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COMPUTABLE ERROR BOUNDS FOR FINITE ELEMENT APPROXIMATIONS TO THE DIRICHLET PROBLEM

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ABSTRACT. The constants bounding the solution of Poisson's equation in terms of the given boundary data are derived. Knowledge of these constants then permits the interpolation remainder theory of Barnhill and Gregory to be used to find computable finite element error bounds.

1. Introduction. The purpose of this paper is to extend the finite element error bounds of Barnhill and Gregory [**2**, **3**] so that they become numerically computable from the data. This section is a very brief review of the necessary facts.

DIRICHLET PROBLEM. If $\Omega \subset \mathbf{R}^2$ is a bounded domain, find $u: \Omega \to \mathbf{R}$ such that

$$\begin{aligned} \Delta u &= f \text{ in } \Omega \ (f \text{ given}) \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

WEAK FORMULATION. Find $u \in \mathring{W}_2^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega fv dx \qquad \forall v \in \mathring{W}_2^1(\Omega).$$

OPERATOR-THEORETIC FORMULATION. Define an unbounded linear operator

$$\Delta_D: L_2(\Omega) \to L_2(\Omega)$$

by

$$\begin{aligned} D(\Delta_D) &= \mathring{W}_2^1(\Omega) \cap \{u: \Delta u \in L_2(\Omega)\} \\ \Delta_D u &= \Delta u \ \forall u \in D(\Delta_D). \end{aligned}$$

LEMMA 1. *u is a weak solution of the Dirichlet problem with $f \in L_2(\Omega)$ if and only if $u \in D(\Delta_D)$ and $\Delta_D u = f$.*

PROOF. ($\Rightarrow$) Take $v \in C_0^\infty(\Omega)$. ($\Leftarrow$) $C_0^\infty(\Omega)$ is dense in $\mathring{W}_2^1(\Omega)$.

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LEMMA 2. *Δ_D is selfadjoint on the Hilbert space $L_2(\Omega)$.*

Ref. [**10**, p. 322]

LEMMA 3. *Let Ω have the segment property* [**1**], *or the "finite tiling property"* [**16**]. *Then $\sigma(\Delta_D)$, the spectrum of Δ_D, is discrete and lies on the negative real axis.*

PROOF. The finite tiling property and boundedness of Ω imply that the Rellich selection theorem is valid for Ω [**16**]. The latter implies that $\sigma(\Delta_D)$ is discrete. If $\lambda \in \sigma(\Delta_D)$ and $u \neq 0$ is a corresponding eigenfunction, then the weak formulation with $f = \lambda u$, $v = u$ implies $\lambda \leqq 0$. Moreover, if $\lambda = 0$, then $\nabla u = 0$ in Ω and hence $u =$ const. But then the boundary condition implies $u = 0$ contrary to hypothesis.

EXISTENCE AND UNIQUENESS OF A SOLUTION. Lemmas 1, 2, and 3 imply the following theorem.

THEOREM 4. *If Ω has the finite tiling property, then there exists a unique solution* (*weak formulation*) *for each $f \in L_2(\Omega)$.*

REGULARITY OF THE SOLUTION. The most precise results are the Schauder estimates [**9**, pp. 331–350, or **11**, pp. 164–9]. Let $C \subset \bar{\Omega}$ be compact, $\partial\Omega \cap C \in C^{n+2+\delta}$ and $f \in C^{n+\delta}(\bar{\Omega})$ for $n = 0, 1, 2, \ldots$ and $0 < \delta < 1$. Then $u \in C^{n+2+\delta}(C)$.

FINITE ELEMENT (GALERKIN) APPROXIMATIONS. Let S_h be a finite dimensional subspace of $\mathring{W}_2^1(\Omega)$ which contains the interpolants of interest. The finite element approximation U to u from S_h is determined by $U = \sum a_j U_j$, $\{U_j\}$ a basis for S_h and $\int_\Omega \nabla U \cdot \nabla V\, dx = -\int_\Omega fV\, dx$ $\forall V \in S_h$.

THE ERROR BOUNDS OF BARNHILL AND GREGORY [**2, 3, 4, 5**].

$$(*) \qquad |u - U|_1 \leqq |u - \tilde{u}|_1 \leqq Kh|u|_2,$$

where $\tilde{u}$ is an interpolant in S_h. Barnhill, Gregory, and Whiteman have given computable expressions for K for various families S_h.

THE COERCIVENESS THEOREM. *If $\partial\Omega$ is smooth, then $\exists M > 0$ such that*

$$(**) \qquad |u|_2 \leqq M\,|\Delta u|_0 = M|f|_0.$$

More generally, if $\partial\Omega$ is piecewise smooth with a finite number of corners $x^{(j)}$ having interior angles α_j, $0 < \alpha_j < \pi$, then the same estimate holds [**8**].

