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THE INVERSE SCATTERING TRANSFORM, NONLINEAR WAVES, SINGULAR PERTURBATIONS AND SYNCHRONIZED SOLITONS

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ABSTRACT. It is not the intention of this article to cover, in detail, all of the material presented in my lecture notes at the Conference. Rather, it is the plan to attempt to point out (a) the universal nature and origin of the partial differential equations which the inverse scattering transform enables one to solve and (b) the rich potential for examining partial differential equations which are close, in some perturbation sense, to integrable ones. Several examples are investigated. We discuss the synchronous response of a nonlinear Schrödinger soliton to an applied field and also examine the effects of density gradients, damping and diffusion. We also consider the behavior of a kink of the sine-Gordon equation in the presence of an impurity and finally develop 4π pulse solutions of the double sine-Gordon equation which are formed by a pair of synchronized 2π pulses.

1. Introduction. One of the significant advances in mathematical physics over the past decade has been the discovery by Gardner, Greene, Kruskal, Miura and Zabusky [1, 2, 3] of (1) a new nonlinear transform and (2) the soliton. The transform (the Inverse Scattering Transform or IST) works in precisely the same way as the Fourier transform does in linear problems; namely, it transforms the dependent variable which satisfies a given partial differential equation to a set of new dependent variables whose evolution in time is described by an infinite sequence of ordinary differential equations. For special classes of partial differential equations, these equations are separable and hence trivially integrable.

Compared with the Fourier transform, there are two major differences. The first is that the basis is no longer fixed (like $e^{\pm ikx}$) but moves in a way which depends on the unknown variable. The second difference is that the spectrum (and here we are considering partial differential equations over infinite spatial intervals) no longer simply consists of the continuum of real wavenumbers k but includes, in addition, a finite number of isolated complex wavenumbers. It is the complex wavenumbers which give rise to the entities known as solitons. They are truly nonlinear quantities and have no linear analogue.

Indeed the general solution of any member of one of the aforemen-

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tioned classes of partial differential equations can be qualitatively described in terms of the various spectral components. The soliton (a term coined by Zabusky and Kruskal) in a solitary wave, that is, a localized, stable, permanent waveform (it may contain internal oscillations), with the crucial additional property that its identity (amplitude, velocity, shape, internal frequency) is preserved even after collision with one of the other solution components. This invariance is derived from the invariance of the eigenvalue with which it is associated. The only interaction memory is a shift in the position of the soliton relative to where it would have been had it traveled without collision. In a two soliton collision, the phase shift is a simple function of the two eigenvalues. On the other hand the "solitary wave" (for example, the hyperbolic tangent solution of the equation $\phi_{tt} - \phi_{xx} - \phi + \lambda\phi^3 = 0$) is subject to distortion on collision. The component of the solution associated with the continuous spectrum is in general not localized, does not have a permanent wave form and, as it disperses, decays algebraically in time in close analogy with the long time behavior of linear dispersive waves. There are, however, some features of this solution component which are distinctly nonlinear, particularly the structure of self similar regions.

One of the major reasons for the widespread and interdisciplinary interest in IST is that the special classes of integrable equations include such a wide variety of useful and universal equations which are central to many areas of mathematical physics and whose solutions are important to our general understanding of nonlinear wave phenomena. To illustrate the point, we will consider a list of these equations and at the same time discuss the historical development of IST over the last ten years.

It is perhaps fitting that the IST method itself was first developed through the studies of the Korteweg-deVries equation (*KdV*),

$$q_t + 6qq_x + q_{xxx} = 0, \tag{1.1}$$

which arises so very naturally as the leading approximation in all conservative wave systems which are *weakly* dispersive and *weakly* nonlinear. It was first suggested by Korteweg and deVries as being relevant to the description of long surface gravity waves whose slope is small and approximately equal to the cube of the depth-to-wavelength ratio. The wave is large enough so that it initially attempts to break, but the water is deep enough so that dispersive effects are eventually important. Imagine then an initial local disturbance. The first response of the system is to divide the arbitrary initial disturbance into leftward and rightward traveling waves according to the D'Alembert solution of the linear wave equation. Both the left and the right traveling profiles (which are now separated) eventually distort due to the combined effects of nonlinearity and dispersion. It is this evolution and this balance which is described by (1.1). There are many natural contexts in which similar dynamics can arise; long internal gravity waves [4], elastic waves [5], ion-acoustic waves in a cold plasma [6], waves on vortex tubes [7], and longitudinal vibrations of a discrete mass string [8]. The equation came to the attention of Zabusky and Kruskal [3] through their studies on lattices and on the Fermi-Pasta-Ulam [9] results on heat conduction in solids.

