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VECTOR POTENTIALS FOR SYMMETRIC HYPERBOLIC SYSTEMS

DAVID S. GILLIAM

0. **Introduction.** Potentials are widely used in the physics literature in the study of a great variety of wave-propagation problems, but it is not altogether clear how (or if) a potential for a particular problem might be discovered. In a recent paper **[4]** J. Schulenberger showed that the Lagrange identities and potentials for Maxwell's equations and the equations of two-dimensional elasticity derive in a natural way from the spectral resolutions of the corresponding operators. More recently, J. Schulenberger and the author have shown [3] that there is a class of symmetric hyperbolic systems, describing most wave-propagation phenomena of classical physics, which is characterized by admitting potentials of a special form, called there "potentials of classical type". These potentials are very useful in studying symmetries, degeneracies, and conserved quadratic forms, but they suffer the disadvantage of being nonlocal.

It is the purpose of this paper to show that this same class of symmetric hyperbolic systems is also characterized by another type of potential decomposition which is very often in applications local and which even more strongly emphasizes the special nature of this class of equations. It is, furthermore, easier to compute these local potentials than those of [3](cf. §3).

1. **Definitions and Notations.** We shall be concerned with first-order symmetric hyperbolic systems

$$i\partial_t u = A(D)u = \sum_{l=1}^{n} A_l D_l u,\ D_l = -i\partial_{\chi_l} \tag{1.1}$$

where A_l is an $m \times m$ hermitian matrix. The operator $A \equiv A(D)$ with domain

$$\mathscr{D}(A) = \{f \in \mathscr{H} : Af \in \mathscr{H}\} \tag{1.2}$$

is self-adjoint in $\mathscr{H} = L(\mathbf{R}^n, \mathbf{C}^m)$ and generates the unitary group $U(t) = \exp(-iAt)$ (cf. [3], [4], [5], [6], [7]).

We shall be concerned with those systems (1.1) with symbol $A(p) = \Phi A$ of the form

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$$A(p) = \begin{bmatrix} O_{k\times k} & \Lambda_{k\times j} \\ \Lambda^*_{j\times k} & O_{j\times j} \end{bmatrix} \tag{1.3}$$

where Λ^* means conjugate transpose, $k + j = m$, $k \geqq j$ and Φ denotes Fourier transform in $\mathbf{R}^n$, that is

$$\Phi f(p) \equiv \hat{f}(p) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \exp(-i\, px) f(x)\, dx$$

and its inverse $\Phi^* f(p) = \Phi f(-p)$.

The symbol $A(p)$ of (1.1), $p \in \mathbf{R}^n \backslash \{0\}$, has real eigenvalues $\lambda_l(p)$ which are homogeneous of degree one and continuous on $\mathbf{R}^n$. Enumerating them in the order

$$\lambda_{-N}(p) \leqq \cdots \leqq \lambda_{-1}(p) \leqq \lambda_0(p) \equiv 0 \leqq \lambda_1(p) \leqq \cdots \leqq \lambda_N(p), \tag{1.4}$$

it was shown in [4] that

$$\lambda_l(p) = -\lambda_{-l}(p) = \lambda_l(-p)$$

and that the multiplicity of each $\lambda_l(p)$ is a constant, say ν_l, on $\mathbf{R}^n \backslash Z$ where Z is a closed set of measure zero in $\mathbf{R}^n$ which intersects the sphere, S^{n-1}, in a set of surface measure zero.

The matrices $\Lambda\Lambda^*(p)$ $(k \times k)$ and $\Lambda^*\Lambda(p)$ $(j \times j)$ have eigenvalues

$$\lambda_0(p) \equiv 0 \leqq \lambda_1^2(p) \leqq \cdots \leqq \lambda_N^2(p) \tag{1.5}$$

again with constant multiplicity on $\mathbf{R}^n \backslash Z$ (Note that $\Lambda^*\Lambda(p)$ may be elliptic (cf. [3])). In $\mathbf{C}^k$ (respectively $\mathbf{C}^j$) we have the resolution of the identity for $\Lambda\Lambda^*(\Lambda^*\Lambda)$ given by

$$I_{k\times k} = \sum_{l=0}^{N} \hat{P}_l^k(p)\ (I_{j\times j} = \sum_{l=0}^{N} \hat{P}_l^j(p)) \tag{1.6}$$

where $\hat{P}_l^k(p)$ $(\hat{P}_l^j(p))$ is the orthogonal projection onto the eigensubspace corresponding to $\lambda_l^2(p)$. These projections are given by

$$\hat{P}_l^k(p) = -(2\pi i)^{-1} \int_{|\lambda_l^2(p)-\xi|=\delta} [\Lambda\Lambda^*(p) - \xi I]^{-1} d\xi$$

