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PRODUCTS OF GENERALIZED METRIC SPACES

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ABSTRACT. In this paper we investigate the productivity of many classes of generalized metric spaces; including M-spaces, wM-spaces, $w\Delta$-spaces, quasi-complete spaces, Σ-spaces and β-spaces. The concept of a weakly-i space is introduced for the purpose of showing that, in the class of weakly-i spaces, each of the above classes of spaces are countably productive. The class of weakly-i spaces is general enough to include both the k-spaces and the weakly $\delta\theta$-refinable spaces. Further, it is shown that the product of two M-spaces (wM-spaces, quasi-complete spaces or Σ-spaces) is an M-space (wM-space, quasi-complete space or Σ-space) if one of the factors is a weakly-i space. Also, examples are cited which show that most of the above mentioned classes are not finitely productive and, even in the presence of fairly strong conditions, are not preserved by uncountable products.

1. **Introduction.** In [19] Isiwata presents an example of two completely regular countably compact spaces whose product is not a q-space. That example when considered in a more general context, shows that many other well-known classes of spaces are not finitely productive. The most general consequence is best illustrated by introducing the following notion:

Let $\mathscr{Q}$ be any class of spaces satisfying the following two conditions:

(a) If X is a completely regular countably compact space, then $X \in \mathscr{Q}$.

(b) If $X \in \mathscr{Q}$, then X is a q-space.

Any such class of spaces will be called a *class of type* $\mathscr{Q}$. Of course, by Isiwata's example, any class of type $\mathscr{Q}$ is not finitely productive.

In light of Isiwata's example, it is natural to consider the following class of problems: Let $\mathscr{P}$ be a class of type $\mathscr{Q}$. Find a property p such that if X, $Y \in \mathscr{P}$ and satisfy property p, then $X \times Y \in \mathscr{P}$. In addition, of course, we would like the same property p to work for numerous

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classes of type $\mathscr{Q}$ and to be a fairly general property which is implied by a number of important or frequently investigated properties. Problems of this sort have previously been investigated by several authors, especially for countably compact and M-spaces (c.f. [10], [18], [20], [32] and [33]) and more recently for several other classes of type $\mathscr{Q}$ in [15]. In this paper we exhibit a particular property p which we use to provide solutions to the above mentioned problems in many classes of type $\mathscr{Q}$. In fact, the property p which we exhibit will be used to provide affirmative answers to the following questions for various classes of spaces $\mathscr{P}$.

QUESTION 1.1. Let $\mathscr{P}$ be a class of spaces. If $\langle X_n \rangle$ is a sequence of spaces such that each $X_n \in \mathscr{P}$ and satisfies property p, does $\prod_{n=1}^{\infty} X_n \in \mathscr{P}$?

QUESTION 1.2. Let $\mathscr{P}$ be a class of spaces. If X, $Y \in \mathscr{P}$ and X satisfies property p, does $X \times Y \in \mathscr{P}$?

The property p which we introduce will be general enough to be implied by an extensive number of properties which have been previously studied in the literature. Unfortunately, as will be readily seen in § 3 and § 4, the author has not succeeded in answering Question 1.2 for some classes of spaces $\mathscr{P}$ for which he has provided affirmative answers to Question 1.1.

In § 2 we introduce the various concepts and terminology used throughout this paper. Affirmative answers to Questions 1.1 and 1.2 for various classes of type $\mathscr{Q}$ are presented in § 3. In § 4 we discuss the productivity of several classes of spaces which are not of type $\mathscr{Q}$. Finally, in § 5 we exhibit a few examples and present a few open questions.

Unless otherwise stated, all spaces are assumed to be T_1. The positive integers will be denoted by N. A sequence $x_1, x_2, x_3, \cdots$ of points will be denoted by $\langle x_n \rangle$, a sequence $\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3, \cdots$ of covers of a space X will be denoted by $\langle \mathscr{U}_n \rangle$ and a sequence $X_1, X_2, X_3, \cdots$ of spaces will be denoted by $\langle X_n \rangle$.

2. **Preliminaries and definitions.** If $\mathscr{U}$ is a collection of subsets of a space X and $x \in X$, we define $St(x, \mathscr{U})$, $St^2(x, \mathscr{U})$ and $c(x, \mathscr{U})$ as follows:

$$St(x, \mathscr{U}) = \cup \{U \in \mathscr{U} : x \in U\},$$

$$St^2(x, \mathscr{U}) = \cup \{U \in \mathscr{U} : U \cap St(x, \mathscr{U}) \neq \phi\} \text{ and}$$

$$c(x, \mathscr{U}) = \cap \{U \in \mathscr{U} : x \in U\}.$$

Also, if $A \subset X$, we define $St(A, \mathscr{U})$ by $St(A, \mathscr{U}) = \cup \{U \in \mathscr{U} : A \cap U \neq \phi\}$. The collection $\{St(U, \mathscr{U}) : U \in \mathscr{U}\}$ will be denoted by $\mathscr{U}^*$.

If $\mathscr{U}$ *and* $\mathscr{V}$ are collections of subsets of a space X, we say that $\mathscr{U}$ is a *refinement* of $\mathscr{V}$, denoted as $\mathscr{U} < \mathscr{V}$ or $\mathscr{V} > \mathscr{U}$, if for every $U \in \mathscr{U}$ there is a $V \in \mathscr{V}$ such that $U \subset V$. A sequence $\langle \mathscr{U}_n \rangle$ of covers of a space X such that $\mathscr{U}_1 < \mathscr{U}_2 < \mathscr{U}_3 < \cdots$ is called a *refining sequence.* Further, if $\mathscr{U}_1 < \mathscr{U}_2{}^* < \mathscr{U}_2 < \mathscr{U}_3{}^* < \cdots$, the sequence $\langle \mathscr{U}_n \rangle$ is called a *normal sequence.*

Let $\langle \mathscr{U}_n \rangle$ be a sequence of covers of a space X, and, for each $x \in X$, consider the following conditions on $\langle \mathscr{U}_n \rangle$:

(1) $x_n \in St(x, \mathscr{U}_n)$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

(2) $x_n \in St^2(x, \mathscr{U}_n)$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

(3) $\{x_i : i \geqq n\} \cup \{x\} \subset U_n \in \mathscr{U}_n$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

(4) $x_n \in c(x, \mathscr{U}_n)$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

A space X with a refining sequence $\langle \mathscr{U}_n \rangle$ of open covers satisfying (1), (2) or (3) is called a *$w\Delta$-space* [4], *wM-space* [17], or a *quasi-complete space* [9], respectively. The sequence $\langle \mathscr{U}_n \rangle$ is called a *$w\Delta$-sequence, wM-sequence,* or a *quasi-complete sequence,* respectively. A space with a normal $w\Delta$-sequence is called an *M-space* [26] and the sequence $\langle \mathscr{U}_n \rangle$ is called an *M-sequence.* A space X is called an *M^*-space* [16] (*$M^\#$-space* [34]) if there is a refining sequence $\langle \mathscr{U}_n \rangle$ of locally finite (closure preserving) closed covers of X satisfying (1). A space X with a refining sequence $\langle \mathscr{U}_n \rangle$ of locally finite (closure preserving) closed covers satisfying (4) is called a *Σ-space* [27] (*$\Sigma^\#$-space* [23]). The sequence $\langle \mathscr{U}_n \rangle$ will be referred to as an *M^*($M^\#$, Σ or $\Sigma^\#$)-sequence,* respectively.

