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INTRODUCTION TO WAVES IN PLASMAS

Plasma physics seems to provide an inexhaustible source of nonlinear problems. In the particular area of controlled thermonuclear fusion one important goal is to focus and to contain an intensely hot plasma. The mathematical formulation of this problem might well involve localized, stable (soliton?) solutions of terribly complicated nonlinear equations. Of course, no one expects the soliton to solve the energy crisis, but it might be a very useful concept in the analysis of simpler model problems.

Some acquaintance with the fluid description of a plasma is helpful for an understanding of the articles in this section. The basic information can be found in any recent text on plasmas (older books seem to pay less attention to the type of waves that evolve into solitons). For a first reading, a few definitions should suffice.

A plasma is an ionized gas (a gas of positively charged ions and negatively charged electrons) whose behavior is dominated by the collective motion of the particles. Even though the gas consists of charged particles, it is electrically neutral on a macroscopic scale. If, for example, electrons are displaced by an applied electric field so that a local charge imbalance results, the ions will attempt to restore neutrality by returning the electrons to their original positions. This results in the so-called *electron-plasma oscillations.* These are not true propagating waves, in the sense that the plasma oscillates in unison: the wavelength is of the order of the plasma dimension. It is simple and instructive to derive this sort of disturbance from the governing equations.

In one space dimension, assume the ions to be fixed (they are, indeed, much heavier than the electrons) with constant density N_0. Let n, v be the electron number density and velocity, m the electron mass, E the applied electric field, $-e$ the electron charge, and ϵ_0 the dielectric constant. Then

$$n_t + (nv)_x = 0 \qquad \textit{(continuity)}$$

$$v_t + vv_x = -\frac{e}{m}E \qquad \textit{(conservation of momentum)}$$

$$E_x = \frac{1}{\epsilon_0}(N_0 - n)e \qquad \left(\textit{Maxwell's equation } \nabla \cdot E = \frac{1}{\epsilon_0}\rho\right).$$

Linearize this system about the equilibrium solution $n = N_0$, $v = E = 0$ and seek solutions proportional to $\exp i(kx - \omega t)$. It is found that ω must satisfy

$$\omega^2 = \frac{N_0 e^2}{m\epsilon_0} \equiv \omega_{pe}^2.$$

ω_{pe} is the plasma electron frequency, which is seen to be independent of the wavelength $2\pi/k$. ($\omega_{pe}/2\pi \sim 9000\sqrt{N_0}$ oscillations per/second.)

This derivation neglected the thermal motion of the electrons. One can incorporate this effect by adding a pressure term $-p_x/nm$ to the right side of the momentum equation (nm is the number of electrons times the electron mass, i.e., the mass density ρ). One may take $p = c^2 mn$ because of the ideal gas law; c^2 is the local sound speed of the gas. Linearization of the resulting system leads to the dispersion relation

$$\omega^2 = \omega_{pe}^2 + c^2 k^2.$$

For large k, the disturbances are high-frequency, short wavelength, electron waves. It is convenient to study modulations of these waves. One is led to envelope equations such as the nonlinear Schrödinger equation, and to the envelope solitons, called *Langmuir solitons* (see Morales and Lee).

The above derivation of Langmuir waves pictured the electrons moving rapidly against a neutralizing background of stationary ions. In the *ion-acoustic* oscillations dealt with by Maxson, the ions move slowly against a background of electrons in rapid thermal fluctuation; in this instance, the electron cloud provides the neutralizing effect. One can obtain both types of waves from the coupled ion-electron fluid equations (capital letters denote quantities referring to ions):

$$n_t + (nv)_x = 0,$$

$$v_t + vv_x = \frac{e}{m}E - \frac{1}{nm}p_x,$$

$$N_t + (NV)_x = 0,$$

$$V_t + VV_x = \frac{e}{M}E - \frac{1}{NM}P_x,$$

$$E_x = \frac{e}{\epsilon_0}(N - n).$$

The last equation is Maxwell's equation $\nabla \cdot E = \rho/\epsilon_0$; the charge density is $\rho = Ne + n(-e)$. Again, set $p = c^2nm$. It is seen by thermodynamic arguments that P, the ion pressure, is proportional to the ion kinetic temperature T_i. The assumption of "cold" ions, made by Maxson for instance, means $T_i = 0$ (a "cold" plasma is still about 11,000° K). Linearization then leads to a fourth-order equation for ω. One branch of solutions reproduces (with more precision) the Langmuir dispersion relation obtained earlier. The other branch gives, approximately,

$$\omega^2 = \frac{k^2c^2\omega_{pi}^2}{\omega_{pe}^2 + k^2c^2},$$

the dispersion relation for *ion-acoustic waves*. (ω_{pi} is the ion-plasma frequency which is analogous to ω_{pe}.) This relation is physically relevant for small k, or long wavelength. The weakly nonlinear, weakly dispersive approximation of these long waves leads, not surprisingly, to the KdV equation once more.

To summarize, we have: Weakly nonlinear, strongly dispersive, short wavelength electron Langmuir waves described by their modulating envelope (nonlinear Schrödinger); weakly nonlinear, weakly dispersive, long wavelength ion-acoustic waves described by the KdV equation.

These are the two basic sources of solitons in plasmas. More complicated examples will be found in the papers which follow.