(1.7) and

$$\hat{P}_l^j(p) = -(2\pi i)^{-1} \int_{|\lambda_l^2(p)-\xi|=\delta} [\Lambda^*\Lambda(p) - \xi I]^{-1} d\xi$$

where the integration goes over a small circle in $\mathbf{C}$ enclosing only the eigenvalue $\lambda_l^2(p)$ in the positive direction. Let $P_l^k = \Phi^* \hat{P}_l^k \Phi$ and $P_l^j = \Phi^* \hat{P}_l^j \Phi$, to obtain the decompsotions

$$\mathscr{H}^k = \sum_{l=0}^{N} L_2^l(\mathbf{R}^n, \mathbf{C}^k),\ \mathscr{H}^j = \sum_{l=0}^{N} L_2^l(\mathbf{R}^n, \mathbf{C}^j) \tag{1.8}$$

where $L_2^l(\mathbf{R}^n, \mathbf{C}^k) \equiv P_l^k \mathscr{H}^k$ and $L_2^l \equiv L(\mathbf{R}^n, \mathbf{C}^j) \equiv P_l^j \mathscr{H}^j$. When the symbol $A(p)$ has the form (1.3), the resolution of the identity for $A(p)$ can be expressed in terms of the above notation in a simple way. First denote by $\mathscr{S}(\mathbf{R}^n, \mathbf{C}^j)$ the Schwartz space of rapidly decreasing, infinitely dif ferentiable functions from $\mathbf{R}^n$ to $\mathbf{C}^j$ and let $\mathscr{S}_l = \mathscr{S}_l(\mathbf{R}^n, \mathbf{C}^j) = P_l^j\mathscr{S}(\mathbf{R}^n, \mathbf{C}^j)$. Simple consequences of the above definitions and notation are

(1.9) (i) $\{A\phi | \phi \in \mathscr{S}_l\}$ is dense in $L_2^l(\mathbf{R}^n, \mathbf{C}^k)$ and

(ii) $\hat{P}_l^k(p) = \lambda_l^{-2}(p)A(p)\hat{P}_l^j(p)A^*(p)$.

The symbol $A(p)$ has resolution of the identity given by

$$I = \hat{Q}_0(p) + \sum_{l=1}^{N} [\hat{Q}_l(p) + \hat{Q}_{-l}(p)] \tag{1.10}$$

where $\hat{Q}_r(p)$ denotes the projection in $\mathbf{C}^m$ onto the eigensubspace corresponding to $\lambda_r(p)$. Now, as an immediate consequence of (1.6) through (1.10), we have

$$\hat{Q}_{\pm l}(p) = \frac{1}{2}\begin{bmatrix} \lambda_l^{-2}(p)A\hat{P}_l^jA^*(p) & \pm\lambda_l^{-1}A\hat{P}_l^j(p) \\ \pm\lambda_l^{-1}\hat{P}_l^jA^*(p) & \hat{P}_l^j(p), \end{bmatrix}, \tag{1.11}$$

$$\hat{Q}_0(p) = \begin{bmatrix} \hat{P}_0^k(p) & 0 \\ 0 & \hat{P}_0^j(p) \end{bmatrix},$$

so that (1.10) becomes

$$I = \sum_{l=0}^{N} \begin{bmatrix} \hat{P}_l^k(p) & 0 \\ 0 & \hat{P}_l^j(p) \end{bmatrix}. \tag{1.12}$$

Defining the projections in $\mathscr{H}$ by $Q_l = \Phi^*\hat{Q}_l\Phi$ we obtain the decomposition

$$\mathscr{H} = \sum_{l=0}^{N} \mathscr{H}_l, \; \mathscr{H}_0 = Q_0\mathscr{H}, \; \mathscr{H}_l = [Q_l + Q_{-l}]\mathscr{H}. \tag{1.13}$$

This decomposition is the same as that obtained in (1.8).