We reiterate: If the dynamical system has the properties of (1) weak dispersion (of order ϵ), (2) weak nonlinearity (of order ϵ), and (3) initial data prescribed on compact support on an order one spatial interval, then: (1) the propagation for times of order one is described by the linear wave equation, and (2) the propagation along each of the (separate) characteristics on time scales of order ϵ^{-1} is described by the *KdV* equation (except in those few cases where the quadratic (and strongest) nonlinearity is not present under which circumstances the equation with the cubic nonlinearity (the modified Korteweg-deVries equation or *MKdV*) obtains).

It is also fitting that the second equation to which IST was applied was also of universal character. In a beautiful 1972 paper, Zakharov and Shabat [10] showed how the nonlinear Schrödinger equation (NLS),

$$q_t - iq_{xx} \pm 2iq^2q^* = 0 \tag{1.2}$$

fits into the formalism. These authors used heavily the ideas of Lax [11], who had reformulated the principal results of GGKM in an operator theoretic notation and in addition had found several of the other integrable equations in the Korteweg-deVries family. This universal equation (discovered in various contexts by authors in the late fifties and early sixties [12, 13, 14, 15, 16, 17]) describes the slow temporal and spatial evolution of the envelope $q(\epsilon(X - c_gT), \epsilon^2T)$ (X, T real space and time, c_g the group velocity) of an *almost monochromatic* wavetrain (centered on wavenumber k) in a weakly nonlinear and strongly dispersive system. For context free and very general derivations, see [16] or [17]. This equation simply comes up everywhere, from the modulation of high intensity electromagnetic signals in media where the refractive index is amplitude dependent, to the break up of deep water gravity wave envelopes. (It should be pointed out that in more than one dimension, waves with the linear dispersion relation $\omega = c|\vec{k}|$ are strongly dispersive; the (vector) group velocity depends on the direction of $\vec{k}$ and the dispersion tensor $\partial^2\omega/\partial k_r\partial k_s$ is nonzero). One of the most dramatic applications of (1.2) is in the context of deep water gravity waves. Indeed, one can show that if the minus sign in (1.2) obtains, then the monochromatic wavetrain solution $q = q(t)$ is unstable to x dependent perturbations (an instability first discovered by Benjamin and Feir [18]) and the wavetrain breaks up into separated and localized pulses. In some sense, this provides an explanation of a fact known to everybody with a surfboard, namely, that every tenth (seventh, eleventh) wave is the largest. Imagine the following experiment. A paddle is oscillated in a periodic manner (say, the Fourier spectrum contains a number of distinct frequencies) and excites several wavepackets (of width ϵ), each being centered on a distinct frequency. Since the system is strongly dispersive, the packets move with different group velocities and separate on a time scale ϵ^{-1}. For times on the order of ϵ^{-2}, each packet feels the combined effects of dispersion (which tends to split up the packet) and nonlinearity (just like a simple weakly nonlinear oscillator, the nonlinearity is manifested as a third order self-modal $\omega + \omega - \omega = \omega$ interaction of strength ϵ^2), and the envelope $q(x, t)$ is modulated according to (1.2). What happens is that the initial envelope $q(x, 0)$ is decomposed into a series of *solitons* (given by

$$q(x, t) = 2\eta \operatorname{sech} 2(\theta_0 - \eta x - 4\xi\eta t)e^{-2i\theta_0 - 4i(\xi^2 - \eta^2)t - 2i\xi x} \tag{1.3}$$

in which the parameter $\zeta = \xi + i\eta$ is one of the complex eigenvalues discussed before) and *radiation* which disperses and decays [19]. [We have assumed that the paddle oscillates for a long but finite time. Should it continue to oscillate, the solitons and radiation regroup into an almost uniform wavetrain, break up again, and so on.] It is probably fortunate, at least for ocean travellers, that the unidirectional soliton solutions of the higher dimensional analogues of (1.2) are unstable to perturbations in directions perpendicular to the x direction [20, 21, 22]. However, there are circumstances in which such instabilities may be inhibited (for example, an ocean current with a large horizontal shear would act as a one-dimensional wave guide) and in these circumstances one can indeed expect the total energy of the system to be concentrated in intense patches in the form of (1.3).

