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A NON-CATENARY, NORMAL, LOCAL DOMAIN

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Ever since Nagata constructed his celebrated example of a non-catenary Noetherian domain, it has been an open question whether or not such an example could be integrally closed. If the "chain conjecture" were valid, it could not be. However, in [3], T. Ogoma showed such an example existed. Precisely, he constructed a Noetherian domain R and showed that the integral closure of R was non-catenary and a finite R-module (thus Noetherian).

The present article is an alternate presentation of Ogoma's example. It is intended to serve two purposes. First, the construction itself has been simplified. It is shorter, requires less machinery, and should be more accessible than the original. Secondly, some new properties of R are observed. Most significantly, R is in fact integrally closed already (Theorem 4). This also simplifies matters.

It should be noted that [3] contains other examples, numerous positive results of interest, and a great deal of creativity which are omitted here. We begin the construction.

Let F be a countable field and $\{a_i, b_i, c_i \mid i \in \mathbf{Z}^+\}$ be indeterminates. Set $K_m = F(\{a_i, b_i, c_i \mid i \leqq m\})$ and $K = \bigcup K_m$. Let x, y, z, w be additional indeterminates.

Select a set $\mathscr{P}$ ($\subset K[x, y, z, w]$) of prime elements, exactly one for each height one prime of $S = K[x, y, z, w]_{(x,y,z,w)}$, such that $w \in \mathscr{P}$ and $\mathscr{P}$ contains infinitely many elements from $F[x, y, z, w]$.

Noting $\mathscr{P}$ is countable, these assumptions allow a numbering $\mathscr{P} = \{p_i \mid i \in \mathbf{Z}^+\}$ with $p_1 = w$ and $p_i \in K_{i-2}[x, y, z, w]$ for every $i \geqq 2$. Set $q_n = \prod_{k=1}^n p_k$, $f_n = x + \sum_{k=1}^n a_k q_k^k$, $g_n = y + \sum_{k=1}^n b_k q_k^k$, $h_n = z + \sum_{k=1}^n c_k q_k^k$, and $P_n = (f_n, g_n, h_n) S$ for $n \geqq 0$. Observe that for each $n > 0$, we have (modulo $((x, y, z, w)S)^2$) $f_n \equiv x + a_1 w$, $g_n \equiv y + b_1 w$, $h_n \equiv z + c_1 w$, and so f_n, g_n, h_n, w is a regular system of parameters. Therefore P_n is a height three prime ideal.

PROPOSITION 1. *$p_n \notin P_i$ where $i \geqq n - 1$.*

PROOF. Use induction on i. Since $w \notin (x, y, z)$, the proposition holds for $i = 0$ Assume in holds for $i - 1$. Thus $p_m \notin p_{i-1}$ for $m \leqq i$ and

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$q_i \notin p_{i-1}$. Let $B_i = K_{i-1}[x, y, z, w, a_i, b_i, c_i]_{(x,y,z,w,) = K_i[x,y,z,w](x,y,z,w)}$. For $2 \leqq k \leqq i$, $p_k \in K_{k-2}[x, y, z, w] \subseteqq K_{i-1}[x, y, z, w]$, and of course $p_1 = w \in K_{i-1}[x, y, z, w]$. So $q_k = \prod_{m=1}^{k} p_m \in K_{i-1}[x, y, z, w]$ and

$$f_i = x + \sum_{k=1}^{i} a_k q_k^k = (x + \sum_{k=1}^{i-1} a_k q_k^k) + a_i q_i^i$$