For each $l = 1, \ldots, N$ define the Beppo-Levi space

$$BL^l \times L_2^l = BL^l(\mathbf{R}^n, \lambda_l, \mathbf{C}^j) \times L_2^l(\mathbf{R}^n, \mathbf{C}^j) \tag{1.14}$$

as the completion of $\mathscr{S}_l(\mathbf{R}^n, \mathbf{C}^j) \times \mathscr{S}_l(\mathbf{R}^n, \mathbf{C}^j)$ in the norm.

$$\|f\|^2 = \int_{\mathbf{R}^n} \{\lambda_l^2(p)|\hat{f}_1(p)|^2 + |\hat{f}_2(p)|^2\}dp. \tag{1.15}$$

The operator

$$H_l = \Phi^*\hat{H}_l\Phi, \; \hat{H}_l = i\begin{bmatrix} 0 & I_{j\times j} \\ -\lambda_l^2(p)I_{j\times j} & 0 \end{bmatrix} \tag{1.16}$$

with domain

$$(1.17) \qquad \mathscr{D}(H_l) = \{f \in BL^l \times L_2^l : f_2 \in BL^l, \lambda_l^2(p)\hat{f}_1(p) \in L_2^l\}$$

is self-adjoint in $BL^l \times L_2^l$, and generates the unitary group $S_l(t) = \exp(-itH_l)$.

For $v = S_l(t)f, f \in \mathscr{D}(H_l)$, the equation

$$(1.18) \qquad i\partial_t v = H_l v$$

is a generalized wave equation; in the case where $A(p)$ is isotropic, i.e., $\lambda_l(p) = \lambda_l(|p|)$, (1.18) is a classical vector wave equation. Denote by A_l and $U_l(t)$, respectively, the operators A and $U(t)$ restricted to $\mathscr{H}_l$.

DEFINITION. *A* collection of unitary maps

$$(1.19) \qquad \sigma_l : BL^l \times L_2^l \to \mathscr{H}_l$$

such that

$$(1.20) \qquad U_l(t) = \sigma_l S_l(t) \sigma_l^{-1}, \; l = 1, \ldots, N,$$

is called a potential decomposition for (1.1). If further, σ_l has the form

$$(1.21) \qquad \sigma_l f = \Phi^*(\Lambda(p)\hat{\hat{f}}_1(p), i\hat{f}_2(p))$$

then σ_l is called a vector potential and the decomposition (1.19), (1.20) a vector-potential decomposition.

2. **The Main Result.** THEOREM 1. *The system* (1.1) *admits a vector-potential decomposition if and only if the symbol $A(p)$ has the form* (1.3).

PROOF. (*of the sufficiency*) Define σ^l: $BL_l \times L_2^l \to \mathscr{H}_l$ by (1.21); the proof of sufficiency is contained in the lemma:

LEMMA. (i) σ_l is unitary, (ii) $\sigma_l \mathscr{D}(H_l) = \mathscr{D}(A_l)$,

$$(2.1) \qquad \text{(iii)} \quad \sigma_l^{-1} U_l(t) \sigma_l = S_l(t)$$

PROOF. For f in $(BL^l \times L_2^l) \cap (\mathscr{S}_l \times \mathscr{S}_l)$ we have

$$\begin{aligned}
\|\Phi\sigma_l f\|^2 &= (\Lambda\hat{f}_1, \Lambda\hat{f}_1) + (i\hat{f}_2, i\hat{f}_2) \\
&= (\Lambda^*\Lambda\hat{f}_1, \hat{f}_1) + (\hat{f}_2, \hat{f}_2) \\
&= \int\{\lambda_l^2(p)\,|\hat{f}_1(p)|^2 + |\hat{f}_2(p)|^2\}dp \\
&= \|f\|^2.
\end{aligned}$$

So σ_l is an isometry.

If $f = (f^k, f^j)$ in $\mathscr{H}_l^k \times \mathscr{H}_l^j = \mathscr{H}_l$ is orthogonal to the range of σ_l, then for all $\phi = (\phi_1, \phi_2)$ in $\mathscr{S}_l \times \mathscr{S}_l$ we have

$$0 = (\hat{f}, \widehat{\sigma_l \phi}) = (\hat{f}^k, \widehat{\Lambda\phi_1}) + (\hat{f}^j, i\hat{\phi}_2)$$

and hence $f^j = 0$ since $\mathscr{S}_l$ is dense in $\mathscr{H}_l^j$. The fact that $f^k = 0$ follows because $\{\Lambda\phi | \phi \in \mathscr{S}_l\}$ is dense in $\mathscr{H}_l^k$, see (1.9).