We emphasize that the NLS equation is the most canonical equation of all. It arises (often in multidimensional form; simply replace x by $\vec{x} = \epsilon(\vec{x} - \vec{c}_g T)$, $\vec{c}_g = \nabla\omega$ and the factor in front of the dispersion term $\partial^2 q/\partial x_r \partial x_s$ is the dispersion tensor $(-i/2)(\partial^2\omega/\partial k_r \partial k_s))$ as the envelope equation in all dynamical systems which are (1) strongly dispersive but where the envelope contains an order ϵ spread of wavenumbers, and (2) weak nonlinearity of order ϵ. It is irreducible; the NLS equation developed from examining an almost monochromatic solution of the NLS equation is again the same NLS equation. Indeed, the *KdV* equation itself, in circumstances where the dispersion is stronger than the nonlinearity (as often happens after sufficient time behind the front $x = (gh)^{1/2}\, t$) and where the wavenumber is locally constant, will reduce to the NLS equation. [It is an interesting turnaround that the envelope-hole solutions of the stable NLS equation, i.e., long wave, small amplitude *perturbations* on a $Ce^{ikx-i\Omega t}$ solution, are described by the KdV equation.] Furthermore, the NLS equation provides a useful approximation for examining much more complicated equations when a low amplitude, almost monochromatic situation prevails, an approximation we have found [23] to be extremely useful.

Once Zakharov and Shabat had demonstrated that the method of GGKM could work for equations other than the Korteweg-deVries equation and therefore was not simply a fluke, there was renewed investigation of many other equations of the class which showed the important property common to (1.1) and (1.2) — namely, an infinite number of conserved quantities. In rapid succession Wadati [24] solved the modified Korteweg-deVries equation,

$$q_t + 6q^2 q_x + q_{xxx} = 0 \tag{1.4}$$

and Ablowitz, Kaup, Newell and Segur (AKNS [25, 26, 27]), Lamb [28] and a little later Faddeev and Takhtadzhyan [29] (see also [29a]), solved the sine-Gordon equation,

$$u_{xt} = +\sin u \tag{1.5a}$$

$$u_{TT} - u_{XX} + \sin u = 0 \tag{1.5b}$$

The method used by AKNS was novel in the sense that it was the spectral analogue of the Lax formulation and had the distinct advantage that all the necessary computations were simple and algebraic. In fact, the method generated, in a most natural way, classes of integrable equations and showed how each member of these classes could be identified with the dispersion relation of its associated linear version [27]. The fact that (1.4) has an infinite number of conservation laws and conserved quantities has been established by Miura [30], who discovered a remarkable transformation between solutions of (1.1) and (1.4). Indeed this transformation not only holds between (1.1) and (1.4) but also between solutions of corresponding members of the Korteweg-deVries and modified Korteweg-deVries families [27], the correspondence being established by the common dispersion relation [31]. The fact that the sine-Gordon equation (1.5) has an infinite number of conservation laws and conserved quantities was established by Kruskal [32] who also found that the sine, sinh and Klein-Gordon equations (which turn out to be the only integrable ones) are the only ones of the class

