ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 4, Fall 1979

COMPACT OPERATORS, WEAKLY COMPACT OPERATORS, AND SMOOTH POINTS

RUSSELL G. BILYEU AND PAUL W. LEWIS

S. Heinrih [9] recently announced the following result.

THEOREM. *If E and F are Banach spaces and $K(E, F)$ is the Banach space of all compact operators from E to F, then a compact operator $L : E \to F$ is a smooth point in $K(E, F)$ if and only if* (a) *there is a unique point (up to scalar multiples) $x_0^* \in S(F^*)$ so that $||L^*x_0^*|| = ||L^*||$ and* (b) *$L^*(x_0^*)$ is a smooth point in E^*.*

In Theorem 2 of this note, we use this theorem to characterize those continuous function spaces $C(H, E)$ whose duals contain smooth points, and we give an explicit representation of those compact linear operators $T : C(H, E) \to F$ which are smooth points in $K(C(H, E), F)$. The paper then concludes with a proposition which shows how a deep geometrical result of James [10]—together with a recent result of Diestel and Seifert [6]—can be used to easily obtain a characterization of weakly compact operators $T : C(H) \to F$.

The general setting is as follows. Each of E and F is a Banach space, H is a compact Hausdorff space, and $C(H, E)$ is the Banach space (sup norm) of all continuous E-valued functions defined on H. If E is the scalar field, we shorten the notation to $C(H)$. If $T : C(H, E) \to F$ is a continuous linear map (= operator), then the Riesz Representation Theorem asserts that there is a unique finitely additive vector measure $m : \Sigma \to B(E, F^{**})$ on the Borel σ-algebra Σ of H with values in the space of operators from E to the bidual of F such that (i) m has finite semivariation [7, Chapter 1], (ii) $|m_z| \in \mathrm{rca}(\Sigma)$ (= the Banach space of all regular countably additive real-valued measures on Σ) for each $z \in F^*$ (here $|m_z|$ is the total variation of the measure $m_z : \Sigma \to E^*$ defined by $m_z(A)(x) = z(m(A)(x))$), and (iii) $T(f) = \int_H f dm$, $f \in C(H, E)$ (the integral converges in the norm). The reader may consult Goodrich [8], Brooks and Lewis [3], and Batt [1] for a full discussion of this setting. In particular we note that (1) if m is the representing measure for T, then $m(A)x = T^{**}(\xi_A x)$, where ξ_A is the characteristic function of the Borel set A and $x \in E$, and (2) m is countably additive and takes its values in $B(E, F)$ if T is compact or weakly compact. Further, if T is a

Received by the editors on April 15, 1977, and in revised form on March 13,1978.
≳Supported in part by NSF grant MCS 77-21857.
AMS (MOS)subject classification: 46G10, 28A45.

linear functional, then m maps to E^*, and the semivariation coincides with the total variation of m [7, Chapter 1]. In fact, $\|T\| = |m|(H)$ and $|m| \in \text{rca}\ (\Sigma)$; consequently, $C(H, E)^*$ may be identified isometrically with rcabv (Σ, E^*) in this case.

A non-zero element x of the Banach space X is called a smooth point if the Gateaux derivative $D(x, y)(= \lim_{t\to 0}(\|x + ty\| - \|x\|)/t,\ t$ real) of the norm at x in the direction y exists for all $y \in X$. Equivalently, $o \neq x$ is a smooth point if there is a unique (real) linear functional $x^* \in S(X^*)$ ($=$ unit sphere of X^*) such that $x^*(x) = \|x\|$. Since the notion of smoothness is essentially a "real" concept, we shall assume all Banach spaces to be defined over the real field. We do note, however, that Theorem 2 is valid (with the same proof) when the scalar field is assumed to be the complex numbers. An exposition of smoothness and differentiability may be found in Chapter II of Diestel [5].

We remark here that the study of Gateaux differentiability of the norm in spaces of measures has recently been profitable in obtaining information about these spaces. We specifically mention [2] in which the connection between differentiability and absolute continuity proves to be the key idea in establishing the relationships between the Hewitt-Yosida decomposition theorem and othogonality and [4] in which differentiability is used to characterize weakly compact operators on $C(H)$.

