ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 1, Winter 1982

CERTAIN FUNCTIONALS ON ℓ_∞

F. S. CATER

1. Introduction. In [**2**] Albert Wilansky observed that if A is a linear functional on the Banach space of bounded sequences, ℓ_∞, such that

$$(*) \qquad Ax \text{ is a limit of some subsequence of } x \text{ for each } x \in \ell_\infty,$$

then A is multiplicative on ℓ_∞. In this note we show that any additive real valued function satisfying $(*)$ on ℓ_∞ must be linear and multiplicative on ℓ_∞. We also show that if G denotes the subgroup of ℓ_∞ composed of all sequences with finite range, then any additive real valued function of G satisfying $(*)$ extends to a unique additive real valued function satisfying $(*)$ on ℓ_∞. We will show that there is a canonical correspondence between the linear functionals on ℓ_∞ satisfying $(*)$ and the nontrivial ultrafilters in the set of positive integers. Finally, we extend all this work from sequences to nets on a directed set.

2. Notation. Throughout this note, D will be a nonvoid set directed by the ordering $<$ such that D has no greatest element. Let S be the set of all bounded real valued nets on D [**1**, p. 65]. We make S a Banach algebra under the sup norm by defining vector addition, multiplication, and scalar multiplication pointwise. Let G_0 denote the additive subgroup of S consisting of those nets that take only integer values. If G is an additive subgroup such that $G_0 \subseteqq G \subseteqq S$, then by a *special* function on G, we mean a real valued function f on G satisfying

$$(*) \qquad fx \text{ is the limit of some subnet of } x \text{ for each } x \in G.$$

Fix a $d \in D$. Then the set of all subsets of D containing d is (trivially) an ultrafilter in D. By a nontrivial ultrafilter in D, we mean an ultrafilter with void intersection. By a *special* ultrafilter in D, we mean a nontrivial ultrafilter every set of which is cofinal in D. (Of course, if D is the set of positive integers with the usual ordering, then any nontrivial ultrafilter in D is a special ultrafilter.)

A simple example of a special ultrafilter in D can be constructed as follows. Let $\mathscr{F}$ be the family of all subsets of D containing sets of the form $\{x : x > d\}$ for $d \in D$. Then $\mathscr{F}$ is a filter in D. Extend $\mathscr{F}$ to an ultrafilter by Zorn's axiom.

Received by the editors on February 20, 1979.

If $F \subseteqq D$, then χ_F will denote the characteristic function of the set F on D.

3. Special functions. Our first order of business is to establish a canonical correspondence between the additive special functions and the special ultrafilters. This will be done in two lemmas.

LEMMA 1. *Let A be an additive special function on an additive subgroup G of S containing G_0. Then there is a unique special ultrafilter $\mathscr{F}$ in D such that for any $\varepsilon > 0$, $g \in G$, we have $g^{-1}(Ag - \varepsilon, Ag + \varepsilon) \in \mathscr{F}$.*

PROOF. Let $F \subseteqq D$. Then $\chi_F \in G_0 \subseteqq G$ and $A\chi_F = 0$ or 1. If $A\chi_F = 1$ put $F \in \mathscr{F}$. If, on the other hand, $A\chi_F = 0$, put $D \backslash F \in \mathscr{F}$. In the latter case, note that

$$1 = A1 = A(\chi_F + \chi_{D\backslash F}) = A\chi_F + A\chi_{D\backslash F} = A\chi_{D\backslash F}.$$

Thus $\mathscr{F}$ is a family of cofinal subsets of D, and for any $F \subseteqq D$, either $F \in \mathscr{F}$ or $D\backslash F \in \mathscr{F}$.

Suppose $F \subseteqq F_0 \subseteqq D$ and $F \in \mathscr{F}$. Then

$$A\chi_{F_0} = A\chi_F + A\chi_{F_0\backslash F} = 1 + A\chi_{F_0\backslash F}.$$

But $A\chi_{F_0}$ and $A\chi_{F_0\backslash F}$ are either 0 or 1, so $A\chi_{F_0\backslash F} = 0$ and $A\chi_{F_0} = 1$. Hence $F_0 \in \mathscr{F}$. Now suppose that $F_1 \in \mathscr{F}$, and $F_2 \in \mathscr{F}$. Then

$$\chi_{F_1 \cap F_2} = \chi_{F_1} + \chi_{F_2} - \chi_{F_1 \cup F_2}$$

and

$$A\chi_{F_1 \cap F_2} = A\chi_{F_1} + A\chi_{F_2} - A\chi_{F_1 \cup F_2} = 1 + 1 - 1 = 1.$$

Hence $F_1 \cap F_2 \in \mathscr{F}$. We have shown that $\mathscr{F}$ is a filter in D. But for any $F \subseteqq D$, either $F \in \mathscr{F}$ or $D\backslash F \in \mathscr{F}$. So $\mathscr{F}$ is in fact an ultrafilter. And every member of $\mathscr{F}$ is cofinal in D, so $\mathscr{F}$ is finally a special ultrafilter.

