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THE INFLUENCE OF THE SECOND COEFFICIENT ON PRESTARLIKE FUNCTIONS

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ABSTRACT. Various properties of prestarlike functions, including when they are univalent, are determined. Using a general coefficient bound, we see how the modulus of the second coefficient influences the remaining coefficients. Similar results are obtained for subclasses of starlike, convex, and close-to-convex functions.

1. **Introduction.** A function $f(z)$, normalized by $f(0) = f'(0) - 1 = 0$, is said to be in the class S if it is analytic and univalent in the unit disk U. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be in the class of functions *starlike of order* α, $0 \leqq \alpha \leqq 1$, denoted by $S^*(\alpha)$, if

$$\mathrm{Re}\{zf'(z)/f(z)\} \geqq \alpha \qquad (z \in U)$$

and in the class of functions *convex of order* α, $0 \leqq \alpha \leqq 1$, denoted by $K(\alpha)$, if

$$\mathrm{Re}\{1 + zf''(z)/f'(z)\} \geqq \alpha \qquad (z \in U).$$

The families $S^*(\alpha)$ and $K(\alpha)$ are known to be in S. The *convolution* or *Hadamard product* of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. A normalized analytic function f is said to be in the class of functions *prestarlike of order* α, $0 \leqq \alpha < 1$, denoted by R_α, if $f * s_\alpha \in S^*(\alpha)$, where $s_\alpha = z/(1-z)^{2(1-\alpha)}$. The class R_α was introduced by Ruscheweyh [4], who showed that $f(z)$ in R_α is characterized by having

$$G(z) = \frac{f(z) * \dfrac{z}{(1-z)^{3-2\alpha}}}{f(z) * \dfrac{z}{(1-z)^{2-2\alpha}}}$$

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satisfy

$$\text{(1)} \qquad \operatorname{Re} G(z) > 1/2 \qquad (z \in U).$$

We thus say that $f(z)$ is prestarlike of order 1 if $\operatorname{Re}\{f(z)/z\} > 1/2$. In **[4]**, Ruscheweyh shows that

$$\text{(2)} \qquad R_\alpha \subset R_\beta \text{ for } 0 \leqq \alpha < \beta \leqq 1,$$

which is essentially due to Suffridge **[8]**. Note that $R_{1/2} = S^*(1/2)$ and $R_0 = K(0)$. Thus a special case of (2) is the well-known result that $K(0) = K \subset S^*(1/2)$.

In Section 2, we will give additional properties of R_α and show how the modulus of the second coefficient in the power series expansion influences the growth of the other coefficients. In Section 3, we will apply these methods to subclasses of starlike, convex, and close-to-convex functions.

2. **Properties of R_α.** For a compact family $\mathscr{F}$, we denote the closed convex hull of $\mathscr{F}$ by cl co $\mathscr{F}$ and the extreme points of cl co $\mathscr{F}$ by $\mathscr{E}$(cl co $\mathscr{F}$). In **[2]**, it is shown that a function is in $R_1 =$ cl co R_1 if and only if it can be expressed in the form

$$\text{(3)} \qquad \int_{|x|=1} \frac{z}{1-xz}\, d\mu(x),$$

where μ varies over the probability measures on the unit circle. Further, a function is in cl co R_0 if and only if it is in the form (3). Since $R_0 \subset R_\alpha \subset R_1$ for $0 < \alpha < 1$, it follows that (3) represents cl co R_α for every α and that

$$\mathscr{E}(\text{cl co } R_\alpha) = \left\{\frac{z}{1-xz},\ |x| = 1\right\}.$$

Even though the closed convex hulls (and consequently their sets of extreme points) are identical, the containment in (2) is strict; this will be demonstrated by showing that

$$f(z) = z + \frac{1}{2(2-\beta)}\, z^2 \in R_\beta - R_\alpha$$

for $\alpha < \beta$. In **[7]**, it is shown that $z + a_n z^n \in S^*(\beta)$ if and only if $|a_n| \leqq (1-\beta)/(n-\beta)$. Thus

$$f * s_\beta = z + \frac{1-\beta}{2-\beta}\, z^2 \in S^*(\beta),$$

while

$$f * s_\alpha = z + \frac{1-\alpha}{2-\beta}\, z^2 \notin S^*(\alpha)$$

for $\alpha < \beta$.

Our first theorem, although not used in the sequel, is of interest because it gives an alternate characterization for the class R_α. We will make use of the fact that the convolution of two functions in R_α is again in R_α [**4**].

THEOREM 1. *A normalized analytic function f is prestarlike of order α, $0 \leqq \alpha < 1$, if and only if $f * g \in S^*(\alpha)$ for all $g \in S^*(\alpha)$.*

PROOF. Since $s_\alpha \in S^*(\alpha)$, the condition is sufficient. On the other hand, suppose $f * s_\alpha \in S^*(\alpha)$ and $g \in S^*(\alpha)$. Then $f \in R_\alpha$ and $g * s_\alpha^{-1} \in R_\alpha$, where s_α^{-1} denotes the inverse of s_α with respect to convolution. Hence, $f * g * s_\alpha^{-1} \in R_\alpha$ or, equivalently, $f * g \in S^*(\alpha)$.