Let $(X, \mathscr{T})$ be a space and let g be a function from $N \times X$ into $\mathscr{T}$. Then g is called a *COC-function for X* if it satisfies the following two conditions: (1) $x \in \bigcap_{n=1}^{\infty} g(n, x)$ for all $x \in X$; (2) $g(n + 1, x) \subset g(n, x)$ for all $n \in N$ and $x \in X$. The notion of a COC-function was introduced in [1].

Now let X be a space with a COC-function g, and for each $x \in X$, consider the following conditions on g:

(a) $x_n \in g(n, x)$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

(b) $x \in g(n, x)$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

(c) $\{x, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, x)$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

(d) $y_n \in g(n, x)$ and $x_n \in g(n, y_n)$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

(e) $g(n, x) \cap g(n, x_n) \neq \emptyset$ for each $n \in N$ implies the sequence $\langle x_n \rangle$ has a cluster point.

If X is a space with *COC*-function g satisfying (a), then X is called a *q-space* [22]; X is called a *β-space* [13] if g satisfies (b); and X is called a *$w\theta$-space, $w\gamma$-space* or *wN-space* if g satisfies (c), (d) or (e), respectively [14]. The *COC*-function g will be referred to as a *q (β, $w\theta$, $w\gamma$ or wN)-function,* respectively. If in (b), (c), and (d) we require the sequence $\langle x_n \rangle$ to cluster to x, then we have a *semi-stratifiable space, θ-space* and *γ-space,* respectively (see [14]).

For the basic implications between the various classes of spaces introduced above, the reader is referred to [11], [14], [25] and [30]. In order to determine which of those classes are classes of type $\mathscr{Q}$, a few comments are necessary. The space X in Example 5.1 shows that the class of semi-stratifiable spaces is not a class of type $\mathscr{Q}$. Since the space X in Example 5.5 is not first countable, X is neither a θ-space nor a γ-space (see [14]). Thus, neither the class of θ-spaces nor the class of γ-spaces are classes of type $\mathscr{Q}$. The space $N \cup \{p\}$, where $p \in \beta N - N$(βN being the Stone-Čech compactification of N) is an example of a Σ-space (hence a $\Sigma^{\#}$-space and a β-space) which is not a q-space. Thus, the classes of Σ-spaces, $\Sigma^{\#}$-spaces and β-spaces are not classes of type $\mathscr{Q}$. Each of the remaining classes of spaces introduced above are defined by requiring that certain sequences have cluster points. Thus, it is immediate that the class of countably compact spaces is contained in each of those classes of spaces. It also follows immediately from the definitions that each of those classes is contained in the class of q-spaces and are therefore classes of type $\mathscr{Q}$.

As will be shown in § 3, the following property p of a space X provides affirmative answers to Questions 1.1 and 1.2 for many classes of type $\mathscr{Q}$. A space X will be called *weakly-i* if given $F \subset X$, $F \cap C$ is finite for all closed isocompact subsets C of X implies that F is closed. According to Bacon [3], a space X is called *isocompact* if every closed countably compact subset of X is compact. The notion of a weakly-i space is motivated by that of a weakly-k space introduced in [32]. Recall that a space X is called *weakly-k* if given $F \subset X$, $F \cap C$ is finite for all closed compact subsets C of X implies that F is closed.

That the property of being a weakly-i space is indeed a very weak condition to impose on a topological space is the content of the following:

THEOREM 2.1. (a) *Every weakly-k space is a weakly-i space.*
(b) *Every isocompact space is a weakly-i space.*

PROOF. The implication in (a) follows immediately from the definitions. To see (b), suppose X is an isocompact space and $F \subset X$ is not closed. Now $\overline{F}$ is isocompact and, since F is not closed, $\overline{F} \cap F = F$ is not finite. Thus X is a weakly-i space.

As an immediate consequence of Theorem 2.1, each of the following classes of spaces are contained in the class of weakly-i spaces. The redundancies are intentional.

(A) First countable spaces; (countably) bi-sequential spaces; Fréchet spaces; sequential spaces; spaces of pointwise-countable type; (strongly) k-spaces; (countably, singly) bi-k spaces; k-spaces.

(B) Paracompact spaces; subparacompact spaces [**6**]; metacompact spaces, (weakly) θ-refinable spaces; meta-Lindelöf spaces; (weakly) $\delta\theta$-refinable spaces.

To see that each of the concepts in (A) implies weakly-i, we note that every k-space is clearly weakly-k and then invoke Theorem 2.1. Since each of the remaining classes of spaces in (A) are contained in the k-spaces [**24**], the implications all follow. For the definitions of the various concepts listed in (A) and the implications between them, the reader is referred to the paper of Michael [**24**] where a beautiful discussion is presented.

That each of the concepts in (B) implies weakly-i, follows from the following result of Wicke and Worrell [**36**, Corollary 2.4]: Every weakly $\delta\theta$-refinable, countably compact space is compact. (The same result was stated without proof in Theorem (iv) of Worrell and Wicke [**37**]). Thus every weakly $\delta\theta$-refinable space is isocompact and hence weakly-i. As the reader will observe in [**36**], weak $\delta\theta$-refinability is implied by all those properties in (B).

Another property of interest in determining when various classes of type $\mathcal{D}$ are productive is the notion of a weakly-subsequential space introduced in [**15**]. A space X is called *weakly-subsequential* if each sequence in X which has a cluster point has a subsequence with compact closure. (When we say a subsequence has a compact closure, we will mean, of course, that the range of the subsequence has compact closure.) This concept is implicit in the work of previous authors ([**18**], [**19**], [**28**], [**33**]) and was also used by the author in [**11**].

House [**15**] observes that every weakly-k, T_2-space is weakly subsequential and so all of those concepts listed in (B) imply weakly-subsequential (at least for T_2-spaces). On the other hand, there are paracompact T_2-spaces (hence weakly i-spaces) which are not weakly-subsequential (see Example 5.2). Thus, while weakly-subsequential spaces do generalize those concepts in (A), they have the disadvantage of not generalizing those concepts in (B). However, as the following result shows, this only becomes important when studying those classes of spaces which are not of type $\mathcal{D}$.