ERROR BOUNDS. *Combining* (*), (**) *gives* $|u - U|_1 \leqq KMh\,|f|_0$.

PROBLEM. Determine computable estimates for M. Combining this with the Barnhill-Gregory results will give computable error bounds for the approximate solution U in terms of the prescribed data f. Convex polygons are the domain of definition Ω for most practical problems. The boundary of a polygon is not smooth, because of the corners, and so the Miranda—Talenti Theorem [11, 12, 14] does not apply directly. It is the purpose of this paper to extend the Miranda—Talenti Theorem to the case of convex polygons.

Natterer [13] has recently developed computable error bounds for piecewise linear finite elements over triangles. He assumed that the Miranda—Talenti Theorem is true for convex polygons. However, as noted above, this proof is the objective of our paper. Natterer's analysis applies only to piecewise linear finite elements. (Hence his analysis cannot be applied to the bilinear example in §5.) For the linear case, Natterer obtains the constant 0.81, whereas Barnhill and Gregory obtain 1.207 for this case. Hence Natterer's error bounds are about 2/3 of Barnhill and Gregory's, for the one case to which Natterer's apply.

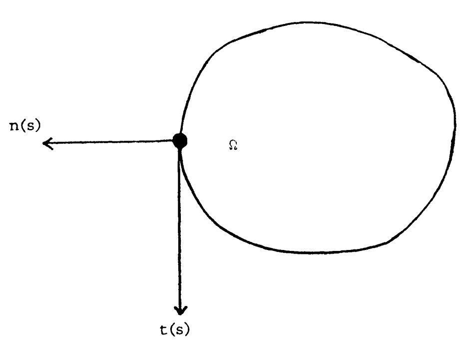
OUTLINE OF THE REMAINDER OF THE PAPER.

2. The Miranda-Talenti Theorem and its applications.
3. Extension of the Miranda-Talenti Theorem to domains with corners.
4. Computable error bounds for domains with corners.
5. Numerical results.

2. The Miranda-Talenti Theorem and its applications. The following result is due to C. Miranda and G. Talenti [11, 12, 14].

THEOREM 5. *If $\partial\Omega$ is smooth, then $\forall f \in L_2(\Omega)$ the weak solution of the Dirichlet problem satisfies $|u|_2^2 = |f|_0^2 - \int_{\partial\Omega} k(\partial u/\partial n)^2\, ds$ where k is the curvature of $\partial\Omega$.*

REMARK. k, the "signed curvature" of $\partial\Omega$, is positive for convex domains Ω. It may be defined as follows. Let $x = x(s) = (x_1(s), x_2(s))$, $0 \leqq s \leqq 1$, be a parametric representation of $\partial\Omega$ in terms of arc-length s. Then $t(s) = dx(s)/ds = (dx_1(s)/ds, dx_2(s)/ds)$ is the unit tangent vector to $\partial\Omega$. Choose the positive sense of s so that Ω always "lies to the left of $t(s)$".



Then $n(s) = (dx_2(s)/ds, - dx_1(s)/ds)$ is the exterior unit normal vector to $\partial\Omega$. Moreover, $\exists$ a unique scalar function $k(s)$ such that $dt(s)/ds = - k(s)n(s)$. It is this $k(s)$ that appears in the Miranda-Talenti Theorem.

APPLICATION TO CONVEX DOMAINS. It is known that a domain Ω with smooth $\partial\Omega$ is convex if and oniy if $k(s) \geqq 0$. Thus Ω convex implies $|u|_2 \leqq |f|_0$; that is, $M \leqq 1$.

REMARK. Actually, $M = 1$ in this case since Theorem 5 holds for every $u \in C_0^\infty(\Omega)$ and these functions satisfy $\partial u/\partial n = 0$ on $\partial\Omega$.

APPLICATION TO NON-CONVEX DOMAINS. In this case $k(s)$ can change sign and the estimates are more complicated. Suppose that $k(s) \leqq 0$ on $\Gamma \subset \partial\Omega$ and $k(s) \geqq -\mu$ on Γ where $\mu > 0$. The Miranda Talenti Theorem implies that

$$\frac{|u|_2^2}{|\Delta u|_0^2} = 1 - \frac{\int_{\partial\Omega} k\left(\frac{\partial u}{\partial n}\right)^2 ds}{|\Delta u|_0^2} \leqq 1 - \frac{\int_\Gamma k\left(\frac{\partial u}{\partial n}\right)^2 ds}{|\Delta u|_0^2}$$

$$\leqq 1 + \mu \frac{\int_\Gamma \left(\frac{\partial u}{\partial n}\right)^2 ds}{|\Delta u|_0^2}.$$