We now discuss the univalence of R_α.

THEOREM 2. *The family R_α is contained in S if and only if $\alpha \leqq 1/2$.*

PROOF. Since $R_{1/2} = S^*(1/2) \subset S$, it follows from (2) that $R_\alpha \subset S$ for $\alpha \leqq 1/2$. If $1/2 < \alpha \leqq 1$, we will show that $g_n(z) = z + (2/n)\, z^n \in R_\alpha$ for $n = n(\alpha)$ sufficiently large. Note that g_n is not even locally univalent in U. If $n \geqq 4$, then $g_n \in R_1$. If $\alpha < 1$, then $g_n * s_\alpha = z + a_n z^n$, where $a_n = 2\prod_{k=2}^n (k - 2\alpha)/n!$. An application of Stirling's formula [**10**, p. 58] shows that

$$a_n \sim \frac{A(\alpha)}{n^{2\alpha}} (n \to \infty), \tag{4}$$

where $A(\alpha)$ is a positive constant. Thus when $1/2 < \alpha < 1$, we have $a_n \leqq (1 - \alpha)/(n - \alpha)$ for n sufficiently large, and the proof is complete.

Since $R_\alpha \subset R_1$, $0 \leqq \alpha \leqq 1$, it follows that the modulus of the coefficients for functions in R_α is bounded by one. The following lemma will enable us to obtain better coefficient bounds in terms of a fixed second coefficient. These bounds will be sharp only when the second coefficient is one.

LEMMA. *If $f(z) = z + \sum_{n=2}^\infty a_n z^n \in R_\alpha$ and $s_\alpha = z + \sum_{n=2}^\infty \gamma(\alpha, n) z^n$, then*

$$|a_n| \leqq \frac{1 + \sum_{k=2}^{n-1} |a_k| \gamma(\alpha, k)}{1 + \sum_{k=2}^{n-1} \gamma(\alpha, k)}.$$

PROOF. In view of (1), we may write

$$f * \frac{z}{(1 - z)^{3-2\alpha}} = (f * s_\alpha)(1 + \sum_{n=1}^\infty b_n z^n), \tag{5}$$

with $|b_n| \leqq 1$. Equating the coefficients of z^n in the power series expansions of (5), we have

$$a_n(1 + \sum_{k=2}^n \gamma(\alpha, k)) = b_{n-1} + \sum_{k=2}^{n-1} a_k \gamma(\alpha, k) b_{n-k} + a_n \gamma(\alpha, n).$$

Hence,

$$|a_n|(1 + \sum_{k=2}^{n-1} \gamma(\alpha, k)) \leqq 1 + \sum_{k=2}^{n-1} |a_k|\gamma(\alpha, k).$$

THEOREM 3. *If $f(z) = z + a_2z^2 + \cdots$ is in R_α, $|a_2| = p$, then*

$$|a_n| \leqq \frac{1 + 2p(1 - \alpha)}{3 - 2\alpha} \quad (n = 3, 4, \cdots).$$

PROOF. We proceed by induction. The bound for $|a_3|$ follows from the lemma, with $n = 3$. Now assume

$$|a_k| \leqq \frac{1 + 2p(1 - \alpha)}{3 - 2\alpha} = \frac{1 + p\gamma(\alpha, 2)}{1 + \gamma(\alpha, 2)}$$

for $k = 3, 4, \ldots, n - 1$. In view of the lemma,

$$|a_n| \leqq \frac{1 + p\gamma(\alpha, 2) + \dfrac{1 + p\gamma(\alpha, 2)}{1 + \gamma(\alpha, 2)}(\gamma(\alpha, 3) + \cdots + \gamma(\alpha, n - 1))}{1 + \sum_{k=2}^{n-1} \gamma(\alpha,k)}$$

$$= \frac{1 + p\gamma(\alpha, 2)}{1 + \gamma(\alpha, 2)},$$

which is what we wanted to prove.

Setting $\alpha = 0$ and $\alpha = 1/2$, respectively, in the theorem we obtain the following corollaries.

COROLLARY 1. *If $f(z) = z + a_2z^2 + \cdots$ is in K, $|a_2| = p$, then $|a_n| \leqq 1 + 2p/3$.*

COROLLARY 2. *If $f(z) = z + a_2z^2 + \cdots$ is in $S^*(1/2)$, $|a_2| = p$, then $|a_n| \leqq (1 + p)/2$.*

3. **Fixed Coefficient Results.** Denote by $S_p^*(\alpha)$ and $K_p(\alpha)$ the subfamilies of $S^*(\alpha)$ and $K(\alpha)$, respectively, for which the modulus of the second coefficient is p. Most of the work done with fixed second coefficients for subfamilies of S seems to have focused on various distortion results (see, for example, [**1**], [**3**], and [**6**]). We now show how the other coefficients can be influenced by the second coefficient.