THEOREM 2.2. *For a regular q-space* $(X, \mathscr{T})$, *the following are equivalent*:

(a) X *is weakly-subsequential.*

(b) X *is weakly-k.*

(c) X *is weakly-i.*

PROOF. That (a) $\Rightarrow$ (b) is the content of [15, Theorem 3.7] and (b) $\Rightarrow$ (c) is Theorem 2.1 (a). Thus it suffices to show (c) $\Rightarrow$ (a). Let $g: N \times X \to T$ be a q-function for X. By regularity, for each $x \in X$ we can construct a q-function $h: N \times X \to \mathscr{T}$ such that, for each $x \in X$, $h(1, x) \subset \overline{h(1, x)} \subset g(1, x)$ and for $n \geqq 2$, $h(n, x) \subset \overline{h(n, x)} \subset h(n-1, x) \cap g(n, x)$. Let $\langle x_n \rangle$ be a sequence in X with a cluster point p. Then for each k, there is an $n_k \geqq k$ such that $x_{n_k} \in h(k, p)$. Since h is a q-function, the sequence $\langle x_{n_k} \rangle$ has a cluster point s which belongs to $\bigcap_{n=1}^{\infty} h(n, p)$. Let $F = \{x_{n_k} : k \in N\}$ and note that if $F - \{s\}$ is closed, there must be a subsequence of $\langle x_{n_k} \rangle$ all of whose terms are s, so that $\langle x_{n_k} \rangle$ has a subsequence with compact closure.

On the other hand, suppose $F - \{s\}$ is not closed. Since X is weakly-i, there exists a closed isocompact subset C of X such that $C \cap (F - \{s\}) = \{z_j : j \in N\}$, where $\langle z_j \rangle$ is an infinite subsequence of $\langle x_{n_k} \rangle$. Now let $K = F \cup (\bigcap_{n=1}^{\infty} h(n, p))$ and note that by the conditions imposed on the q-function h, K is a closed countably compact subset of X. Since $\overline{\{z_j : j \in N\}} \subset K$, $\overline{\{z_j : j \in N\}}$ is closed and countably compact. But $\overline{\{z_j : j \in N\}} \subset C$ and, since C is isocompact, $\{z_j : j \in N\}$ has compact closure. Thus X is weakly-subsequential.

We note that regularity is only needed for (c) $\Rightarrow$ (a). In fact, (a) and (b) are equivalent for T_2-spaces [11].

COROLLARY 2.3. *Let* $\mathscr{P}$ *be any class of regular spaces of type* $\mathscr{Q}$; *then the following are equivalent for any* $X \in \mathscr{P}$:

(a) X *is weakly-subsequential.*

(b) X *is weakly-k.*

(c) X *is weakly-i.*

It follows that for regular spaces of type $\mathscr{Q}$, the presence of any of the properties listed in (B) implies the space is weakly subsequential. In addition, for spaces of type $\mathscr{Q}$, each of the properties listed below implies first countability [21] and hence implies the space is weakly-i and weakly subsequential:

(C) Point-countable separating open cover [13]; G_δ-diagonal; $G_\delta{}^*$-diagonal [13]; points are G_δ's.

The author wishes to thank F. Siwiec for his suggestion on using weakly-i spaces. Although the proofs of the theorems remain unaltered, this concept improves upon that of weakly-θk originally used in [11].

3. **Products of classes of type $\mathscr{D}$.** The results of this section provide affirmative answers to Questions 1.1 and 1.2 for various classes of type $\mathscr{D}$ when property p is weak subsequentiality. For some classes of type $\mathscr{D}$ where weak subsequentiality provides an affirmative answer to Question 1.1 we were not able to provide affirmative answers to the corresponding Question 1.2. However, by using a stronger concept than weak subsequentiality in the hypothesis of Theorem 3.3, affirmative answers are provided for some of those classes. We wish to impress upon the reader that, although weak subsequentiality appears in the hypothesis of Theorems 3.1 and 3.2, it could validly be replaced by regular weakly-i space or by any of those concepts listed in (A), (B), or (C) of Section 2 (at least for regular spaces). Even when some of these stronger concepts are used, many of the results are new.

The major results of this section are listed below; the proofs will be given after the discussion following Theorem 3.3.

THEOREM 3.1. *Let $\langle X_n \rangle$ be a sequence of weakly-subsequential spaces and let $X = \prod_{n=1}^{\infty} X_n$.*

(1) *If each X_n is countably compact, then so is X.*

(2) *If each X_n is an M-space (M^*-space, $M^{\#}$-space), then so is X.*

(3) *If each X_n is a wM-space, then so is X.*

(4) *If each X_n is a $w\Delta$-space, then so is X.*

(5) *If each X_n is quasi-complete, then so is X.*

(6) *If each X_n is a wN-space, then so is X.*

(7) *If each X_n is a $w\gamma$-space, then so is X.*

(8) *If each X_n is a $w\theta$-space, then so is X.*

(9) *If each X_n is a q-space, then so is X.*

REMARKS. The result in (1) is due to Saks and Stephenson [33]; in fact, they show we may take a product of up to $\aleph_1$ factors instead of only countably many. On the other hand, Example 5.6 shows that an uncountable product of metrizable spaces need not be a q-space. Thus we cannot take more than countably many factors in (2)–(9). The results in (4), (6) and (9) were obtained in [15, Theorem 3.11] with (4) being obtained independently by the author in [11]. The technique used in [15] is somewhat different than that used in this paper and in [11].

THEOREM 3.2. *Let X and Y be spaces with X weakly subsequential.*

(1) *If X and Y are countably compact, then so is $X \times Y$.*

(2) *If X and Y are M-spaces (M^*-spaces, $M^{\#}$-spaces), then so is $X \times Y$.*

(3) *If X and Y are wM-spaces, then so is $X \times Y$.*

(4) *If X and Y are quasi-complete, then so is $X \times Y$.*

(5) *If X and Y are q-spaces, then so is $X \times Y$.*

In (4) and (5), Y is assumed to be a regular space.

We note that missing from the peceding theorem are the analogous results for $w\Delta$-spaces, $w\gamma$-spaces, $w\theta$-spaces and wN-spaces. The reason for this is that the technique used to prove (2)–(6) requires constructing a certain closed countably compact subset of Y. The analogous construction fails in the other situations. However, using a concept from [15], we obtain a somewhat weaker result in Theorem 3.3.

A space X is called *subsequential* if each sequence in X which has a cluster point has a convergent subsequence. This concept was also used in [11] but was not given a name.