is a linear polynomial in a_i with coefficients in $K_{i-1}[x, y, z, w]$. Similarly g_i is linear in b_i and h_i is linear in c_i. Consequently, we may define a homomorphism $\phi: B_i \to K_{i-1}(x, y, z, w)$ by $\phi_{K_{i-1}[x,y,z,w]} =$ identity and $\phi(f_i) = \phi(g_i) = \phi(h_i) = 0$. [This uniquely defines $\phi(a_i)$, $\phi(b_i)$, $\phi(c_i)$.] To show ϕ is well-defined on the localization B_i, we must show (Kernel ϕ) $\subseteqq (x, y, z, w)$. Since $q_i \notin p_{i-1}$; f_{i-1}, g_{i-1}, h_{i-1}, q_i^i is a system of parameters for $(x, y, z, w)R$ and so is analytically independent [2, p. 68]. The inclusion follows immediately. Since (Kernel ϕ) $\cap K_{i-1}[x, y, z, w] = (0)$, we obtain $(f_i, g_i, h_i)B_i \cap K_{i-1}[x, y, z, w] = (0)$. Also $B_i \subseteqq S$ is a faithfully flat extension. Hence $(f_i, g_i, h_i) S \cap B_i = (f_i, g_i, h_i) B_i$ and $(f_i, g_i, h_i) S \cap K_{i-1}[x, y, z, w] = (0)$. Finally, for $n \leqq i + 1$, $p_n \in K_{i-1}[x, y, z, w]$ demonstrates $p_n \notin P_i$.

Next set $\omega_n = f_n g_n / q_n^n$ and $\mu_n = f_n h_n / q_n^n$. Recalling $f_{n+1} = f_n + a_{n+1} q_{n+1}^{n+1}$, etc., and $q_{n+1} = q_n p_{n+1}$, a simple calculation gives $\omega_n \in q_{n+1} S[\omega_{n+1}]$ and $\mu_n \in q_{n+1} S[\mu_{n+1}]$.

Set $R = \bigcup S[\omega_n, \mu_n]$, a direct union.

Proposition 2. *There is a one-to-one correspondence between the nonzero primes of R and prime ideals of S which contain either (p_n, f_n) or (p_n, g_n, h_n) for some n. The correspondence is given by contraction and extension. Thus, R is a (Noetherian) local domain.*

Proof. As R and S have the same quotient field, for every nonzero prime Q of R, $Q \cap S \neq (0)$ and so $p_n \in Q$ for some n. Since $f_n g_n / q_n^n, f_n h_n / q_n^n \in R$ and $p_n | q_n$, either $(p_n, f_n) \subseteqq Q \cap S$ or $(p_n, g_n, h_n) \subseteqq Q \cap S$. Further $p_n | \omega_m, \mu_m$ for large m; thus $\omega_m, \mu_m \in Q$ and $R/Q \cong S/(Q \cap S)$. Further $Q = (Q \cap S)R$ and the contraction map is necessarily one-to-one.

Conversely, suppose P is a prime of S minimal over $(p_n, g_n, h_n)S$. P is the contraction of a prime of R if and only if $PR \cap S = P$, which holds if and only if $PR_{S-P} \cap S_P = PS_P$. Now, for every $m \geqq n$, $p_n | g_m - g_n, h_m - h_n$ and so $g_m, h_m \in P$. As $p_n \notin (f_m, g_m, h_m)S$ by Proposition 1, P cannot contain f_m (height $P = 3$). Thus $f_m^{-1} \in S_P$ and $R_{S-P} = \bigcup S_P[g_m/q_m^m, h_m/q_m^m]$. If $PR_{S-P} \cap S_P \neq PS_P$, then $1 \in PR_{S-P}$ and so $1 \in PS_P[g_m/q_m^m, h_m/q_m^m]$ for some m. However, as $p_k \notin (g_m, h_m)S$ for $k \leqq m$, $q_m \notin (g_m, h_m)S$ and so q_m^m, g_m, h_m is a system of parameters for PS_P. Thus q_m^m, g_m, h_m are analytically independent [2, p. 68] and $PS_P[g_m/q_m^m, h_m/q_m^m] \cap S_P = PS_P$. The proof in the case P minimal over (p_n, f_n) is essentially the same. Finally $S/P \cong R/PR$ yields the going up property for nonzero primes of S and so all primes containing $(p_n, f_n)S$ or $(p_n, g_n, h_n)S$ extend to primes of R.