$$u_{tt} - u_{xx} + V(u) = 0 \tag{1.6}$$

which possess this property. In a sense, then, the sine-Gordon equation is the simplest nonlinear model of type (1.6) which possesses the property of Lorentz invariance and possesses soliton solutions (the sinh-Gordon equation does not have any soliton solutions), properties which suggest its relevance as a field theory model. It also arises in many other contexts (see the article by Scott, Chu and McLaughlin [33]), and is a particularly useful model in those systems which are intrinsically one dimensional, whose kinetic energy is of the form $\int (1/2)\,\phi_t^2\,dx$, and whose potential energy consists of a strain energy $\int (1/2)\,\phi_x^2\,dx$ and a periodic potential $\int V(\phi)\,dx$. In the marvelously simple mechanical analogue demonstrated by Al Scott at the conference, $\phi(x, t)$ is the local displacement angle of the pendulae with respect to the torsion spring to which each pendulum is affixed, $(1/2)\,\phi_x^2$ is the torsion potential and the gravitational potential is proportioned to $1\text{-}\cos\phi$. The resulting Lagrangian leads directly to the equation of motion (1.6). Phenomenological models incorporating these simple ingredients have been proposed to explain crystal dislocation [34] and also some of the remarkable electrical conducting properties of the TTF-TCNQ organic compounds [35] and based on these models several interesting predictions have been made [23].

It is not widely appreciated that (in some sense) the *MKdV* equation is also of universal character. It is often of interest to examine the long time spatial behavior of a dynamical system to which a forced oscillation $e^{i\omega t}$ is applied. Solutions of the form $\psi(x, y, t) = \operatorname{Re}(\psi(x, y)e^{i\omega t})$ can lead to a partial differential equation in x, y which is hyperbolic in character and takes the form

$$\psi_{xx} - \psi_{yy} + \epsilon a(\psi_x{}^2\psi_x{}^*)_x + \epsilon b(\psi_{xxxx}) = 0, \tag{1.7}$$

in which the nonlinear term has arisen from a cubic $e^{i\omega t}\cdot e^{-i\omega t}\cdot e^{i\omega t} = e^{i\omega t}$ interaction. Indeed just such an equation describes the bending of the lower hybrid cones, the study of which is central in the problem of plasma heating [see the article by Morales and Lee, this volume]. In this context, the nonlinear term is generated by the interaction of the ponderomotive force (the d.c. rectification of the convection terms in the ion and electron momentum equations — analogous to Reynolds stresses) with the fluctuating density field. The higher derivative dispersive terms are due to thermal dispersion. The usual perturbation methods ($\psi = \psi_0(x - y, \epsilon y) + \epsilon\psi_1 + \cdots$) will reduce (1.3) to the complex modified Korteweg-deVries equation (*CMKdV*) for $\psi_x = q$, $\xi = x - y$, $\tau = \epsilon y$

$$q_\tau + (q^2q^*)_\xi + q_{\xi\xi\xi} = 0. \tag{1.8}$$

In the event that the initial data are real, (1.8) reduces to the integrable *MKdV* equation (1.4). In general then, the *MKdV* equation will arise in those circumstances in which the fundamental solution contains a sinusoidal factor (which can only reproduce itself by a third order interaction) depending on time or another direction. For example, long atmospheric waves in a horizontal shear flow with vertical density stratification [72] (which introduces a vertical dependence of sinusoidal character) will be governed by the *MKdV* equation, whereas with no density stratification these waves are described by the *KdV* equation [73].

One particularly valuable application [36] of the AKNS approach, closely related to the sine-Gordon equation, is the problem of coherent pulse propagation, mathematically described by the Maxwell-Bloch equations,

$$(1.9)\qquad \begin{gathered} E_x = \int_{-\infty}^{\infty} g(\alpha)\lambda(\alpha, x, \tau)\, d\alpha \\ \lambda_\tau + 2i\alpha\lambda = -EN,\ N_\tau = -\frac{1}{2}(E\lambda^* + E^*\lambda), \end{gathered}$$

which describe the passage of an optical pulse envelope (electric field $E(x, \tau)$) of an optical wave through a resonant, inhomogeneously broadened ($g(\alpha)$) medium where the difference in energy levels corresponds (almost) to the frequency of the carrier wave. McCall and Hahn [37] had discovered that pulses with very special profiles could propagate through these resonant media in a lossless and coherent fashion. As one can guess, these pulses turn out to be solitons, and there are two types. One corresponds to a single eigenvalue and leads to a pulse-like solution whose area is 2π. The other corresponds to an eigenvalue pair, has zero area, and has been named the 0π pulse or breather in the literature. Although McCall and Hahn had a good qualitative understanding of the nature of the general solution, the IST solution proved to be very useful in predicting the precise behavior in a quantitative manner.