THEOREM 3.3. *Let X and Y be spaces with X subsequential.*

(1) *If X and Y are $w\Delta$-spaces, then so is $X \times Y$.*

(2) *If X and Y are $w\gamma$-spaces, then so is $X \times Y$.*

(3) *If X and Y are $w\theta$-spaces, then so is $X \times Y$.*

(4) *If X and Y are wN-spaces, then so is $X \times Y$.*

REMARKS. (1) The author does not know exactly which of the properites in (A), (B) and (C) of Section 2 can validly replace subsequentiality; however, the following replacements are possible. First, note that each of the classes of spaces in (1)–(4) of Theorem 3.3 are spaces of type $\mathscr{Q}$, and so any of those properties listed in (C) could replace subsequentiality. Also, it is clear that every Fréchet space is subsequential, so that any of those properties preceding Fréchet spaces in (A) could replace subsequentiality.

(2) The following remark holds only for $w\Delta$-spaces as Example 5.3 will show. If we assume both X and Y are regular θ-refinable spaces, it follows from Remark 1.9 of [5] and Theorem 4.10 in [15] that $X \times Y$ is a $w\Delta$-space. Further, we can replace θ-refinability by any of those properties preceding it in (B).

We now turn to the proofs of Theorems 3.1–3.3. In the proofs of those theorems, we will present a proof of only one part and simply indicate how the other parts follow using the same technique. Three comments concerning notational conventions used in the proofs are necessary. By a point x in the product space $X = \prod_{n=1}^{\infty} X_n$ we will always mean $x = (x_1, x_2, x_3, \cdots)$ where $x_j \in X_j$ for each $j \in N$. By a sequence $\langle x(n) \rangle$ in the product space $X = \prod_{n=1}^{\infty} X_n$ we will mean the sequence $x(1), x(2), x(3), \cdots$ of points of X. By the first convention, for each $n \in N$, $x(n) = (x_1(n), x_2(n), x_3(n), \cdots)$ where $x_j(n) \in X_j$ for each $j \in N$. Also, recall that a sequence is said to have compact closure if the range of the sequence has compast closure. A similar convention is adopted when we say a sequence has countably compact closure.

PROOF OF THEOREM 3.1. We give a proof of the result for wN-spaces in (6). For (6)–(9) let g_n be a COC-function for X_n. For each $i \in N$ and

$x \in X$, put

$$g(i, x) = g_1(i, x_1) \times g_2(i, x_2) \times \cdots \times g_i(1, x_i) \times \prod_{j>i} X_j.$$

It is easy to verify that g is a COC-function for X.

(6) Suppose g_i is a wN-function for X_i and let $\langle x(n) \rangle$ be a sequence in X such that $g(n, x) \cap g(n, x(n)) \neq \emptyset$ for each $n \in N$ and some point $x \in X$. It follows that $g_1(n, x_1) \cap g_1(n, x_1(n)) \neq \emptyset$ for each $n \in N$ and, since g_1 is a wN-function for X_1, that the sequence $\langle x_1(n) \rangle$ has a cluster point in X_1. Since X_1 is weakly-subsequential, there is an infinite subset $N_1 \subset N$ such that $\langle x_1(j) : j \in N_1 \rangle$ has compact closure in X_1. Let $n_1 = \inf \{j : j \in N_1\}$ and consider the subsequence $\langle x_2(j) : j \in N_1 - \{n_1\} \rangle$ of $\langle x_2(n) : n \in N \rangle$. Since $g_2(j, x_2) \cap g_2(j, x_2(j)) \neq \emptyset$ for every $j \in N_1 - \{n_1\}$, the sequence $\langle x_2(j) : j \in N_1 - \{n_1\} \rangle$ has a cluster point in X_2. (It is essential here to observe that, since g_2 is a wN-function for X_2, $g_2 \mid (N - \{n_1\}) \times X_2$ can be considered as a wN-function for X_2). Since X_2 is weakly-subsequential, there is an infinite subset $N_2 \subset N_1 - \{n_1\}$ such that $\langle x_2(j) : j \in N_2 \rangle$ has compact closure in X_2. Let $n_2 = \inf \{j : j \in N_2\}$. Continuing by induction, we can select for each $k > 2$ an infinite subset $N_k \subset N_{k-1} - \{n_{k-1}\}$, with $n_{k-1} = \inf \{j : j \in n_{k-1}\}$, such that $\langle x_k(j) : j \in N_k \rangle$ has compact closure in X_k.

Now, with $n_i = \inf \{j : j \in N_i\}$, consider the subsequence $\langle x(n_i) : i \in N \rangle$ of the sequence $\langle x(n) \rangle$. Let

$$C_1 = Cl_{X_1}\{x_1(j) : j \in N_1\};$$

and

$$C_k = Cl_{X_k}\{x_k(j) : j \in n_k\} \cup \{x_k(j) : j < n_k \text{ and } j \in N_1\} \text{ for } k \geqq 2.$$

Since C_k is a compact subset of X_k for each k, $C = \prod_{k=1}^{\infty} C_k$ is a closed, compact subset of X which contains $\{x(n_i) : i \in N\}$. It follows that $\langle x(n_i) : i \in N \rangle$ is a subsequence of $\langle x(n) \rangle$ with compact closure and so has a cluster point in X. Hence, the sequence $\langle x(n) \rangle$ has a cluster point showing that X is a wN-space.

(7) Suppose g_i is a $w\gamma$-function for X_i and let $\langle x(n) \rangle$ and $\langle y(n) \rangle$ be sequences in X such that, for some point $x \in X$, $y(n) \in g(n, x)$ and $x(n) \in g(n, y(n))$ for each $n \in N$. Then the sequences $\langle y_1(n) \rangle$ and $\langle x_1(n) \rangle$ satisfy $y_1(n) \in g_1(n, x_1)$ and $x_1(n) \in g_1(n, y_1(n))$ for each $n \in N$. Thus, since X_1 is a $w\gamma$-space, the sequence $\langle x_1(n) \rangle$ has a cluster point. The remainder of the argument proceeds exactly as in (6) so that X is a $w\gamma$-space.

(8) Suppose g_i is a $w\theta$-function for X_i and let $\langle x(n) \rangle$ and $\langle y(n) \rangle$ be sequences in X such that, for some point $x \in X$, $\{x, x(n)\} \subset g(n, y(n))$

and $y(n) \in g(n, x)$ for each $n \in N$. Then the sequences $\langle y_1(n)\rangle$ and $\langle x_1(n)\rangle$ satisfy $\{x_1, x_1(n)\} \subset g_1(n, y_1(n))$ and $y_1(n) \in g_1(n, x_1)$ for each $n \in N$. Hence, since X_1 is a $w\theta$-space, the sequence $\langle x_1(n)\rangle$ has a cluster point. The remainder of the argument proceeds exactly as in (6) so that X is a $w\theta$-space.