Consider the functional

$$J(u) = \frac{\int_\Gamma \left(\frac{\partial u}{\partial n}\right)^2 ds}{|\Delta u|_0^2}, \quad u \in \mathring{W}_2^1(\Omega) \cap W_2^2(\Omega).$$

Standard techniques of the calculus of variations show that J is stationary at the solutions of the following eigenvalue problem (eigenvalue λ):

$$\Delta^2 u = 0 \text{ in } \Omega, \ \chi_\Gamma \frac{\partial u}{\partial n} = \lambda \Delta u \text{ and } u = 0 \text{ on } \partial\Omega \tag{2.1}$$

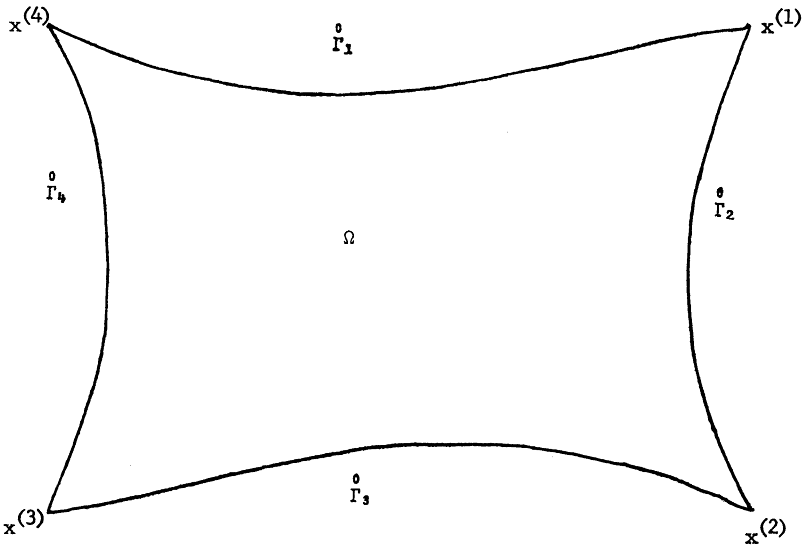
where χ_Γ is the characteristic function of Γ. Moreover if u_j is a solution of this problem with eigenvalue λ_j, then $J(u_j) = \lambda_j$. Thus if $\lambda_{\max}$ is the largest eigenvalue, then $J(u) \leqq \lambda_{\max}$, $\forall u \in \mathring{W}_2^1(\Omega) \cap W_2^2(\Omega)$, and hence $|u|_2^2/|\Delta u|_0^2 \leqq 1 + \mu\lambda_{\max}$, i.e., $M^2 \leqq 1 + \mu\lambda_{\max}$.

3. Extension of the Miranda-Talenti Theorem to domains with corners. For many problems of practical interest $\partial\Omega$ is not smooth but is piecewise smooth with a finite number of corners. If the interior angles at the corners are all less that π, then the Sobolev 2-norm of solutions of the Dirichlet problem is still finite and the coerciveness Theorem holds [**8**]. However, the Miranda-Talenti Theorem is known only for domains with smooth boundaries. The purpose of this section is to extend the Miranda-Talenti Theorem to a class of domains with corners.

DOMAINS WITH CORNERS. A bounded domain $\Omega \subset \mathbf{R}^2$ will be said to be a $C^{n+\delta}$ -domain with corners if and only if

1) $\partial\Omega = \bigcup_{j=1}^m \Gamma_j$, m a positive integer;
2) $\Gamma_j = \{(x_1, x_2) = (\phi_1^{(j)}(\tau), \phi_2^{(j)}(\tau)): 0 \leqq \tau \leqq 1\}, j = 1, 2, \ldots, m$;
3) $\phi_k^{(j)} \in C^{n+\delta}([0, 1]), k = 1, 2,\ j = 1, 2, \ldots, m, \delta > 0\ (n = 0, 1, 2, \ldots)$;
4) $\phi_1^{(j)\prime}(\tau)^2 + \phi_2^{(j)\prime}(\tau)^2 > 0$ for $0 \leqq \tau \leqq 1$ and $j = 1, 2, \ldots, m$;
5) the arcs Γ_j intersect only at their end-points and the set of boundary points so determined forms a set $(x^{(1)}, x^{(2)}, \ldots, x^{(m)})$ where $x^{(j)} = (x_1^{(j)}, x_2^{(j)})$; and
6) The internal angles α_j at the points $x^{(j)}$ satisfy $0 < \alpha_j < \pi$, $j = 1, 2, \ldots, m$.