If f is in $\mathscr{S}_l \times \mathscr{S}_l$ then we have

$$\begin{aligned}\|A\sigma_l f\|^2 &= \|A(p)\widehat{\sigma_l f}\|^2 \\ &= \|i\Lambda(p)\hat{f}_2(p)\|^2 + \|\lambda_l^2(p)\hat{f}_1(p)\|^2 \\ &= \|f_2\|_{BL_l}^2 + \|\lambda_l^2(p)\hat{f}_1(p)\|_{L_2^l}^2.\end{aligned}$$

Hence f is in the domain of H_l if and only if $\sigma_l f$ is in the domain of A.

For $g = (g^k, g^j)$ in an appropriate dense subset of $\mathscr{H}_l = \mathscr{H}_l^k \times \mathscr{H}_l^j$ we have

$$\sigma_l^{-1} g = \Phi^*(\lambda_l^{-2}(p)\Lambda^*(p)\hat{g}^k(p), -i\hat{g}^j(p)).$$

Using this, the result (iii) follows from the identity

$$\begin{aligned}\sigma_l^{-1}A\sigma_l f &= \Phi^* i \begin{bmatrix} \lambda_l^{-2}\Lambda^*\Lambda\hat{f}_2 \\ -\Lambda\Lambda^*\hat{f}_1 \end{bmatrix} = \Phi^* \begin{bmatrix} i\hat{f}_2 \\ -i\lambda_l^2\hat{f}_1 \end{bmatrix} \\ &= \Phi^*\hat{H}_l\hat{f} = H_l f.\end{aligned}$$

Thus the proof of the sufficiency of the theorem is complete.

The proof of necessity is the same as the proof of the necessity part of Theorem 1 in [**3**], therefore will not be reproduced here, but rather the reader is referred to [**3**].

It should be commented that in the case of the vector potentials, the assumptions for necessity (in contrast to those in [**3**]) are that $A(p)$ has the form

$$A(p) = \begin{bmatrix} \Gamma(p) & \Lambda(p) \\ \Lambda^*(p) & \Pi(p) \end{bmatrix}$$

and there is a vector potential decomposition of the form $\sigma_l \colon BL^l \times L^l \to \mathscr{H}_l$,

$$\sigma_l f = \Phi^*(\Lambda(p)\hat{f}_1(p), i\hat{f}_2(p))$$

for f in $BL^l \times L_2^l$, $l = 1, \ldots, N$.

Just as in [**3**], the existence of a vector potential decomposition for (1.1), (1.3) affords a simple characterization of smooth data of a special form in each $\mathscr{H}_l$, $l = 1, \ldots, N$. In contrast to [**3**] though, in the case when $A(p)$ is isotropic, this characterization delivers smooth, compactly supported data in each $\mathscr{H}_l$.

Define the bounded measurable matrix-valued functions $\mathscr{A}_l(\omega)$ on S^{n-1} for $l = 1, \ldots, N$ by

$$\mathscr{A}_l(\omega) = 2^{-1/2}\lambda_l^{-1}(\omega)\hat{P}_l^j(\omega)\Lambda^*(\omega) \tag{2.2}$$

then, using the definitions and remarks in Section 1, we have

$$\hat{P}_l^k(\omega) = 2\mathscr{A}_l^*\mathscr{A}_l,\ \hat{P}_l^j = 2\mathscr{A}_l\mathscr{A}_l^*$$

$l = 1, \ldots, N$. The Lagrange Identity in [3] and the statement (1.10) become

$$I - \hat{Q}_0 = 2\sum_{l=1}^{N} \begin{bmatrix} \mathscr{A}_l^*\mathscr{A}_l & 0 \\ 0 & \mathscr{A}_l\mathscr{A}_l^* \end{bmatrix} \tag{2.3}$$

Define the operators on $\mathscr{D}(\mathbf{R}^n, \mathbf{C}^k)$ and $\mathscr{D}(\mathbf{R}^n, \mathbf{C}^j)$, respectively, by

$$\mathscr{A}_l^*(D) = \Phi^*\mathscr{A}_l^*(\omega)\,|p|^{S_a}\Phi$$
$$\mathscr{A}_l(D) = \Phi^*\mathscr{A}_l(\omega)\,|p|^{S_a}\Phi$$

where $\mathscr{D}(\mathbf{R}^n, \mathbf{C}^r)$ is the space of compactly supported smooth functions from $\mathbf{R}^n$ to $\mathbf{C}^r$ and $S_a = [\deg\mathscr{A}_l(\omega)/2]$ if $\mathscr{A}_l(\omega)$ has polynomial entries and zero otherwise. Here $[r]$ means the smallest integer $\geqq r$.