(9) This is the content of [15, Theorem 3.11 (3)]. An argument analogous to that used in (6) could also be given.

For (2)–(5), let $\langle \mathscr{U}_{n,j}; j \in N\rangle$ be a sequence of covers of X_n. For each $i \in N$, put

$$\mathscr{W}_i = \{U_1 \times U_2 \times \cdots \times U_i \times \textstyle\prod_{k>i} X_k : U_j \in \mathscr{U}_{j,i}, j = 1, 2, \cdots, i\}.$$

(2) The result for M-spaces follows from [18, Theorem 1.3 in II]. Suppose $\langle \mathscr{U}_{n,j} : j \in N\rangle$ *is an* M^*-sequence ($M^\#$-sequence) for X_n. It is easy to verify that $\langle \mathscr{W}_i\rangle$ is a refining sequence of locally finite (closure preserving) closed covers of X. The remainder ot the argument follows exactly as in [18, Theorem 1.3 in II]. (It is essential here to know that, for each $n \in N$, the sequence $\langle \mathscr{U}_{n,i} : i \in N\rangle$ is a refining sequence).

(3) Let $\langle \mathscr{U}_{n,i}\rangle$ be a wM-sequence for X_n and let $\langle x(n)\rangle$ be a sequence in X such that, for some point $x \in X$, $x(n) \in St^2(x, \mathscr{W}_n)$ for each $n \in N$. It is easy to verify that the sequence $\langle x_1(n)\rangle$ satisfies $x_1(n) \in St^2(x_1, \mathscr{U}_{1,n})$ for each $n \in N$ and thus has a cluster point in X_1. The remainder of the argument proceeds exactly as in (6) so that X is a wM-space.

(4) This is the content of [15, Theorem 3.11 (2)]. An argument analogous to that used in (6) could also be given. This result was also obtained independently by the author in [11].

(5) Let $\langle \mathscr{U}_{n,i}\rangle$ be a quasi-complete sequence for X_n and let $\langle x(n)\rangle$ be a sequence in X such that, for some point $x \in X$, $\{x(j) : j \geqq i\} \cup \{x\} \subset W_i \in \mathscr{W}_i$ for each $i \in N$. It is easy to see that the sequence $\langle x_1(n)\rangle$ satisfies $\{x_1(j) : j \geqq i\} \cup \{x_1\} \subset U_1 \in \mathscr{U}_1$ for each $i \in N$ and thus has a cluster point in X_1. The remainder of the argument proceeds exactly as in (6) so that X is a quasi-complete space.

Proof of Theorem 3.2. (1) Let $\langle s_n\rangle$ be a sequence in $X \times Y$. For each n, put $s_n = (x_n, y_n)$ where $x_n \in X$ and $y_n \in Y$. Since X is weakly subsequential $\langle x_n\rangle$ has a subsequence $\langle x_{n_k}\rangle$ with compact closure in Y. Since Y is countably compact, the subsequence $\langle y_{n_k}\rangle$ of $\langle y_n\rangle$ has countably compact closure in Y. It follows that $Cl_X\{x_{n_k} : k \in N\} \times Cl_Y\{y_{n_k} : k \in N\}$ is a closed, countably compact subset of $X \times Y$, and so the subsequence $\langle s_{n_k} : k \in N\rangle$, and hence also $\langle s_n\rangle$ has a cluster point in $X \times Y$. Thus $X \times Y$ is countably compact.

For (2)–(4), let $\langle \mathscr{U}_n\rangle$ and $\langle \mathscr{V}_n\rangle$ be sequences of covers of X and Y, respectively. For each $n \in N$, put $\mathscr{W}_n = \{U \times V : U \in \mathscr{U}_n, V \in \mathscr{V}_n\}$.

(2) The result for M-spaces follows from [18, Theorem 1.1 in II]. Suppose $\langle \mathscr{U}_n \rangle$ and $\langle \mathscr{V}_n \rangle$ are M^*-sequences ($M^\#$-sequences) for X and Y, respectively. It is easy to see that $\langle \mathscr{W}_n \rangle$ is a sequence of locally finite (closure preserving) closed covers of $X \times Y$. That $X \times Y$ is an M^*-space ($M^\#$-space) follows exactly as in [18]. (It is essential to know that since $\langle \mathscr{V}_n \rangle$ is a sequence of closure preserving closed covers of Y, $\bigcap_{n=1}^{\infty} St(y, \mathscr{V}_n)$ is countably compact in Y).

(3) Suppose $\langle \mathscr{U}_n \rangle$ and $\langle \mathscr{V}_n \rangle$ are wM-sequences for X and Y, respectively. Let $\langle s_n \rangle$ be a sequence in $X \times Y$ such that, for each $n \in N$, $s_n \in St^2(s_0, \mathscr{W}_n)$ for some point $s_0 \in X \times Y$. Since $x_n \in St^2(x_0, \mathscr{U}_n)$ for each $n \in N$, the sequence $\langle x_n \rangle$ has a cluster point and, since X is weakly-subsequential, has a subsequence $\langle x_{n_k} \rangle$ with compact closure. On the other hand, the subsequence $\langle y_{n_k} \rangle$ has a cluster point in Y and any such cluster point is in $\bigcap_{n=1}^{\infty} Cl_Y St^2(y_0, \mathscr{V}_n)$ which is closed and countably compact in Y. To see this, note that $Cl_Y St^2(y_0, \mathscr{V}_n) \subset St^3(y_0, \mathscr{V}_n)$ for each $n \in N$ and use [17, Lemma 2.5 in I]. It follows that $Cl_Y \{y_{n_k} : k \in N\}$ is countably compact in Y. It follows, as in (1), that $X \times Y$ is a wM-space.

(4) Let $\langle \mathscr{U}_n \rangle$ and $\langle \mathscr{V}_n \rangle$ be quasi-complete sequences for X and Y, respectively. Since Y is regular, we may assume without loss of generality that $\{Cl_Y V : V \in \mathscr{V}_{n+1}\} < \mathscr{V}_n$ for each $n \in N$. Let $\langle s_n \rangle$ be a sequence in $X \times Y$ such that, for each $n \in N$, $\{s_i : i \geqq n\} \cup \{s_0\} \subset W_n \in \mathscr{W}_n$ for some point $s_0 \in X \times Y$. It is easy to see that the sequence $\langle x_n \rangle$ satisfies $\{x_i : i \geqq n\} \cup \{x_0\} \subset U_n \in \mathscr{U}_n$ for each $n \in N$ and thus has a subsequence $\langle x_{n_k} \rangle$ with compact closure. On the other hand, the subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ has a cluster point in Y, and any such cluster point is in $\bigcap_{n=1}^{\infty} Cl_Y V_n$ which is easily seen to be closed and countably compact in Y. It follows, as in (1), that $X \times Y$ is a quasi-complete space.