REMARK. If $\alpha_j \geqq \pi$, then $u \notin W_2^2(\Omega)$, in general, and the M-T Theorem does not hold. The points $x^{(j)}$ will be called the *corners* of Ω. The notation $\mathring{\Gamma}_j = \{(x_1, x_2) = (\phi_1^{(j)}(\tau), \phi_2^{(j)}(\tau)): 0 < \tau < 1\}$ will be used to denote the open boundary arcs. The closure of Ω, denoted by $\bar{\Omega}$, is the disjoint union of the sets $\Omega, \mathring{\Gamma}_1, \ldots, \mathring{\Gamma}_m$ and $\{x^{(1)}, \ldots, x^{(m)}\}$.



PROOF OF THE MIRANDA-TALENTI THEOREM FOR $C^{n+\delta}$-DOMAINS WITH CORNERS, $n \geqq 4$. The Miranda-Talenti Theorem relates the quantities

$$|u|_2^2 = \int_\Omega \left\{\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2\right\} dx_1 dx_2$$

and

$$|f|_0^2 = |\Delta u|_0^2 = \int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right)^2 dx_1 dx_2$$
$$= \int_\Omega \left\{\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + 2\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2\right\} dx_1 dx_2.$$

Hence the theorem is equivalent to the equation

$$(*) \quad \int_\Omega \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 dx_1 dx_2 = \int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2}\right)\left(\frac{\partial^2 u}{\partial x_2^2}\right) dx_1 dx_2 - \frac{1}{2}\int_{\partial\Omega} k\left(\frac{\partial u}{\partial n}\right)^2 ds.$$

Moreover, the two integrals in $(*)$ can be transformed into one another by two integrations by parts, using Gauss's Theorem

$$\int_\Omega \frac{\partial v}{\partial x_j} dx_1 dx_2 = \int_{\partial\Omega} n_j v ds \quad (j = 1, 2).$$

This will be carried out and the boundary terms will be shown to yield precisely the last term in $(*)$. Gauss's Theorem cannot be applied directly to derivatives of u in Ω because of possible singularities at the corners. To avoid these the integration will be carried out in the subdomain $\Omega_\varepsilon = \Omega - \bigcup_{j=1}^m N_j(\varepsilon)$, $N_j(\varepsilon) = \{x : |x - x^{(j)}| \leqq \varepsilon\}$.

If Ω is a $C^{3+\delta}$-domain with corners and $f \in C^{1+\delta}(\bar{\Omega})$, then the Schauder estimates imply that $u \in C^{3+\delta}(\bar{\Omega} - \{x^{(1)}, \ldots, x^{(m)}\})$. Hence Green's formula can be applied in Ω_ε $(\varepsilon > 0)$ with v replaced by any derivative of u of order $\leqq 3$. Thus the following relations are valid:

$$\int_{\Omega_\varepsilon} \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 dx_1 dx_2 = \int_{\Omega_\varepsilon} \left(\frac{\partial u}{\partial x_1}\left\{\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial u}{\partial x_2}\right\} - \left(\frac{\partial^3 u}{\partial x_1^2 \partial x_2}\frac{\partial u}{\partial x_2}\right)\right) dx_1 dx_2$$
$$= -\int_{\Omega_\varepsilon} \left(\frac{\partial^3 u}{\partial x_2 \partial x_1^2}\right)\frac{\partial u}{\partial x_2} dx_1 dx_2 + \int_{\partial\Omega_\varepsilon} n_1 \frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial u}{\partial x_2} ds$$
$$= \int_{\Omega_\varepsilon} \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} dx_1 dx_2$$
$$+ \int_{\partial\Omega_\varepsilon} \left\{n_1 \frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial u}{\partial x_2} - n_2 \frac{\partial^2 u}{\partial x_1^2}\frac{\partial u}{\partial x_2}\right\} ds.$$

Now $(-n_2, n_1) = (x_1'(s), x_2'(s)) = t$ and hence

$$-n_2 \frac{\partial^2 u}{\partial x_1^2} + n_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} = x_1'(s)\frac{\partial^2 u}{\partial x_1^2} + x_2'(s)\frac{\partial^2 u}{\partial x_2 \partial x_1} = \frac{d}{ds}\frac{\partial u}{\partial x_1}.$$

Thus

$$\int_{\Omega_\varepsilon} \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 dx_1 dx_2 = \int_{\Omega_\varepsilon} \left(\frac{\partial^2 u}{\partial x_1^2}\right)\left(\frac{\partial^2 u}{\partial x_2^2}\right) dx_1 dx_2 + \int_{\partial\Omega_\varepsilon} \frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right) ds.$$