Finally, as primes of R are extensions of primes of S, they are all finitely generated and so, by Cohen's Theorem, R is Noetherian. The going up property for nonzero primes assures us that $(x, y, z, w)R$ is the unique maximal of R.

COROLLARY 3. *R is not catenary.*

PROOF. We claim that the only prime of S which contains both (p_n, f_n) and (p_k, g_k, h_k) is the maximal $M = (x, y, z, w)$. For, if (p_n, f_n) and $(p_k, g_k, h_k) \subseteqq P$, we obtain as before $f_m, g_m, h_m \in P$ for any $m > n, k$. Then $p_n \notin (f_m, g_m, h_m)S$ forces $P = M$.

Next, if P is minimal over (p_n, f_n), ht. $PR = 1$ and ht.$(MR/PR) =$ ht.$(M/P) = 2$. Also, if P is minimal over (p_n, g_n, h_n), ht. $PR = 1$ and ht.$(MR/PR) =$ ht.$(M/P) = 1$.

THEOREM 4. *R is integrally closed.*

PROOF. To prove this, it suffices [1, p. 125] to verify two conditions:
$(R1)$ R_Q is a regular local ring when ht. $Q = 1$.
$(S2)$ depth $R_Q \geqq$ min.(2, ht. Q) for every prime Q.
We do this with two lemmas.

LEMMA 5. *If Q is a non-maximal prime of R, then R_Q is a regular local ring.*

PROOF. We assume $(p_n, f_n) \subseteqq Q$ for some n—the argument when $(p_n, g_n, h_n) \subseteqq Q$ has the same flavor. It is enough to consider the case when ht. $Q = 2$ (and ht. $Q \cap S = 3$) since localizations of UFD's are UFD's. Now $S_{Q\cap S}$ is a three dimensional regular local ring and so has a regular system of parameters f_n, p_i, p_j for some i, j. (Since S/f_nS is regular, f_n is part of a regular system of parameters of any prime that contains it.) If $m > n$,

$$f_m - f_n \in q_n^{n+1} S \subseteqq p_n^{n+1} S \subseteqq p_n^2 S \subseteqq (Q \cap S)^2.$$

Thus, using Nakayama's Lemma, f_m, p_i, p_j will also be a regular system of parameters; we choose $m \geqq i$.

Now $p_i \notin (f_m, g_m, h_m)S$ and so $Q \cap S \neq (f_m, g_m, h_m)S$. Thus either $g_m \notin Q$ or $h_m \notin Q$. The two cases are handled identically; we assume $g_m \notin Q$. Then $f_m g_m / q_m^m \in R$ implies $f_m g_m / p_i \in R$, which implies $f_m / p_i \in R_Q$. Thus

$$QR_Q = (Q \cap S)S_{Q\cap S} R_Q = (f_m, p_i, p_j)R_Q = (p_i, p_j)R_Q,$$

and R_Q is regular as desired.

LEMMA 6. Depth $R = 2$.

PROOF. Recall $p_1 = w, f_1 = x + a_1w, g_1 = y + b_1w$, and $h_1 = z + c_1w$. The primes minimal over $(p_1, f_1)S$ and $(p_1, g_1, h_1)S$ are $(w, x)S$ and $(w, y, z)S$. Thus $(w, x)R$ and $(w, y, z)R$ are primes of R such that $(w, x)R \cap S = (w, x)S$ and $(w, y, z)R \cap S = (w, y, z)S$. Also f_1g_1/w, $f_1h_1/w \in R$ implies $xy, xz \in wR$ and so

$$\begin{aligned} wR \cap S \supseteq (w, xy, xz)S &= (w, x)S \cap (w, y, z)S \\ &= (w, x)R \cap (w, y, z)R \cap S \supseteq wR \cap S. \end{aligned}$$

Thus $R/wR \cong S/(w, x) \cap (w, y, z)$ has depth $= 1$ and so depth $R = 2$.

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