Besides the obvious value of providing a way to compute the exact solution to the initial value problem, there are three further key features of the general IST method which this example serves to illustrate. First, unlike all previous examples of integrable system, this problem does not contain an infinite number of conserved quantities. There are an infinite number of local conservation laws, but the only globally conserved quantities in the electromagnetic field are the shapes of the solitons (2π and 0π pulses) into which the initial profile is decomposed. The reason for that is that the radiation is trapped by the medium. Unlike previous examples, there are also two distinct dispersion relations connected with this problem [for more discussion of this point see [38], 39]], and the difference between the dispersion relations is the factor which causes irreversible behavior. The second feature which this example illustrates is that much information about the system may be obtained without solving each stage of the inversion procedure explicitly. Kaup's article in this volume will illustrate this point very clearly. The third point is that in the sharp line limit (where $g(\alpha)$ is the Dirac delta function), (1.9) reduces to the sine-Gordon equation (1.5), in which limit the two nonsingular dispersion relations relax to the singular dispersion relation

$$\omega = \frac{1}{k}.$$

The fact that the dispersion relation of the Goursat problem (1.5a) is singular has important consequences which are discussed in reference [40]. In a sense, then, we can also think of the sine-Gordon equation as typifying the class of integrable systems which have dispersion relations $\omega = 1/k$ (or $(K^2 + 1)^{1/2}$).

At approximately the same time, a parallel study on differential-difference equations was being developed by Flaschka [41] who solved the equations for the Toda lattice

$$Q_{ntt} = e^{-(Q_n - Q_{n+1})} - e^{-(Q_{n-1} - Q_n)}$$

where Q_n is the position of the nth particle in the lattice, one of several models vital to our eventual understanding of the heat conduction process. These ideas have been extended by Moser [42], Calogero [43], and Ablowitz and Ladik [44] who have also developed the theory for partial difference equations, a development which could be very significant in the theory of numerical computation.

In late 1973, Zakharov and Manakov [45] introduced a matrix operator of higher order and found a way to express the 3-wave interaction problem

$$\frac{\partial A_j}{\partial t} + \vec{C}_j \cdot \nabla A_j = \theta_j A^*_k A^*_\ell \tag{1.11}$$

(j, k, ℓ cycled over 1, 2, 3) in the Lax formalism, and once again an exact solution was found for a set of canonical equations. These equations are central to all weakly nonlinear systems which support a continuum of dispersive waves, since the quadratic nonlinearities can cause a resonance between three resonant wavetrains with amplitudes, A_j, $j = 1, 2, 3$, whose wavevectors $\vec{k}_j$ and frequencies ω_j satisfy conservation of energy and momentum (the "resonant conditions"): $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$, $\omega_1 + \omega_2 + \omega_3 = 0$. Interactions of this kind are important in Rossby and Baroclinic waves [46–49], in internal gravity waves [50], plasma waves [51] and many other areas of continuum physics. Both Kaup [52] and Zakharov and Manakov independently developed the inverse formulae for solving these equations in the one-dimensional case, and Kaup describes many of the interesting consequences in his second article in this volume.

A closely related system is the model which simulates the interaction between long waves (amplitude $A(x, t)$) and short waves (envelope amplitude $B(x, t)$). The equations are an extension of the nonlinear Schrödinger equation to account for the effects of a long wave and take the form

$$\frac{\partial A}{\partial t} = 2s \frac{\partial}{\partial x} BB^*, \quad \frac{\partial B}{\partial t} - i \frac{\partial^2 B}{\partial x^2} = - \frac{\partial A}{\partial x} B + iA^2B - 2isB^2B^*. \tag{1.12}$$

This exactly solvable model was developed [53] in order to investigate further the very novel idea of Benney [54], who suggested that long waves might be driven by the recurring instability of (say, wind driven) short waves which, together with the long wave, form a resonant triad.