Consider the integral of $(\partial u/\partial x_2)(d(\partial u/\partial x_1)/ds$ over $\Gamma_j^\varepsilon = \Gamma_j - \bigcup_{k=1}^m N_j(\varepsilon)$. On Γ_j^ε the boundary condition $u = 0$ implies that $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2) = \lambda(n_1, n_2) = \lambda n$ where $\lambda = \nabla u \cdot n = \partial u/\partial n$. Thus on Γ_j^ε

$$\frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right) = \lambda n_2 \frac{d}{ds}(\lambda n_1) = \lambda n_2\left(\frac{d\lambda}{ds}n_1 + \lambda\frac{dn_1}{ds}\right).$$

Combining this with Frenet's formula $dn/ds = kt$ gives

$$\begin{aligned}\frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right) &= \lambda n_2\left(\frac{d\lambda}{ds}n_1 + \lambda k t_1\right) = \lambda n_2\left(\frac{d\lambda}{ds}n_1 - \lambda k n_2\right)\\ &= \lambda\frac{d\lambda}{ds}n_1n_2 - \lambda^2 k n_2^2 = \frac{1}{2}\frac{d\lambda^2}{ds}n_1n_2 - \lambda^2 k n^2.\end{aligned}$$

Thus

$$\begin{aligned}&\int_{\Gamma_j^\varepsilon}\frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right)ds\\ &\quad = \int_{\Gamma_j^\varepsilon}\left(\frac{1}{2}\frac{d\lambda^2}{ds}n_1n_2 - \lambda^2 k n_2^2\right)ds\\ &\quad = \int_{\Gamma_j^\varepsilon}\left(-\frac{1}{2}\lambda^2\frac{d(n_1n_2)}{ds} - \lambda^2 k n_2^2\right)ds + \frac{1}{2}\lambda^2 n_1n_2\Big]_{x^{j-}(\varepsilon)}^{x^{j+}(\varepsilon)}\end{aligned}$$

where $x^{j-}(\varepsilon)$ and $x^{j+}(\varepsilon)$ are initial and final end points of the oriented arc Γ_j^ε.

Now $d(n_1n_2)/ds = n_1(dn_2/ds) + n_2(dn_1/ds) = k(n_1t_2 + n_2t_1) = k(n_1^2 - n_2^2)$ by Frenet's formulas. Thus

$$\begin{aligned}&\int_{\Gamma_j^\varepsilon}\frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right)ds\\ &\quad = \int_{\Gamma_j^\varepsilon}\left(\frac{1}{2}\lambda^2 k(n_2^2 - n_1^2) - \lambda^2 k n_2^2\right)ds + \frac{1}{2}\lambda^2 n_1n_2\Big]_{x^{j-}(\varepsilon)}^{x^{j+}(\varepsilon)}\\ &\quad = -\frac{1}{2}\int_{\Gamma_j^\varepsilon}\lambda^2 k(n_1^2 + n_2^2)ds + \frac{1}{2}\lambda^2 n_1n_2\Big]_{x^{j-}(\varepsilon)}^{x^{j+}(\varepsilon)}\\ &\quad = -\frac{1}{2}\int_{\Gamma_j^\varepsilon}k\left(\frac{\partial u}{\partial n}\right)^2 ds + \frac{1}{2}n_1n_2\left(\frac{\partial u}{\partial n}\right)^2\Big]_{x^{j-}(\varepsilon)}^{x^{j+}(\varepsilon)}\end{aligned}$$

because $\lambda^2 = |\nabla u|^2 = (\partial u/\partial n)^2$. Combining these results gives

$$\begin{aligned}&\int_{\Omega_\varepsilon}\left(\frac{\partial^2 u}{\partial x_1\partial x_2}\right)^2 dx_1dx_2 - \int_{\Omega_\varepsilon}\left(\frac{\partial^2 u}{\partial x_1^2}\right)\left(\frac{\partial^2 u}{\partial x_2^2}\right)dx_1dx_2\\ &\qquad + \frac{1}{2}\sum_{j=1}^m\int_{\Gamma_j^\varepsilon}k\left(\frac{\partial u}{\partial n}\right)^2 ds \qquad (3.1)\\ &= \frac{1}{2}\sum_{j=1}^m n_1n_2\left(\frac{\partial u}{\partial n}\right)^2\Big]_{x^{j-}(\varepsilon)}^{x^{j+}(\varepsilon)} + \sum_{j=1}^m\int_{\partial\Omega_\varepsilon\cap N_j(\varepsilon)}\frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right)ds.\end{aligned}$$

To complete the proof of the Miranda-Talenti formula for $C^{n+\delta}$-domains with corners it is enough to make $\varepsilon \to 0$ and show that the terms on the right-hand side of (3.1) have the limit zero. For this it is necessary to examine the next topic.

