m and k are positive constants. Clearly, (a), (b) and (c) of Theorem 3 are satisfied. To satisfy (d), we need $-\psi' > \ell/k\theta$ and $-\theta'' > m\psi$.

Consider for $C > 0$ the problem

$$\Phi''' = C^3\Phi,\ \Phi(0) = 0,\ \Phi(1) = 0,\ \Phi''(1) = 0. \tag{18}$$

Let $D = \sqrt{3}\,C/2$. Then (18) has the family of solutions

$$\Phi(t) = Ee^{Ct}\{\sin D + \sqrt{3}\cos D + e^{-3Ct/2}[(2e^{3C/2}\sin D - \sqrt{3}) \cos D(t-1) + (2e^{3C/2}\cos D + 1)\sin D(t-1)]\},$$

where E is arbitrary, provided that

$$2\cos(D - \pi/3) = e^{-\sqrt{3}D}. \tag{19}$$

Let D be the smallest positive solution of (19). If $E > 0$, one can show $\Phi(t) > 0$ and $\Phi''(t) < 0$ for $0 < t < 1$. Now Φ can be used to construct a function $\phi(t)$ so that $\phi''' > (\ell m/k)\phi$, $\phi > 0$ and $\phi'' < 0$, on $[0, 1]$, provided $\ell m/k < C^3 \cong 27.46$.

Define $\psi = -\phi''/m$ and $\theta = \phi - \delta(t+1)^2$, $\delta > 0$. Then $\psi > 0$ and $\theta > 0$ for small $\delta > 0$ on $[0, 1]$. Also,

$$-\psi' = \frac{\phi'''}{m} > \frac{\ell}{k}(\phi - \delta(t+1)^2) = \frac{\ell}{k}\theta$$

and

$$-\theta'' = -\phi'' + 2\delta = m\psi + 2\delta < m\psi,$$

on $[0, 1]$, so (d) is satisfied, and Theorem 3 may be applied.

The procedure used in this example to construct ψ and θ is optimal for the case $q = 0$ (see Example 3). For other cases, an optimal procedure is more difficult to find.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OK 73019