(5) Let g and h be q-functions for X and Y, respectively. Since Y is regular, we may assume without loss of generality that $Cl_Y h(n+1, y) \subset h(n, y)$ for each $n \in N$ and $y \in Y$. For each $s = (x, y) \in X \times Y$ and $n \in N$ define $k(n, s) = g(n, x) \times h(n, y)$ and let $s_n \in k(n, s_0)$ for each $n \in N$. It follows that the sequence $\langle x_n \rangle$ has a subsequence $\langle x_{n_k} \rangle$ with compact closure in X. On the other hand, the subsequence $\langle y_{n_k} \rangle$ has a cluster point in Y and any such cluster point is in $\bigcap_{n=1}^{\infty} Cl_Y h(n, y_0)$ which is easily seen to be closed and countably compact in Y. It follows, as in (1), that $X \times Y$ is a q-space.

Before proceeding to the proof of Theorem 3.3, we note that the essential step in the proofs of (2)–(5) of Theorem 3.2 was the construction of a countably compact subset of the space Y. That construction does not seem to work for the concepts in (1)–(4) of Theorem 3.3.

PROOF OF THEOREM 3.3. Since the same technique is used in (1)–(4), we give a proof for (1) and leave (2)–(4) to the reader.

(1) Let $\langle \mathscr{U}_n \rangle$ and $\langle \mathscr{V}_n \rangle$ be $w\Delta$-sequences for X and Y, respectively. For each $n \in N$, let $\mathscr{W}_n = \{U \times V : U \in \mathscr{U}_n, \ V \in \mathscr{V}_n\}$ and let $s_n \in St(s_0, \mathscr{W}_n)$, where $s_n = (x_n, \ y_n) \in X \times Y$ for $n \geqq 0$. The sequence $\langle x_n \rangle$ satisfies $x_n \in St(x_0, \mathscr{U}_n)$ for each $n \in N$ and, since X is a $w\Delta$-space, has a cluster point in X. Since X is subsequential, there is a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ converging to some point $x \in X$. Also, because Y is a $w\Delta$-space, the subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ has a cluster point $y \in Y$. It follows easily that (x, y) is a cluster point of the subsequence $\langle s_{n_k} \rangle$ and hence of $\langle s_n \rangle$. Thus, $X \times Y$ is a $w\Delta$-space.

4. **Products of other generalized metric spaces.** The purpose of this section is to investigate the productivity of some classes of generalized metric spaces which are not of type $\mathscr{D}$. As indicated in Theorem 4.2, many of those classes are countably productive. Except for the concepts of developable space and $\sigma^\#$-space, the concepts discussed in Theorem 4.2 have either been defined in this paper or can be found in the appropriate references referred to in the proof of that theorem. A definition of developable space may be found in [13]; the concept of a $\sigma^\#$-space is defined below.

A space X is called a *$\sigma^\#$-space* [34] if there exists a closed cover $\mathscr{F} = \bigcup_{n=1}^\infty \mathscr{F}_n$ of X, where each $\mathscr{F}_n$ is closure preserving, such that if given distinct points x and y of X, there is an $F \in \mathscr{F}$ such that $x \in F$ and $y \notin F$. In the proof of Theorem 4.2 we use the following characterization of a $\sigma^\#$-space communicated to the author by R. W. Heath. Since a proof of this characterization has not appeared in the literature, a proof is indicated here.

LEMMA 4.1. *A space X is a $\sigma^\#$-space if and only if there is a* COC-*function g for X satisfying*: (1) $\bigcap_{n=1}^\infty g(n, x) = \{x\}$ *for every* $x \in X$; *and* (2) *if* $y \in g(n, x)$, *then* $g(n, y) \subset g(n, x)$.

PROOF. A space with a COC-function g satisfying (1) and (2) is called an α-space in [13]. Proposition 4.3 of [13] shows that every $\sigma^\#$-space is an α-space. For the converse, suppose g is a COC-function for X satisfying (1) and (2). For each $x \in X$ and $n \in N$, let $B_x(n) = X - \bigcup \{g(n, y) : x \notin g(n, y)\}$ and note that each $B_x(n)$ is a closed subset of X. For each $n \in N$, put $\mathscr{B}_n = \{B_x(n) : x \in X\}$. To see that $\mathscr{B}_n$ is closure preserving let $A \subset X$ and suppose $z \notin \bigcup \{B_x(n) : x \in A\}$. For each $x \in A$, $z \notin B_x(n)$ and so there exists some $y \in X$ such that $z \in g(n, y)$ and $x \notin g(n, y)$. By (2), $g(n, z) \subset g(n, y)$ and hence $g(n, z) \subset \bigcup \{g(n, y) : x \notin g(n, y)\}$. Thus, $g(n, z) \cap B_x(n) = \Phi$. It follows that

$g(n, z) \cap (\bigcup \{B_x(n) : x \in A\}) = \phi$ and so $z \notin \overline{\bigcup \{B_x(n) : x \in A\}}$ showing that $\mathscr{B}_n$ is closure preserving.

To complete the proof that X is a $\sigma^\#$-space it remains to show that $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$ is a cover of X which separates distinct points. Since $x \in B_x(n)$ for each $n \in N$, $\mathscr{B}$ is a cover of X. To see that $\mathscr{B}$ separates distinct points, suppose $x, y \in X$ with $x \neq y$. By (1), $x \notin g(n, y)$ for some $n \in N$ and so $y \notin B_x(n)$. Thus X is a $\sigma^\#$-space.

A COC-function g satisfying conditions (1) and (2) of Lemma 4.1 will be called a *$\sigma^\#$-function.*

THEOREM 4.2. *The following classes of spaces are countably productive*: (1) *σ-spaces*; (2) *M_i-spaces* (i = 1, 2, 3) *and Nagata spaces*; (3) *semi-stratifiable spaces*; (4) *semimetrizable spaces*; (5) *developable spaces*; (6) *strict p-spaces*; (7) *p-spaces*; (8) *quasi-metrizable spaces*; (9) *γ-spaces*; (10) *θ-spaces*; *and* (11) *$\sigma^\#$-spaces.*

PROOF. The proofs of (1), (2), (3) and (4) may be found in [30], [8] and [9], respectively. The truth of (5) is well-known and can be proved using the techniques used in the proof of Theorem 3.1. (Note that a developable space is weakly-subsequential). Fairly easy proofs of (6) and (7) can be given using the internal characterizations found in [5] and [7] (see [11, Theorems 4.5.2 and 4.5.4] and [15, Corollary 4.11]). Actually, (7) is the content of [2, Theorem 15]. The truth of (8) is well-known; in fact, by using the characterization of Ribeiro [31] this can be proved in the same fashion as (11).