BEHAVIOR AT CORNERS OF SOLUTIONS OF THE DIRICHLET PROBLEM. The results of N. M. Wigley [15] will be used. To state them let $z = x_1 + ix_2$, $z^{(j)} = x_1^{(j)} + ix_2^{(j)}$. The following is a special case of [15, Th. 5].

THEOREM 6. *Let Ω be a $C^{4+\delta}$-domain with corners and let $f \in C^{2+\delta}(\bar{\Omega})$ (so that $u \in C^{4+\delta}(\bar{\Omega} - \{x^{(1)}, \ldots, x^{(m)}\})$ by the Schauder Theorems). Two cases are distinguished.*

CASE 1. *α_j/π irrational. Then to each $\mu > 0$ there corresponds a polynomial $P_j = P_j^\mu$ in $z - z^{(j)}$, $\overline{z - z^{(j)}}$, $(z - z^{(j)})^{\pi/\alpha_j}$ and $\overline{(z - z^{(j)})^{\pi/\alpha_j}}$ such that for all $x = (x_1, x_2) \in \Omega \cup \bigcup_{j=1}^m \mathring{\Gamma}_j$*

$$(+) \qquad u(x) = P_j + o(|x - x^{(j)}|^{3-\mu}),\ x \to x^{(j)}.$$

CASE 2. *$\alpha_j/\pi = p/q$ rational, $(p, q) = 1$. Then to each $\mu > 0$ there corresponds a polynomial $P_j = P_j^\mu$* in *$z - z^{(j)}$, $\overline{z - z^{(j)}}$, $(z - z^{(j)})^{\pi/\alpha_j}$, $\overline{(z - z^{(j)})^{\pi/\alpha_j}}$, $(z - z^{(j)})^q\log(z - z^{(j)})$, $\overline{(z - z^{(j)})^q\log(z - z^{(j)})}$ such that $(+)$ holds.*

Moreover, expansions for the derivatives of $u(x)$ of order $\leqq 2$ may be obtained from $(+)$ by formal differentiation, i.e., if

$$D^\alpha = \frac{\partial_1^\alpha}{\partial x_1^2} \frac{\partial_2^\alpha}{\partial x_2^{\alpha_2}},$$

then

$$D^\alpha u(x) = D^\alpha P_j + o(|x - x^{(j)}|^{3-|\alpha|-\mu}),\ x \to x^{(j)}$$

for $0 \leqq |\alpha| = \alpha_1 + \alpha_2 \leqq 2$.

REMARK. In [15] Theorem 6 is proved under the additional hypothesis that $u(x) = o(|x - x^{(j)}|^p)$, $p > \mathrm{Max}(-1, -\pi/\alpha_j)$. However, here $u \in C(\bar{\Omega})$ and hence this hypothesis can be omitted. To prove this recall that we know that $u \in W_2^2(\Omega)$. Moreover, Sobolev's embedding theorem holds for $C^{4+\delta}$-domains with corners. It follows that $u \in C(\bar{\Omega})$ (and, in fact, that $u = 0$ on $\partial\Omega$).

COROLLARY 7. *Under the hypotheses of Theorem 6 $\exists$ a number $\beta > 1$ such that $u(x) = O(|x - x^{(j)}|^\beta)$, $\partial u(x)/\partial x_k = O(|x - x^{(j)}|^{\beta-1})$ and $\partial^2 u/\partial x_k \partial x_\ell = O(|x - x^{(j)}|^{\beta-2})$ for $j = 1, 2, \ldots, m$, $x \to x^{(j)}$.*

PROOF. It is enough to show the existence of an exponent $\beta_j > 1$ for each corner. Then $\beta = \min(\beta_1, \beta_2, \ldots, \beta_m)$. Introduce polar coordinates $x_1 = x_1^{(j)} + r \cos\theta$ and $x_2 = x_2^{(j)} + r \sin\theta$. Then by Theorem 6, in the worst case (Case 2),

$$u(x) = P_j\Big(r\cos\theta,\ r\sin\theta,\ r^{\pi/\alpha}\cos\frac{\pi\theta}{\alpha},\ r^{\pi/\alpha}\sin\frac{\pi\theta}{\alpha},\quad (1)$$
$$r^q e^{iq\theta}(\ln r + i\theta),\ r^q e^{-iq\theta}(\ln r - i\theta)\Big) + o(r^{3-\mu})$$