This partial list of extremely important equations illustrates the tremendous relevance and importance of the inverse scattering transform. It also very strongly suggests that the soliton is ubiquitous in nature (in the oceans, atmosphere, plasma, lattices, superconductors, superfluids, elementary particles), at least in those situations which are to a good approximation one dimensional. There has been much effort [55, 56, 57] (reference [55] is important; in it Zakharov and Shabat describe their formalism for deriving larger classes of integrable equations) to extend the ideas to higher space dimensions, and while there has been some success in writing interesting three-dimensional equations in the Lax formalism, neither a higher dimensional soliton (local) nor a fully satisfactory inverse scattering theory has been worked out. Nevertheless, some remarkable properties associated with two dimensional local solutions have been reported. Zakharov and Synakh [22] have described how the radially symmetric solutions of the two dimensional NLS equation are unstable, and their numerical work predicts that the nonlinearity dominates and that the amplitudes become infinite in finite time. This behavior contrasts markedly with what one might expect from a two-dimensional water wave. Here one would conclude that the focusing effect of the nonlinearity would be unable to sustain a local solution against the combined effects of wave and geometric dispersion.

The aim of the remainder of this paper is not only to summarize the IST method (the detailed lecture notes can be found in reference [39]) and to point out how and why it works, but to emphasize that it is very natural to describe systems which are "close" to integrable ones in terms of the normal mode parameters (the scattering functions) of their closest integrable neighbor.

2. **Summary of the Method.** In this section we will outline very briefly the most important features of the method and lay the necessary groundwork for examining the singular perturbation approach. For simplicity, we take the generalized Zakharov-Shabat eigenvalue problem

$$\begin{aligned} v_{1x} + i\zeta v_1 &= q(x, t)v_2 \\ v_{2x} - i\zeta v_2 &= r(x, t)v_1, \end{aligned} \tag{2.1}$$

on the interval $-\infty < x < \infty$. Both $q(x, t)$ and $r(x, t)$ are assumed to be absolutely integrable.

The inverse scattering transform is simply the mapping between the potentials q and r and the scattering data S defined by (2.1), the latter consisting of the *spectrum* of (2.1) (the whole real ζ axis and a finite number of complex ζ points) and the *asymptotic* behavior of the corresponding eigenfunctions. More precisely, if ϕ and $\bar{\phi}$ are solutions of (2.1) with asymptotic behavior $(1, 0)^T e^{-i\zeta x}$ and $(0, -1)^T e^{i\zeta x}$ as $x \to -\infty$, then the scattering data for real ζ are the coefficients describing the respective asymptotic behaviors $(a(\zeta, t)e^{-i\zeta x}, b(\zeta, t)e^{i\zeta x})^T$ and $(\bar{b}(\zeta, t)e^{-i\zeta x}, -\bar{a}(\zeta, t)e^{i\zeta x})^T$ of ϕ and $\bar{\phi}$ as $x \to +\infty$. For an absolutely integrable r and q, a and $\bar{a}$ are analytic in the upper and lower complex ζ plane respectively and their zeros $(\zeta_k, \bar{\zeta}_k)$ are the complex eigenvalues. If q and r have compact support (which for this discussion we will assume), then all the scattering functions, a, $\bar{a}$, b and $\bar{b}$ possess analytic extensions for all complex ζ and the values of $b(\zeta, t)$ $(b_k(t))$ and $\bar{b}(\zeta, t)$ $(\bar{b}_k(t))$ at ζ_k and $\bar{\zeta}_k$, respectively, provide the remainder of the scattering data. The determination of the set

$$S = \{(b_k, \zeta_k)_{k=1}^N, (\bar{b}_k, \bar{\zeta}_k)_{k=1}^N, a(\zeta), b(\zeta), \bar{a}(\zeta), \bar{b}(\zeta)\}$$

is called the direct scattering problem and the reader may verify many of the properties of the transform by explicitly calculating the scattering data for the potentials

$$-r = q = \begin{Bmatrix} 0 & x < 0, x > L \\ Q & 0 < x < L \end{Bmatrix}. \tag{2.2}$$