In the sequel Σ will denote the Borel σ-algebra of the compact Hausdorff space H By fabv(Σ, F) we shall mean the Banach space (total variation norm) of all finitely additive F-valued measures of bounded variation defined on Σ.

LEMMA 1. *If μ, $\nu \in$ fabv(Σ, F) and $D(\mu, \nu)$ exists, then $\nu \ll \mu$.*

PROOF. The Lebesgue decomposition of Rickart [11], $\nu = \nu_a + \nu_s$, has the property that for t real,

$$\|\mu + t\nu\| = \|\mu + t\nu_a\| + \|t\nu_s\|.$$

The one sided derivatives of the norm are

$$D^+(\mu, \nu) = D^+(\mu, \nu_a) + \|\nu_s\|,$$

$$D^-(\mu, \nu) = D^-(\mu, \nu_a) - \|\nu_s\|.$$

The conclusion then follows from the observation that $D^- \leqq D^+$ due to the convexity of the norm.

We remark that if F is the real numbers, then $D(\mu, \nu)$ exists if and only if $\nu \ll \mu$ [2]. In the remainder of the paper, we let μ denote the representing measure of T.

THEOREM 2. *The dual of $C(H, E)$ possesses smooth points if and only if H is countable and E^* possesses smooth points. If T is a smooth point in $K(C(H, E), F)$, then $H = \{h_n\}$ is countable, and there is a countable, set $\{L_n\}$ of compact operators from E to F so that $T(f) = \Sigma L_n(f(h_n))$, $f \in C(H, E)$, ΣL_n is unconditionally convergent, and $\|T\| = \sup\{\|L_n(x_n)\| \leqq 1$ for each $n\}$.*

PROOF. Suppose that T is a smooth point in $K(C(H, E), F)$ and that y^* is the essentially unique element of $S(F^*)$ so that $\|T^*y^*\| = \|T^*\|$. Then $T^*(y^*) = y^*\mu$ is smooth point in $C(H, E)^*$. Let ν be a vector measure in $C(H, E)^*$. Then ν has finite variation $|\nu|$ and $|\nu| \ll |y^*\mu|$ by the lemma. Now let $h \in H$, let δ_h be the point mass at h, and let x^* be a norm one functional in E^*. Then $\delta_h x^* \in C(H, E)^*$ and $|\delta_h x^*| \ll |y^*\mu|$; hence $|y^*\mu|(h) > 0$. Since $|y^*\mu|$ is bounded and countably additive, $H = \{h_n\}$ must be countable. Now let $L_n = \mu(h_n) : E \to F$. Since T is a compact operator, L_n is compact for each n; e.g., see [1] or [3]. The countable additivity of m (since T is compact), the strong convergence of the integral [8], and the countability of H imply that ΣL_n is unconditionally convergent and that $T(f) = \Sigma L_n(f(h_n))$, $f \in C(H, E)$. The statement about the norm of T follows from Corollary 1 of Batt [1].

Our next assertion is that $L_n^*(y^*)$ is a smooth point in E^* for each n for which $L_n^*(y^*) \neq 0$. Suppose to the contrary that $0 \neq L_1^*(y^*)$ and $L_1^*(y^*)$ is not smooth. Hence the measure $\delta_{h_1} L_1^*(y^*)$ is not smooth in rcabv(Σ, E^*). Let u_1 and v_1 be different norm one functionals such that $u_1(\delta_{h_1} L_1^*(y^*)) = v_1(\delta_{h_1} L_1^*(y^*)) = \|L_1^*(y^*)\|$; for $n > 1$, let u_n be a norm one functional such that $\|u_n(\delta_{h_n} L_n^*(y^*)) = \|L_n^*(y^*)\|$. For $\nu \in$ rcabv(Σ, E^*), define $\phi(\nu)$ to be $\Sigma_{n=1}^{\infty} u_n(\delta_{h_n}\nu(h_n))$, and define $\tau(\nu)$ to be $v_1(\delta_{h_1}\nu(h_1)) + \Sigma_{n>1} u_n(\delta_{h_n}\nu(h_n))$. Then $\|\phi\| = \|\tau\| = 1$, $\phi \neq \tau$, and $\phi(T^*(y^*)) = \tau(T^*(y^*)) = \|T^*(y^*)\|$, a contradiction. The smoothness of $L_n^*(y^*)$ follows, $n = 1, 2, \cdots$.