THEOREM 4. *If $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$ is in $S_p^*(\alpha)$, then*

$$|a_n| \leqq \left(\frac{1 + p}{3 - 2\alpha}\right)\frac{\prod_{k=2}^{n}(k - 2\alpha)}{(n - 1)!} \quad (n = 3, 4, \cdots).$$

PROOF. It is known [**5**] that functions in $S^*(\alpha)$ satisfy the coefficient inequality

$$(n - 1)|a_n| \leqq (2 - 2\alpha)[1 + |a_2| + \cdots + |a_{n-1}|]$$

for $n = 2, 3, \cdots$. Assume

$$|a_m| \leqq \left(\frac{1+p}{3-2\alpha}\right)\frac{\prod_{k=2}^{m}(k-2\alpha)}{(m-1)!}$$

for $m = 3, 4, \cdots, n-1$. Then

$$(n-1)|a_n| \leqq (2-2\alpha)\left[1+p+\left(\frac{1+p}{3-2\alpha}\right)\left(\sum_{m=3}^{n-1}\frac{\prod_{k=2}^{m}(k-2\alpha)}{(m-1)!}\right)\right]$$

$$= (2-2\alpha)\left[1+p+\left(\frac{1+p}{3-2\alpha}\right)\left(\frac{\prod_{k=3}^{n}(k-2\alpha)}{(n-2)!}-(3-2\alpha)\right)\right]$$

$$=\left(\frac{1+p}{3-2\alpha}\right)\frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-2)!},$$

and the result follows.

Since $f(z)$ is in $S^*(\alpha)$ if and only if $\int_0^z f(t)/t\,dt$ is in $K(\alpha)$, we obtain the following corollary.

COROLLARY. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $K_p(\alpha)$, then*

$$|a_n| \leqq \left(\frac{1+2p}{3-2\alpha}\right)\frac{\prod_{k=2}^{n}(k-2\alpha)}{n!} \qquad (n = 3, 4, \cdots).$$

REMARK 1. The special cases $\alpha = 0$ in the corollary and $\alpha = 1/2$ in the theorem reduce to the corollaries of Theorem 3.

REMARK 2. Since $p \leqq 2(1-\alpha)$ for $f \in S^*(\alpha)$ and $p \leqq 1-\alpha$ for $f \in K(\alpha)$, the theorem and its corollary represent improvements on tne known bounds when $|a_2|$ does not assume its maximum.

REMARK 3. In **[9]**, Suffridge obtained bounds on $|a_n|$ for $f \in S^*(\alpha)$ when $a_2 = 0$.

Denote, as in **[6]**, by $C_p(\alpha, \beta)$, the class of normalized functions $f(z)$ for which there exists a function $\phi(z)$ in $K_p(\alpha)$ such that

$$\operatorname{Re}\{f'(z)/\phi'(z)\} \geqq \beta \quad (0 \leqq \beta \leqq 1,\, z \in U).$$

Note that functions in this family are close-to-convex and that $C_p(1, \beta) = C_0(1, \beta)$ consists of functions $f(z)$ for which $\operatorname{Re} f'(z) \geqq \beta$. We close with a coefficient bound for $C_p(\alpha, \beta)$.

THEOREM 5. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $C_p(\alpha, \beta)$, then $|a_2| \leqq 1 - \beta + p$* and

$$|a_n| \leqq 2[n(1-\beta)+\beta-\alpha]\left(\frac{1+2p}{3-2\alpha}\right)\frac{\prod_{k=3}^{n}(k-2\alpha)}{n!} \quad (n = 3, 4, \cdots).$$

PROOF. For some function $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $K_p(\alpha)$,

$$f'(z) = \phi'(z)p(z), \tag{6}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ satisfies $\operatorname{Re} p(z) \geqq \beta$. Equating the coefficients of z^{n-1} in the series expansions of (6), we obtain

$$na_n = c_{n-1} + \sum_{m=2}^{n-1} mb_m c_{n-m} + nb_n.$$

Since $|c_k| \leqq 2(1 - \beta)$ for all k, we get

$$n|a_n| \leqq 2(1 - \beta)[1 + \sum_{m=2}^{n-1} mb_m] + nb_n.$$

This gives the bound for a_2. When $n \geqq 3$, the bounds from the corollary to Theorem 4 and $|b_2| = p$ yield

$$\begin{aligned} n|a_n| &\leqq 2(1 - \beta)\left(\frac{1 + 2p}{3 - 2\alpha}\right)\frac{\prod_{k=3}^{n}(k - 2\alpha)}{(n - 2)!} + \left(\frac{1 + 2p}{3 - 2\alpha}\right)\frac{\prod_{k=2}^{n}(k - 2\alpha)}{(n - 1)!} \\ &= \left(\frac{1 + 2p}{3 - 2\alpha}\right)\frac{\prod_{k=3}^{n}(k - 2\alpha)}{(n - 1)!}[2(1 - \beta)(n - 1) + 2 - 2\alpha], \end{aligned}$$

and the proof is complete.

Remark. When $\alpha = 1$, this reduces to $|a_n| \leqq 2(1 - \beta)/n$, a well-known result.

Applying (4) to the theorem, we obtain, for all p, the following corollary.

COROLLARY. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_p(\alpha, \beta)$, *then* $a_n = O(n^{1-2\alpha})$.

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