where P_j is a polynomial and $\alpha = \alpha_j$. Note that $q \geqq 2$ because $0 < \alpha < \pi$ and $\alpha/\pi = p/q$. It will be assumed that $0 < \mu < 1$. (1) may be written

$$\begin{aligned} u(x) &= c_0 + c_1 r\cos\theta + c_2 r\sin\theta + c_3 r^{\pi/\alpha}\cos\frac{\pi\theta}{\alpha} + c_4 r^{\pi/\alpha}\sin\frac{\pi\theta}{\alpha} \\ &\quad + r^q\{c_5(\ln r)\cos q\theta + c_6(\ln r)\sin q\theta + c_7\theta\cos q\theta + c_8\theta\sin q\theta\} \\ &\quad + \text{terms with products of 2 or more factors} + o(r^{3-\mu}). \end{aligned} \quad (2)$$

Note that the product of two or more of the factors $r\cos\theta$, $r\sin\theta$, $r^{\pi/\alpha}\cos(\pi\theta/\alpha)$, $r^{\pi/\alpha}\sin(\pi\theta/\alpha)$, $r^q\ln r\cos q\theta$, $r^q\ln r\sin q\theta$, $r^q\theta\cos q\theta$, $r^q\theta\sin q\theta$ will also be $O(r^{2-\nu})$ for any $\nu > 0$, because $q \geqq 2$ and $\pi/\alpha > 1$. Thus

$$\begin{aligned} u(x) &= c_0 + c_1(x_1 - x_1^{(j)}) + c_2(x_2 - x_2^{(j)}) + O(r^{\pi/\alpha}) \\ &\quad + O(r^{2-\nu}) + o(r^{3-\mu}). \end{aligned} \quad (3)$$

It follows that $u(x)$ has a unique limit $c_0(= c_0^j)$ when $x \to x^{(j)}$ through $\Omega \cup \bigcup_{j=1}^m \mathring{\Gamma}_j$. But we know that $u \in C(\Omega \cup \bigcup_{j=1}^m \mathring{\Gamma}_j)$ and $u = 0$ on $\mathring{\Gamma}_j$. Thus $c_0 = 0$ and $u \in C(\bar{\Omega})$. Now differentiate (2), using

$$\frac{\partial}{\partial x_1} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial x_2} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

These operators, applied to the terms in (2), yield finite sums of similar terms with the powers of r reduced by 1. Thus

$$\frac{\partial u}{\partial x_k} = c_k + O(r^{\pi/\alpha - 1}) + O(r^{1-\nu}) + o(r^{2-\mu}). \quad (4)$$

It follows that $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2)$ has a unique limit (c_1, c_2) $(=(c_1^j, c_2^j))$ when $x \to x^{(j)}$ through $\Omega \cup \bigcup_{j=1}^m \mathring{\Gamma}_j$. But we know that $u \in C^1(\Omega \cup \bigcup_{j=1}^m \mathring{\Gamma}_j)$. Thus $u \in C^1(\bar{\Omega})$. Moreover, $u = 0$ on $\mathring{\Gamma}_j$ implies $\nabla u = |\nabla u| n$ on $\mathring{\Gamma}_j$. Thus at $x^{(j)}$ $(c_1, c_2) = |\nabla u(x^{(j)})| n_+ = |\nabla u(x^{(j)})| n_-$ where n_+ and n_- are the unit normal vectors at $x^{(j)}$ to the two boundary arcs that meet at $x^{(j)}$. Since by assumption $n_+ \neq n_-$ $(0 < \alpha_j < \pi)$, it follows that $|\nabla u(x^{(j)})| = 0$, i.e., $c_1 = c_2 = 0$. Thus (4) becomes

$$\frac{\partial u}{\partial x_k} = O(r^{\pi/\alpha - 1}) + O(r^{1-\nu}) + o(r^{2-\mu}). \quad (5)$$

Finally, differentiating (2) twice gives a finite sum of terms in which each power of r is reduced by 2. Thus

$$\frac{\partial^2 u}{\partial x_k \partial x_\ell} = O(r^{\pi/\alpha - 2}) + O(r^{-\nu}) + o(r^{1-\mu}). \tag{6}$$

To complete the proof of Corollary 7 we need only take $\beta = \beta_j = \mathrm{Min}(\pi/\alpha, 2 - \nu)$.