Let

$$S_l = \{\mathscr{A}_l(D)\phi \colon \phi \in \mathscr{D}(\mathbf{R}^n, \mathbf{C}^j)\}$$

and

$$T_l = \{\mathscr{A}_l^*(D)\psi \colon \psi \in \mathscr{D}(\mathbf{R}^n, \mathbf{C}^k)\}.$$

COROLLARY. *$S_l \times T_l$ is dense in $\mathscr{H}_l$.*

The proof is just like the corresponding result in [3] and follows from the Lagrange identity (2.3) and properties of the Fourier transform.

3. **Examples.** Many of the wave-propagation problems of classical physics can be written in the form (1.1), (1.3). Among these are the equations of acoustics, elasticity, magnetohydrodynamics, crystal optics, Maxwell's equations, and the equations governing elastic wave-propagation in fiber-reinforced media. For the description of these equations in the form (1.1), (1.3) and the pertinent information necessary to construct the projections and vector potentials, the reader is referred to [3].

It is worth commenting that in the case of Maxwell's equations and the equations of elasticity in $\mathbf{R}^3$, the vector potentials are the classical vector potentials found in the physics literature; also, in [3] only the uniaxial case in crystal optics was considered. This was partly due to the difficulty encountered in constructing orthonormal eigenvectors for the symbol $A(p)$. In the case of vector potentials, the eigenvectors are unnecessary; only the projections are needed and these are many times easier to compute. Below, the symbol and projections for the general biaxial case of crystal optics are given. I would like to thank John Schulenberger for presenting me with this example. In order to present this example in a more notationally convenient setting, a slight modification of the development given in

(1.1) through (1.21) is given. This modification also shows how simply the preceding can be extended to more general situations.

Consider a symmetric hyperbolic system of the form

$$i\partial_t u = A(D)u = E^{-1}A_0(D)u, \tag{3.1}$$

with symbol

$$A(p) = E^{-1}A_0(p) = E^{-1}\begin{bmatrix} 0 & \Lambda(p) \\ {}^t\Lambda(p) & 0 \end{bmatrix}$$

where E is a positive-definite, symmetric matrix of the form

$$E = \begin{bmatrix} E_0 & 0 \\ 0 & I \end{bmatrix}.$$

Here $\Lambda(p)$ is a $k \times j$ matrix, $j + k = m$, ${}^t\Lambda$ is the transpose of Λ, E_0 is a positive-definite, symmetric $k \times k$ matrix and I is the $j \times j$ identity matrix. Replacing the system (1.1) by (3.1), one must also replace $\mathscr{H} = L_2(\mathbf{R}^n, \mathbf{C}^m)$ by $\mathscr{H}_E = L_2(\mathbf{R}^n, \mathbf{C}^m)$ with inner product

$$(f, g)_E = (f, Eg). \tag{3.2}$$

Note that (3.1) can be put in the form (1.1) with an appropriate choice of dependent variables, although, in the specific application given below, this change of variables would greatly complicate the notation. The only important changes encountered in Section 1 with (3.1) replacing (1.1) are: The statement (1.11) becomes

$$Q_l(\omega) = 2^{-1}\begin{bmatrix} \lambda_l^{-2}E_0^{-1}\Lambda P_l^{j\,t}\Lambda(\omega) & \lambda_l^{-1}E_0^{-1}\Lambda P_l^j(\omega) \\ \lambda_l^{-1}P_l^{j\,t}\Lambda(\omega) & P_l^j(\omega) \end{bmatrix} \tag{3.3}$$

and the vector potentials (1.21) become

$$\sigma_l f = \Phi^*(E_0^{-1}\Lambda(p)\hat{f}_1(p), i\hat{f}_2(p)). \tag{3.4}$$