Take any $\varepsilon > 0$ and $g \in G$. Suppose that $F = g^{-1}(Ag - \varepsilon, Ag + \varepsilon) \notin \mathscr{F}$. We assume, without loss of generality, that $\varepsilon < 1/2$. Then $A\chi_F = 0$. Put $f = g + \chi_F$. Then $Af = Ag + A\chi_F = Ag$. Clearly f is bounded away from Ag on F and on D. Thus f has no subnet that converges to $Ag = Af$, contrary to hypothesis. This contradiction proves that $g^{-1}(Ag - \varepsilon, Ag + \varepsilon) \in \mathscr{F}$.

Now suppose that $\mathscr{F}'$ is a special ultrafilter in D and $\mathscr{F}' \neq \mathscr{F}$. Say $F \in \mathscr{F}\backslash\mathscr{F}'$. Then $A\chi_F = 1$. But $\chi_F^{-1}(1 - 1, 1 + 1) = F \notin \mathscr{F}'$. This proves the uniquencess of $\mathscr{F}$.

LEMMA 2. *Let $\mathscr{F}$ be a special ultrafilter in D. Then there exists a unique special function A on G related to $\mathscr{F}$ as in Lemma 1. Moreover, A is additive on G.*

PROOF. Take any $g \in G$ and any integer $n > 0$. Then exactly one of the sets

$$\ldots g^{-1}(-2\cdot 2^{-n}, -2^{-n}], g^{-1}(-2^{-n}, 0], g^{-1}(0, 2^{-n}], \\ g^{-1}(2^{-n}, 2\cdot 2^{-n}], g^{-1}(2\cdot 2^{-n}, 3\cdot 2^{-n}], g^{-1}(3\cdot 2^{-n}, 4\cdot 2^{-n}], \ldots$$

lies in $\mathscr{F}$. Call it E_n. Clearly $E_1 \supseteqq E_2 \supseteqq E_3 \supseteqq \ldots$ and each E_n is cofinal in D. By [**1**, Lemma 5, p. 70], there is a subnet of g which is eventually in $g\ E_n$ for all n. This subnet evidently converges to a real number. Let Ag denote its limit. For any $\varepsilon > 0$, clearly $g^{-1}(Ag - \varepsilon, Ag + \varepsilon) \in \mathscr{F}$.

Suppose that B is a real valued function on G and $A \neq B$. Say $g \in G$ and $Ag \neq Bg$. Then there is an $\varepsilon > 0$ such that $|u - Ag| < \varepsilon$ implies that $|u - Bg| > \varepsilon$. Then $g^{-1}(Ag - \varepsilon, Ag + \varepsilon) \cap g^{-1}(Bg - \varepsilon, Bg + \varepsilon) = \emptyset$. Since $g^{-1}(Ag - \varepsilon, Ag + \varepsilon) \in \mathscr{F}$, it follows that $g^{-1}(Bg - \varepsilon, Bg + \varepsilon) \notin \mathscr{F}$. This proves the uniqueness of A.

It remains only to prove that A is additive on G. Suppose, on the contrary, that there exist $g \in G, f \in G$, such that $A(f + g) \neq Af + Ag$. There is an $\varepsilon > 0$ such that $|u - Af| < \varepsilon$, $|v - Ag| < \varepsilon$ imply that $|u + v - A(f + g)| > \varepsilon$. Consequently

$$f^{-1}(Af - \varepsilon, Af + \varepsilon) \cap g^{-1}(Ag - \varepsilon, Ag + \varepsilon) \\ \cap (f + g)^{-1}(A(f + g) - \varepsilon, A(f + g) + \varepsilon) = \emptyset,$$

contrary to the fact that this intersection is in $\mathscr{F}$.

Lemmas 1 and 2 establish our canonical correspondence between the additive special functions on S and the special ultrafilters in D.

LEMMA 3. *Let $\mathscr{F}$ and A be related as in Lemmas* 1 *and* 2. *Let U be a continuous real valued function on the Euclidean plane. Let $f \in G$, $g \in G$, such that $U(f, g) \in G$. Then $AU(f, g) = U(Af, Ag)$. Thus in particular, $A(fg) = (Af)(Ag)$ if $fg \in G$, and $A(cg) = cAg$ if c is real and $cg \in G$.*

PROOF. Suppose, on the contrary, $AU(f, g) \neq U(Af, Ag)$. Put $w = AU(f, g)$. Since U is continuous, there is an $\varepsilon > 0$ such that $|r - Af| < \varepsilon$, $|s - Ag| < \varepsilon$ imply that $|w - U(r, s)| > \varepsilon$. Thus

$$f^{-1}(Af - \varepsilon, Af + \varepsilon) \cap g^{-1}(Ag - \varepsilon, Ag + \varepsilon) \cap (U(f, g))^{-1}(w - \varepsilon, w + \varepsilon) = \emptyset.$$

But this is impossible since the intersection lies in $\mathscr{F}$. Hence $AU(f, g) = U(Af, Ag)$.