Indeed it can be shown [39, 58, 60] that the direct transform is simply a canonical transformation between the conjugate coordinates q and

r in physical space and the sets of conjugate coordinates $(2i\zeta_k, \ln b_k)_{k=1}^N$, $(2i\bar{\zeta}_k, \ln \bar{b}_k)_{k=1}^{\bar{N}}$, $((1/\pi) \ln a\bar{a}, \ln b(\xi))$ in scattering space. The following two-form is preserved:

$$\int_{-\infty}^{\infty} \delta r \wedge \delta q \, dx = \int_{-\infty}^{\infty} \delta \frac{1}{\pi} \ln a\bar{a} \wedge \delta \ln b(\xi) \, d\xi + \sum_{k=1}^{N} \delta(2i\zeta_k) \wedge \delta \ln b_k + \sum_{k=1}^{\bar{N}} \delta(2i\bar{\zeta}_k) \wedge \delta \ln \bar{b}_k. \tag{2.3}$$

For a special class of Hamiltonians, the flow represented by q and r is integrable and in this case the conjugate coordinates in scattering space are simply action-angle variables. All this is described in [39]. Let us just emphasize here that the scattering function $\ln a(x, t)$ plays a central role in the Hamiltonian formulation. It should be mentioned here that the Hamiltonian formulation of the Korteweg-deVries equation was first developed by Gardner [59] and Zakharov and Faddeev [60].

The inverse problem consists of a set of formulae (which are *linear* integral equations) from which the potentials q and r may be recovered in terms of the scattering data S. There is one note of caution. The scattering data cannot be prescribed arbitrarily, as in general this may lead to nonunique and nonabsolutely integrable solutions. However, for those cases for which the scattering data correspond to the constraint $r = \pm q^*$ or $r = \pm q$ on the potentials, the problem is uniquely invertible. For more discussion on this question see [27, 39, 58].

The next question is naturally: how can one find the time evolution of the scattering data S, and for which classes of flows $q(x, t)$ and $r(x, t)$ is the time evolution of S simple and integrable? To answer this we simply take the variation of (2.1), and determine the infinitesimal changes δv_1, δv_2 due to the infinitesimal changes δq, δr and $\delta\zeta$. The behavior of δv_1, δv_2 as $x \to +\infty$ will give us the variation of the scattering data. It turns out that the variation of the scattering data is the natural inner product between the vector $(\delta r, -\delta q)^T$ and the squared eigenfunctions $(\psi_1{}^2, \psi_2{}^2)^T$ and their ζ derivatives. Dual expressions for the variation of the scattering data as inner products between the vector $(\delta q, \delta r)^T$ and the squared eigenfunctions $(\phi_2{}^2, -\phi_1{}^2)^T$ and their ζ derivatives can also be found. (The eigenfunctions $\psi = (\psi_1, \psi_2)$ and $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ are defined as solutions of (2.1) with respective asymptotic behavior $(0, 1)^T e^{i\zeta x}$ and $(1, 0)^T e^{-i\zeta x}$ as $x \to +\infty$.) Now, Kaup [61] has shown that for certain classes of functions the squared eigenfunctions form a basis, and thus we may invert these expressions (with the help of the orthogonality relations between $(\psi_1^2, \psi_2^2)^T$ and $(\phi_2^2, -\phi_1^2)^T$) and obtain the following generalized Fourier expansions for $(\delta r, -\delta q)^T$ and $(\delta q, \delta r)^T$ respectively.