THEOREM 8. (MIRANDA-TALENTI THEOREM FOR DOMAINS WITH CORNERS). *Assume that Ω is a $C^{4+\delta}$-domain with corners, and $f \in C^{2+\delta}(\bar{\Omega})$. Then the solution of the Dirichlet problem satisfies*

$$|u|_2^2 = |f|_0^2 - \frac{1}{2}\int_{\partial\Omega} k\left(\frac{\partial u}{\partial n}\right)^2 ds.$$

PROOF. Returning to equation (3.1) it must be shown that when $\varepsilon \to 0$ the terms on the right have the limit 0. By Corollary 7, $\partial u/\partial n = \nabla u \cdot n = O(|x - x^{(j)}|^{\beta-1})$, $x \to x^{(j)}$, where $\beta - 1 > 0$. Since $x^{j\pm}(\varepsilon) \to x^{(j)}$ when $\varepsilon \to 0$ it follows that

$$\lim_{\varepsilon \to 0} \sum n_1 n_2 \left(\frac{\partial u}{\partial n}\right)^2 \Big]_{x^{j-}(\varepsilon)}^{x^{j+}(\varepsilon)} = 0.$$

Moreover, $\partial\Omega_\varepsilon \cap N_j(\varepsilon)$ is an arc of a circle of radius ε. Thus, using polar coordinates (r, θ) with origin at $x^{(j)}$, $ds = \varepsilon\, d\theta$ and

$$\frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right) = \frac{\partial u}{\partial x_2}\left\{t_1 \frac{\partial^2 u}{\partial x_1^2} + t_2 \frac{\partial^2 u}{\partial x_1 \partial x_2}\right\}$$

where $t = (t_1, t_2)$ is the unit tangent vector to the arc. Hence, by Corollary 7

$$\frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right) = O(r^{\beta-1})\, O(r^{\beta-2}) = O(r^{2\beta-3}),\ r \to 0,$$

and therefore

$$\int_{\Omega_\varepsilon \cap N_j(\varepsilon)} \frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right) ds = \int_{\theta_1(\varepsilon)}^{\theta_2(\varepsilon)} O(\varepsilon^{2\beta-3})\varepsilon\, d\theta = O(\varepsilon^{2\beta-2})$$

since $0 \leqq \theta_1(\varepsilon), \theta_2(\varepsilon) \leqq 2\pi$. Since $\beta > 1$ it follows that

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon \cap N_j(\varepsilon)} \frac{\partial u}{\partial x_2}\frac{d}{ds}\left(\frac{\partial u}{\partial x_1}\right) ds = 0.$$

4. Computable error bounds for domains with corners.

THEOREM 9. *If the hypotheses of Theorem 8 hold and the arcs Γ_j are all convex ($k \geqq 0$) then $M = 1$, i.e., $|u|_2 \leqq |\Delta u|_0$.*

COROLLARY 10. *$M = 1$ for all convex polygons.*

THEOREM 11. *Under the hypotheses of Theorem 8, $M \leqq (1 - (k_{\min})\lambda_{\max})^{1/2}$ where $k_{\min} \leqq 0$ is the minimum of k on $\partial\Omega$ and $\lambda_{\max}$ is the largest eigenvalue of problem* (2.1).

5. Numerical results. Numerical experiments to test the sharpness of these bounds appear in [**6**, **7**]. We illustrate these numerical results with an example. Let $R = [0, 1] \times [0, 1]$ and $-\Delta u = f$ in R, $u = 0$ on ∂R. Let $f(x, y) = 2[x(1 - x) + y(1 - y)]$. Consider piecewise bilinear finite elements and let $h = 1/2$. Then the finite element approximation U is $U(x, y) = (5/16)xy$ on $[0, 1/2] \times [0, 1/2]$ and symmetrically defined on the rest of R. The error bound of Barnhill and Gregory for this problem and approximation is

$$(+) \qquad |u - U|_1 \leqq KMh\,|f|_0 = 0.7906\,h|f|_0$$

since $K = 0.7906$ [7] and $M = 1$ (Corollary 10). $|f|_0 = 0.6992$, so (+) becomes $|u - U|_1 \leqq 0.2764$. The solution of this Dirichlet problem is $u(x, y) = x(1 - x)y(1 - y)$, so the actual error can be computed. It is $|u - U|_1 = 0.07711$. Thus the error bound is 3.58 times the actual error, in this example. For completeness, we remark that Nitsche's trick enables one to change the error bound $|u - U|_1 \leqq c|f|_0$ to $|u - U|_0 \leqq c^2\,|f|_0$. That is, the L_2 error can be estimated from the energy norm error. Of course, this magnifies the conservativeness of the error bound. For this example, Nitsche's trick applied to (+) yields $|u - U|_0 \leqq 0.6250\,h^2\,|f|_0 = 0.10925$. The actual L_2 error is $|u - U|_0 = 0.009709$. Hence the L_2 error bound is 11.25 times the actual L_2 error in this example.

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