To complete the proof of Theorem 1, we suppose that $H = \{h_n\}$ is countable and that x^* is a smooth point in E^*, $\|x^*\| = 1$, and we show that the measure defined by $\eta \otimes x^*(A) = \eta(A)x^*$ is smooth in $C(H, E)^*$, where $\eta(h_n) > 0$ for each n and $\Sigma\eta(h_n) = 1$. Let ϕ, $\tau \in C(H, E)^{**}$ so that $\|\phi\| = \|\tau\| = \phi(\eta \otimes x^*) = \tau(\nu \otimes x^*) = \eta \otimes x^* = 1$. Since H is countable (we suppose that H is infinite for definiteness), it is not difficult to see that $C(H, E)^*$ is isometrically isomorphic to $\ell^1(E^*)$, the space of absolutely summable sequences from E^*. Hence $C(H, E)^{**} \simeq \ell^\infty(E^{**})$. Let (x_i^{**}) and (y_i^{**}) be bounded sequences from E^{**} such that (1) $\Sigma x_i^{**}(\mu(h_i)) = \phi(\mu)$ and $\Sigma y_i^{**}(\mu(h_i)) = \tau(\mu)$ for each $\mu \in C(H, E)^*$ and (2) $\sup\{\|x_i^{**}\|\} = \sup\{\|y_i^{**}\|\} = 1$. Then $\phi(\eta \otimes x^*) = \Sigma\eta(h_i)x_i^{**}(x^*) = 1$ and $\tau(\eta \otimes x^*) = \Sigma\eta(h_i)y_i^{**}(x^*) = 1$. Since $|x_i^{**}(x^*)| \leqq 1$, $|y_i^{**}(y^*)| \leqq 1$, and $\eta(h_i) > 0$ for $i \geqq 1$, the fact that $\Sigma\eta(h_i) = 1$ forces $x_i^{**}(x^*) = y_i^{**}(x^*) = 1$ for each i. The smoothness of x^* then implies that $x_i^{**} = y_i^{**}$ for all i, and $\phi = \tau$. Thus $\eta \otimes x^*$ is smooth, and the theorem follows.

We now use the preceding results to construct an infinite dimensional example of spaces $C(H, E)$ and F for which there is a smooth point in $K(C(H, E), F)$. Let $H = \{h_n\}$ be a countably infinite compact Hausdorff space, let $E = F = \ell^2$, suppose that $\mu(h_n) > 0$ for each n, $\Sigma\mu(h_n) = 1$, and put $\nu(A) = \Sigma_{n \in A}\mu(h_n)P_n$, where P_n is the projection onto the first n components of elements of ℓ^2. Let T be the compact operator represented by ν, and suppose that $y^* \in \ell^2$, $\|y^*\| = 1$. Then $\|T^*y^*\| = |y^*\nu|(H) = \Sigma\mu(h_n)\|P_n(y^*)\|$. It is clear that this last infinite series is maximized precisely when $\|P_n(y^*)\| = 1$ for each n. Hence $y^* = (1, 0, 0, \cdots)$ or $(-1, 0, 0, 0, \cdots)$(suppose the former), and T^* achieves its norm at an essentially unique $y^* \in S(F^*)$. To see that $T^*(y^*)$ is smooth in $C(H, E)^*$, we note that $y^*\nu = \mu \otimes (1, 0, 0, \cdots)$ and refer to the proof of Theorem 1.

We remark that this example also shows that the operators $L_n : E \to F$, which appear in the representation of the operator T in Theorem 2, need not be smooth points in $K(E, F)$, i.e., $\mu(h_n)P_n$ fails to be smooth for each $n > 1$.

We conclude with a characterization of weakly compact operators.

PROPOSITION 3. *The following are equivalent*:

(i) *the operator T from $C(H)$ to F is weakly compact*;

(ii) *x^{***} attains its supremum on $\mu(\Sigma)$ for each $x^{***} \in F^{***}$*;

(iii) *each sequence contained in $\mu(\Sigma)$ has a subsequence whose arithmetic means converge.*

PROOF. We show that (i) is equivalent to (ii) and (i) is equivalent to (iii).