For $A(fg) = (Af)(Ag)$, let $U(r, s) = rs$. For $A(cg) = cAg$, let $U(r, s) = cs$.

It is worth noting that additive special functions on G (as in Lemmas 1, 2, 3) have norm 1 on G. For $F \in \mathscr{F}$, $|A\chi_F| = 1$. And for $g \in G$, $|Ag| \leqq$

$\|g\|$ because a subnet of g converges to Ag.

If $G = S$ in Lemma 3, then A is linear and multiplicative on S. Thus we have the following theorem.

THEOREM 1 (WILANSKY). *Let A be an additive special function on S. Then A is linear and multiplicative on S.*

PROOF. A is associated with a special ultrafilter as in Lemma 1. By Lemmas 2 and 3, A is linear and multiplicative on S.

Now we have the extension theorem we promised in the introduction.

THEOREM 2. *Let G be an additive subgroup of S containing G_0, and let A be an additive special function on G. Then A has a unique special extension A_0 on S. Moreover, A_0 is linear and multiplicative on S.*

PROOF. A is related to a special ultrafilter $\mathscr{F}$ as in Lemma 1. Then $\mathscr{F}$ in turn is related to a special function A_0 on S by Lemma 2, and indeed A_0 coincides with A on G by uniqueness. By Lemmas 2 and 3, A_0 is linear and multiplicative on S.

Our next result will show, among other things, that if f is a fixed net in S and if w is the limit of some subnet of f, then there is a linear special function A on S such that $Af = w$.

THEOREM 3. *Let X be a subset of S such that for each $f \in X$ there is a real number $w(f)$ satisfying* (i) *for any $\varepsilon > 0$ and any finite number of members of $X, f_1, \dots, f_n$, we have*

$$f_1^{-1}(w(f_1) - \varepsilon, w(f_1) + \varepsilon) \cap \cdots \cap f_n^{-1}(w(f_n) - \varepsilon, w(f_n) + \varepsilon)$$

is cofinal in D. Then there is a linear special function A on S such that $Af = w(f)$ for all $f \in X$.

PROOF. Let $\mathscr{F}'$ be the smallest filter in D containing all the sets of the form $f^{-1}(w(f) - \varepsilon, w(f) + \varepsilon), f \in X, \varepsilon > 0$, and of the form $\{x \in D: x > d\}$ $d \in D$. We extend $\mathscr{F}'$ to an ultrafilter $\mathscr{F}$ by Zorn's axiom. Then $\mathscr{F}$ is a special ultrafilter. Let A be the linear special function given by Lemma 2.

Let f be any member of X. It remains only to show that $Af = w(f)$. Suppose, on the contrary, that $Af \neq w(f)$. Then for some $\varepsilon > 0$,

$$f^{-1}(w(f) - \varepsilon, w(f) + \varepsilon) \cap f^{-1}(Af - \varepsilon, Af + \varepsilon) = \emptyset.$$

Since $f^{-1}(w(f) - \varepsilon, w(f) + \varepsilon) \in \mathscr{F}' \subseteqq \mathscr{F}$, we have that $f^{-1}(Af - \varepsilon, Af + \varepsilon)$ $\notin \mathscr{F}$, which is impossible.

In conclusion we show that there must be uncountably many linear special functions on S.

THEOREM 4. *There are at least c linear special functions on S.*

PROOF. By transfinite induction we construct a cofinal subset E of D well ordered by $\ll$ such that

(i) if $x, y \in E$ and x is a $\ll$ limit point and $y \ll x$, then $x \not< y$, and

(ii) if $x, y \in E$ and y is the $\ll$ successor of x, then $x < y$.

By a type 0 element in E we mean the $\ll$ first element of E or any $\ll$ limit point in E. By a type 1 element of E we mean the $\ll$ successor of a type 0 element. In general, by a type $n + 1$ element of E we mean the $\ll$ successor of a type n element.

Let $E_0 \subseteqq E$ consist of all the type 0, type 2, type 4, type 6, type 8, etc., elements. Let E_1 consist of all the type 1, type 5, type 9, type 13, etc., elements. Let E_2 consist of all type 3, type 11, type 19, type 27, etc., elements. Let E_3 consist of all the type 7, type 23, type 39, etc., elements. We continue in this way to construct a sequence $E_1, E_2, E_3, \ldots, E_n, \ldots$ of pairwise disjoint cofinal subsets of E and of D.

We construct $f \in S$ by making f constant on each E_n so that $f(E)$ is dense in $(0, 1)$ and $f(D \backslash E) = \{0\}$. By [**1**, Theorem 6, p. 71], for each number $w \in (0, 1)$ there is a subnet of f converging to w. And by Theorem 3 there is a linear special function A on S satisfying $Af = w$. Thus there are at least as many linear special functions on S as there are real numbers between 0 and 1.

A possible topic for further study would be to find exactly how many linear special functions on S there are. This will depend, naturally, on D and its ordering.

Note that complex scalars will suffice in this work as well as real scalars.

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PORTLAND STATE UNIVERSITY, PORTLAND, OR 97207