For the sake of brevity, only the pertinent information for constructing the potentials is given and the reader is referred to [**1**, **2**, **3**, **8**] for a more thorough discussion of the equations of crystal optics. Choosing the coordinate system so that the axes coincide with the principal axes of the dielectric tensor, the equations of crystal optics (general biaxial case) can be written in the form (3.1) where the symbol $A(p)$ is given by

$$A(p) = \begin{bmatrix} 0 & E_0^{-1}\omega\lambda \\ -\omega\lambda & 0 \end{bmatrix},$$

the Hilbert space $\mathscr{H}_E$ is $L_2(\mathbf{R}^3, \mathbf{C}^6)$ with energy form, $E = \operatorname{diag}(E_0, I_{3\times 3})$, $E_0^{-1} = \operatorname{diag}(\varepsilon_1^{-1}, \varepsilon_2^{-1}, \varepsilon_3^{-1}) \equiv \operatorname{diag}(c_1, c_2, c_3) \equiv C$, where

$$\omega\lambda = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

The characteristic polynomial for $A(p)$ is

$$\begin{aligned} \det(A(p) - \xi I) &= \xi^2(\xi^4 - R_2(\omega)\xi^2 + R_4(\omega)), \\ R_2(\omega) &= (c_2 + c_3)\omega_1^2 + (c_1 + c_3)\omega_2^2 + (c_1 + c_2)\omega_3^2 \\ &= \lambda_1^2(\omega) + \lambda_2^2(\omega), \\ R_4(\omega) &= c_2c_3\omega_1^2 + c_1c_3\omega_2^2 + c_1c_2\omega_3^2 = \lambda_1^2(\omega)\lambda_2^2(\omega). \end{aligned}$$

The eigenvalues are thus

$$\lambda_0(\omega) \equiv 0,\ \lambda_1^2(\omega) = 2^{-1}[R_2(\omega) + D(\omega)^{1/2}],\ \lambda_2^2(\omega) = 2^{-1}[R_2(\omega) - D(\omega)^{1/2}]$$

where

$$\begin{aligned} [\lambda_1^2(\omega) - \lambda_2^2(\omega)]^2 &= D(\omega) \\ &= [(d_{32}\omega_1 - d_{21}\omega_3)^2 + d_{31}^2\omega_2^2][(d_{32}\omega_1 + d_{21}\omega_3)^2 + d_{31}\omega_2^2] \\ &= R_2^2(\omega) - 4R_4(\omega),\ d_{ij}^2 = c_i - c_j. \end{aligned}$$

The projections from (1.7) are

$$P_0^j(\omega) = \omega \otimes \omega,\ P_0^k(\omega) = \omega \otimes \omega c',$$

where $c' = \operatorname{diag}(c_2c_3, c_1c_3, c_1c_2)$ and for $\xi \in \mathbf{R}^3$, $\omega \otimes \omega(\xi) = \omega(\omega \cdot \xi)$,

$$\begin{aligned} \hat{P}_1^j(\omega) &= (-\omega\lambda c\omega\lambda + \lambda_2^2\omega\lambda\omega\lambda)/D^{1/2} \\ \hat{P}_2^j(\omega) &= (\omega\lambda c\omega\lambda - \lambda_1^2\omega\lambda\omega\lambda)/D^{1/2}. \end{aligned}$$

The projections (3.3) are

$$\hat{Q}_1(\omega) = 2D^{-1/2}\begin{bmatrix} -c\omega\lambda\omega\lambda + \lambda_1^{-2}c\omega\lambda c\omega\lambda & -c\lambda_1^{-1}(\omega\lambda\omega\lambda c\omega\lambda + \lambda_2^2\omega\lambda) \\ \lambda_1^{-1}(\omega\lambda c\omega\lambda\omega\lambda + \lambda_2^2\omega\lambda) & -\omega\lambda c\omega\lambda + \lambda_2^2\omega\lambda\omega\lambda \end{bmatrix},$$

$$\hat{Q}_2(\omega) = 2D^{-1/2}\begin{bmatrix} c\omega\lambda\omega\lambda - \lambda_2^{-2}c\omega\lambda c\omega\lambda & c\lambda_2^{-1}(\omega\lambda\omega\lambda c\omega\lambda + \lambda_1^2\omega\lambda \\ -\lambda_2^{-1}(\omega\lambda c\omega\lambda\omega\lambda + \lambda_1^2\omega\lambda) & \omega\lambda c\omega\lambda - \lambda_1^2\omega\lambda\omega\lambda \end{bmatrix}.$$

Finally, the vector potentials, σ_l, are given by

$$\sigma_l f = \Phi^*(c\omega\lambda\hat{f}_1(p), i\hat{f}_2(p))$$

for f in $BL^l \times L_2^l$, $l = 1, 2$ and $\omega = p/|p|$, and the corresponding wave operators are given by

$$H_l = i\Phi^*\begin{bmatrix} 0 & I_{3\times 3} \\ -\lambda_l^2(p)I_{3\times 3} & 0 \end{bmatrix}\Phi, \quad l = 1, 2.$$

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DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409