Suppose that $T : C(H) \to F$ is weakly compact and that $x^{***} \in F^{***}$. Then $\mu : \Sigma \to F$ $(\subset F^{**})$ is countably additive [3], and consequently there is a Hahn decomposition $H^+ \cup H^-$ of H relative to $x^{***}\mu$. Clearly $x^{***}\mu(H^+) = \sup\{x^{***}\mu(A) : A \in \Sigma\}$.

Conversely, suppose that (ii) holds. But then each x^{***} attains its supremum on $\overline{\text{co}}(\mu(\Sigma))$, and $\overline{\text{co}}(\mu(\Sigma))$ is weakly compact by James' theorem [10]. Since $T(U) \subset \overline{\text{co}}(\mu(\Sigma))$, where U is the unit ball in $C(H)$, it then follows that T is weakly compact.

That (i) implies (iii) is precisely the content of Diestel and Seifert [6]. Therefore, to complete the proof, it suffices to demonstrate that (iii) implies (i). And to achieve this, we use an argument similar to the proof of Theorem 2, p. 82 of Diestel [5] to verify the assertion: if K is a bounded subset of a Banach space X so that each sequence from K has a subsequence whose arithmetic means converge, then K is conditionally weakly compact. In fact, if $x^* \in X^*$, $\alpha = \sup\{x^*(x) : x \in K\}$, and $(x_n) \subset K$ so that $x^*(x_n) \longrightarrow \alpha$, then we may assume that $(1/m)(\sum_{n=1}^{m} x_n) \xrightarrow{m} x_0 \in \overline{\mathrm{co}}(K)$. Then $x^*(x_0) = \alpha = \sup\{x^*(x) : x \in \overline{\mathrm{co}}(\mathrm{K})\}$; by another application of James' theorem, $\overline{\mathrm{co}}(K)$ is weakly compact.

REMARK. If the Banach spaces are defined over the complex field $\mathbf{C}$, then one may replace (ii) by the following statement: (ii)$'$ if $x^{***} \in F^{***}$, then there is an $A \in \Sigma$ such that $|x^{***}\mu(A)| = \sup\{|x^{***}\mu(B)| : B \in \Sigma\}$. To see that (i) implies (ii)$'$ one may use Liapounoff's theorem which asserts that the range of $x^{***}\mu$ is a compact, convex set. And to see that (ii)$'$ implies (i) we note that (ii)$'$ implies that every continuous real functional achieves its sup on $\{\alpha\mu(A) : A \in \Sigma, \alpha \in \mathbf{C}, |\alpha| = 1\}$.

REFERENCES

1. J. Batt, *Applications of the Orlicz-Pettis theorem to operator valued measures and compact and weakly compact linear transformations on the space of continuous functions*, Rev. Roum. Math. **14** (1969), 907–935.

2. R. Bilyeu and P. Lewis, *Orthogonality and the Hewitt-Yosida theorem in spaces of measures*, Rocky Mtn. J. Math. **7** (1977), 629–638.

3. J. Brooks and P. Lewis, *Linear operators and vector measures*, Trans. Am. Math. Soc. **192** (1974), 139–162.

4. ———, *Operators on continuous function spaces and convergence in the space of operators*, Advances in Math. **29** (1978, 157–177.

5. J. Diestel, *Geometry of Banach spaces: selected topics*, Lecture Notes in Math. No. 485, Springer-Verlag, 1975.

6. J. Diestel and C. Seifert, *An averaging property of the range of a vector measure*, Bull. Am. Math. Soc. **82** (1976), 907–909.

7. N. Dinculeanu, *Vector measures*, Pergamon Press, Berlin, 1967.

8. K. Goodrich, *A Riesz representation theorem*, Proc. Am. Math. Soc. **24** (1970), 629–636.

9. S. Heinrih, *The differentiability of the norm in spaces of operators*, Funkcional. Anal. i Prilozen (4) **9** (1975), 93–94. (English translation: Functional Anal. Appl. (4) **9** (1975), 360–362.

10. R. James, *Reflexivity and the supremum of linear functionals*, Israel J. Math. **13** 1972), 289–300.

11. C. Rickart, *Decompositions of additive set functions*, Duke Math. J. **10** (1943), 353–665.

DEPARTMENT OF MATHEMATICS, NORTH TEXAS STATE UNIVERSITY,
DENTON, TEXAS 76203