(11) Let $\langle X_n \rangle$ be a sequence of $\sigma^\#$-spaces and let g_i be a $\sigma^\#$-function for X_i. Let $X = \prod_{n=1}^{\infty} X_n$ and let $x = (x_1, x_2, \cdots) \in X$. Define $g(n, x) = g_1(n, x_1) \times g_2(n, x_2) \times \cdots \times g_n(n, x_n) \times \prod_{j>n} X_j$ for each $n \in N$. It is easy to verify that g is a $\sigma^\#$-function for X.

The proofs of (9) and (10) follow exactly as in the proof of (11). Also, (9) was obtained in [1] using mapping techniques.

It seems to be an open question whether the classes of Σ-spaces, $\Sigma^\#$-spaces or β-spaces are productive. However, we can obtain results similar to those obtained in § 3. In order to do so we introduce the following concepts: A space X is called a *productively Σ-space* (*productively $\Sigma^\#$-space*) if there exists a refining sequence $\langle \mathscr{F}_n \rangle$ of locally finite (closure preserving) closed covers of X such that, for each $x \in X$, if $\langle x_n \rangle$ is a sequence with $x_n \in c(x, \mathscr{F}_n)$ for each $n \in N$, then $\langle x_n \rangle$ has a subsequence with compact closure. The sequence $\langle \mathscr{F}_n \rangle$ will be called a *productively Σ-sequence* (*productively $\Sigma^\#$-sequence*).

THEOREM 4.3. *If $\langle X_n \rangle$ is a sequence of productively Σ-spaces (productively $\Sigma^\#$-spaces), then $X = \prod_{n=1}^{\infty} X_n$ is a productively Σ-space (productively $\Sigma^\#$-space).*

PROOF. Let $\langle \mathscr{F}_{n,i} : i \in N \rangle$ be a productively Σ-sequence (productively $\Sigma^{\#}$-sequence) for X_n. For each $i \in N$, let $\mathscr{H}_i = \{F_1 \times F_2 \times \cdots \times F_i \times \prod_{n>i} X_n : F_j \in \mathscr{F}_{j,i}, j = 1, 2, \cdots, i\}$. It is easy to see that $\langle \mathscr{H}_i \rangle$ is a sequence of locally finite (closure preserving) closed covers of X. Let $\langle x(n) \rangle$ be a sequence in X such that $x(n) \in c(x, \mathscr{H}_n)$ for some $x \in X$ and each $n \in N$. Using the notation of the proof of Theorem 3.1 (6), the sequence $\langle x_1(n) \rangle$ satisfies $x_1(n) \in c(x_1, \mathscr{F}_{1,n})$ and thus has a subsequence with compact closure. The remainder of the argument, showing that X is a productively Σ-space (productively $\Sigma^{\#}$-space), proceeds exactly as in the proof of Theorem 3.1 (6).

COROLLARY 4.4. *Let $\langle X_n \rangle$ be a sequence of weakly-subsequential or weakly-i spaces and let $X = \prod_{n=1}^{\infty} X_n$. If each X_n is a Σ-space ($\Sigma^{\#}$-space), then X is a Σ-space ($\Sigma^{\#}$-space).*

PROOF. We consider only Σ-spaces since the proof for $\Sigma^{\#}$-spaces is identical. It is obvious that every weakly-subsequential, Σ-space is a productively Σ-space; thus it suffices to show that every weakly-*i*, Σ-space is a productively Σ-space. To see this, let Y be a weakly-*i*, Σ-space and let $\langle \mathscr{F}_n \rangle$ be a Σ-sequence for Y. Let $\langle y_n \rangle$ be a sequence in Y such that $y_n \in c(y, \mathscr{F}_n)$ for some point $y \in Y$. Since Y is a Σ-space, the sequence $\langle y_n \rangle$ has a cluster point q and any such cluster point is in $\bigcap_{n=1}^{\infty} c(y, \mathscr{F}_n)$. Let $F = \{y_n : \mathrm{n} \in N\}$ and note that if $F - \{q\}$ is closed, the sequence $\langle y_n \rangle$ has a constant subsequence. So we may assume $F - \{q\}$ is not closed. Since Y is weakly-*i*, there exists a closed isocompact subset C of X such that $\mathrm{C} \cap (F - \{q\}) = \langle y_{n_k} : k \in N \rangle$ is an infinite subsequence of $\langle y_n \rangle$. Now, let $K = F \cup \bigcap_{n=1}^{\infty} c(y, F_n)$ and note that K is a closed, countably compact subset of Y. Since $Cl_Y \{y_{n_k} : k \in N\} \subset C \cap K$, $\langle y_{n_k} \rangle$ has compact closure. Hence Y is a productively Σ-space.

The reader may wonder why we chose to introduce the concepts of productively Σ-space and productively $\Sigma^{\#}$-space in this section, not having introduced analogous definitions for those concepts studied in § 3. The reason, as the proof of Theorem 2.2 indicates, is that such definitions would actually have been equivalent to assuming the spaces were weakly subsequential, i.e. if, for example, we defined productively M-spaces in a manner analogous to our definition for productively Σ-spaces the following would hold: A space is a productively M-space if and only if it is a weakly-subsequential M-space. (The situation for Σ-spaces is discussed in Example 5.2).

THEOREM 4.5. *Let X and Y be topological spaces with X weakly-subsequential (or weakly-i).*

(1) *If X and Y are Σ-spaces, then so is $X \times Y$.*
(2) *If X and Y are $\Sigma^{\#}$-spaces, then so is $X \times Y$.*

PROOF. (1) As the proof of Theorem 4.4 indicates, it is sufficient to consider X to be a productively Σ-space. Let $\langle \mathscr{F}_n \rangle$ be a productively Σ-sequence for X and let $\langle \mathscr{G}_n \rangle$ be a Σ-sequence for Y. For each $n \in N$, put $\mathscr{H}_n = \{F \times G : F \in \mathscr{F}_n, G \in \mathscr{G}_n\}$ and observe that $\langle \mathscr{H}_n \rangle$ is a sequence of locally finite closed covers of $X \times Y$. The argument showing that $X \times Y$ is a Σ-space proceeds in a similar fashion to that of [**18**, Theorem 1.1 in II] and is left to the reader. (It is essential to know that $\bigcap_{n=1}^{\infty} c(y, \mathscr{G}_n)$ is countably compact in Y).

(2) The proof of this result is obtained by making the appropriate changes in the proof of (1) above.

REMARKS. We may validly replace weak-subsequentiality or weakly-i space in the hypotheses of Corollary 4.4. and Theorem 4.5 by any of those properties listed in (A) or (B) of § 2. The author does not know if such replacements are possible for those properties appearing in (C).

We now turn our attention to a brief discussion of β-spaces. Since the class of β-spaces is not of type $\mathscr{Q}$ and since an argument similar to that used in Corollary 4.4 does not give the desired result for β-spaces, we do not know if the countable product of weakly-i, β-spaces is a β-space. However, the following result was obtained in [**15**, Theorem 3.11 (5)] and independently by the author in [**11**].

THEOREM 4.6. *The countable product of weakly-subsequential β-spaces is a β-space.*

THEOREM 4.7. *If X is a subsequential β-space and Y is a β-space, then $X \times Y$ is a β-space.*

PROOF. Let g and h be β-functions for X and Y, respectively. For each $s = (x, y) \in X \times Y$, define $k(n, s) = g(n, x) \times h(n, y)$. Let $\langle s_n \rangle$ be a sequence in $X \times Y$ such that $s \in g(n, s_n)$. It follows, just as in the proof of Theorem 3.3 (1), that $\langle s_n \rangle$ has a cluster point. Hence $X \times Y$ is a β-space.

REMARKS. (1) Since each of the concepts listed in (A) of § 2 implies weak-subsequentiality, Theorem 4.6 holds if we replace weak-subsequentiality by any of those concepts.

(2) Since every Fréchet space is subsequential, Theorem 4.7 holds if we replace subsequentiality by any of those properties listed in (A) of § 12 which precede Fréchet space.

(3) The author does not know if Theorems 4.6 and 4.7 hold if the appropriate replacements are made using those properties listed in (B) or (C) of § 2.

5. Examples and open questions.

EXAMPLE 5.1. A weakly-i, countably compact space which is not compact (hence not isocompact).

Let $X = [0, \Omega)$ where Ω is the first uncountable ordinal. Since X is first countable, it follows from Theorem 2.2 that X is weakly-i. However, X is not compact and, since it is countably compact, can't be isocompact.

In [23], Michael defined the concept of a strong Σ-space and showed that the countable product of strong Σ-spaces is a strong Σ-space. Since every strong Σ-space is subparacompact [23], it follows that $[0, \Omega)$ is not a strong Σ-space. However, as we just observed, $[0, \Omega)$ is a weakly-i, countably compact space and hence a productively Σ-space. This shows that our results in Theorem 4.3 and Corollary 4.4 are not a consequence of Michael's results.

The following example shows that the analogue of Theorem 2.2 does not hold for Σ-spaces, $\Sigma^{\#}$-spaces or β-spaces. Also, this example shows that a productively Σ-space need not be weakly-subsequential. (See the remark following the proof of Corollary 4.4).

EXAMPLE 5.2. A weakly-i, Σ-space which is not weakly subsequential.

Let X be the Arens-Fort Space (Example 26 of [35]). It is easy to see that X is a Σ-space which is not weakly-subsequential (this was observed in [15]). Since X is clearly paracompact, X is weakly-i. It follows that X is a productively Σ-space.

It should be noted that Example 3.8 of [15] shows that a weakly-i, weakly-subsequential space need not be weakly-k. Hence both of these concepts are strictly weaker than weakly-k.

The next example shows why Remark (2) following Theorem 3.3 was restricted to $w\Delta$-spaces.

EXAMPLE 5.3. A paracompact $w\gamma$-space (hence $w\theta$-space) which is not a strict p-space.

The familiar Sorgenfrey line has all the necessary properties (see [14, Example 4.14]). We note that for completely regular spaces, the concepts of a strong $w\Delta$-space and a strict p-space are equivalent.

As the following example shows, it is not possible to appeal directly to Theorem 3.3 when attempting to replace subsequentiality by any of those properties from (B).

EXAMPLE 5.4. A paracompact $w\Delta$-space which is not subsequential.

Let $X = \prod \{I_i : i \in I\}$, where $I = [0, 1]$ and $I_i = [0, 1]$ for each $i \in I$. The space X is clearly compact and hence a paracompact $w\Delta$-space. Consider the sequence $\langle a_n \rangle$ of X, where $a_n(x)$ is defined to be the nth digit of the binary expansion of $x \in I$. Since X is a compact T_2-

space, the sequence $\langle a_n \rangle$ has a cluster point. Now suppose $\langle a_{n_k} \rangle$ is any subsequence of $\langle a_n \rangle$, and let $y \in I$ be such that $a_{n_k}(y) = 0$ if k is odd and $a_{n_k}(y) = 1$ if k is even. It follows easily that the subsequence $\langle a_{n_k} \rangle$ does not converge. We note that, by Theorem 2.2, X is weakly-subsequential.

EXAMPLE 5.5. A completely regular countably compact space which is not subsequential, weakly-subsequential or weakly-i.

Let X be one of the factors in Isiwata's example [19]; the example mentioned in § 1. By Theorems 2.2 and 3.2, X is the desired example.

As the next example shows, none of the results in Theorem 3.1 (2)–(9), Theorem 4.2 (1)–(10), Theorem 4.3, Corollary 4.4 and Theorem 4.6 hold for uncountable products.

EXAMPLE 5.6. Let $X = \Pi \{N_\alpha : \alpha \in A\}$, where A is uncountable and each N_α is a copy of the natural numbers with the discrete topology.

As observed in [15, Example 3.13], X is not a q-space nor a β-space. Since each N_α is metrizable, N_α satisfies the hypotheses of all of those results mentioned preceding this example. However, it is known that each of those properties imply the space is either a q-space or a β-space. Further, this example shows that no replacements can be made from those properties in (A), (B) or (C) of § 2 to obtain results for uncountable products.

Next, let us mention a few open questions concerning products of generalized metric spaces.

QUESTION 1. If X and Y are Σ-spaces is $X \times Y$ a Σ-space?

QUESTION 2. Same as Question 1 for $\Sigma^{\#}$-spaces and β-spaces?

QUESTION 3. Does Theorem 3.3 hold if subsequentiality is replaced by either weak-subsequentiality or the property of being a weakly-i space?

QUESTION 4. Do any of the theorems in this paper hold if weak-subsequentiality is replaced by either normality or collectionwise normality?

Arhangel'skii [2] observes that every p-space is a k-space and thus a weakly subsequential, quasi-complete space. Thus, by Theorem 3.2 (4), the product of a p-space and a quasi-complete space is quasi-complete. However, the following related question seems to be open.

QUESTION 5. Is the product of a strict p-space and a $w\Delta$-space a $w\Delta$-space?

Of course, an affirmative answer to Question 3 would provide an affirmative answer to this question.

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