$$
\begin{aligned}
\begin{pmatrix} \delta r \\ -\delta q \end{pmatrix} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\Bigg\{ &\delta\left(\frac{\bar b}{a}\right)\begin{pmatrix} \phi_2^2 \\ -\phi_1^2 \end{pmatrix} \\
&- \delta\left(\frac{b}{\bar a}\right)\begin{pmatrix} \bar\phi_2^2 \\ -\bar\phi_1^2 \end{pmatrix}\Bigg\} dx \\
&+ 2i\sum_{k=1}^{N}\delta\left(\frac{1}{b_k a'_k}\right)\begin{pmatrix} \phi_2^2 \\ -\phi_1^2 \end{pmatrix}_k \\
&+ \frac{1}{b_k a'_k}\,\delta\zeta_k\,\frac{\partial}{\partial\zeta}\begin{pmatrix} \phi_2^2 \\ -\phi_1^2 \end{pmatrix}_k \\
&+ 2i\sum_{k=1}^{N}\delta\left(\frac{1}{\bar b_k \bar a_k'}\right)\begin{pmatrix} \bar\phi_2^2 \\ -\bar\phi_1^2 \end{pmatrix}_k \\
&+ \frac{1}{\bar b_k \bar a_k'}\,\delta\bar\zeta_k\,\frac{\partial}{\partial\zeta}\begin{pmatrix} \bar\phi_2^2 \\ -\bar\phi_1^2 \end{pmatrix}_k
\end{aligned}
\tag{2.4a}
$$

and

$$
\begin{aligned}
\begin{pmatrix} \delta q \\ \delta r \end{pmatrix} = \frac{1}{\pi}\int_{-\infty}^{\infty}\Bigg\{ &\delta\left(\frac{b}{a}\right)\begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix} \\
&- \delta\left(\frac{\bar b}{\bar a}\right)\begin{pmatrix} \bar\psi_1^2 \\ \bar\psi_2^2 \end{pmatrix}\Bigg\} dx \\
&-2i\sum^{N}\delta\left(\frac{b_k}{a'_k}\right)\begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix}_k \\
&+ \frac{b_k}{a'_k}\,\delta\zeta_k\,\frac{\partial}{\partial\zeta}\begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix}_k \\
&-2i\sum^{\bar N}\delta\left(\frac{\bar b_k}{\bar a'_k}\right)\begin{pmatrix} \bar\psi_1^2 \\ \bar\psi_2^2 \end{pmatrix}_k \\
&+ \frac{\bar b_k}{\bar a'_k}\,\delta\bar\zeta_k\,\frac{\partial}{\partial\zeta}\begin{pmatrix} \bar\psi_1^2 \\ \bar\psi_2^2 \end{pmatrix}_k.
\end{aligned}
\tag{2.4b}
$$

In addition, we may also write expansions for the vectors $(r, q)^T$ and $(q, -r)^T$:

$$\begin{pmatrix} r \\ q \end{pmatrix} = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{\bar{b}}{a} \begin{pmatrix} \phi_2^2 \\ -\phi_1^2 \end{pmatrix} + \frac{b}{\bar{a}} \begin{pmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{pmatrix} \right\} dx$$

$$\text{(2.4c)} \qquad -2i \sum_1^N \frac{1}{b_k a'_k} \begin{pmatrix} \phi_2^2 \\ -\phi_1^2 \end{pmatrix}_k$$

$$+2i \sum_1^{\bar{N}} \frac{1}{\bar{b}_k \bar{a}'_k} \begin{pmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{pmatrix}_k,$$

$$\begin{pmatrix} q \\ -r \end{pmatrix} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{b}{a} \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix} + \frac{\bar{b}}{\bar{a}} \begin{pmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{pmatrix} \right\} dx$$

$$\text{(2.4d)} \qquad +2i \sum^N \frac{b_k}{a'_k} \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix}_k$$

$$-2i \sum^{\bar{N}} \frac{\bar{b}_k}{\bar{a}'_k} \begin{pmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{pmatrix}_k$$

The squared eigenfunctions satisfy the equation,

$$L \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix} \equiv \frac{1}{2i} \begin{pmatrix} -\frac{\partial}{\partial x} - 2q \int_x^{\infty} dy\, r & -2q \int_x^{\infty} dy\, q \\ 2r \int_x^{\infty} dy\, r & \frac{\partial}{\partial x} + 2r \int_x^{\infty} dy\, q \end{pmatrix}$$

$$\text{(2.5a)} \qquad \cdot \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix} = \zeta \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix},$$

and its adjoint,

$$L^A \begin{pmatrix} \phi_2^2 \\ -\phi_1^2 \end{pmatrix} \equiv \frac{1}{2i} \begin{pmatrix} \frac{\partial}{\partial x} - 2r \int_{-\infty}^{x} dy\, q & 2r \int_{-\infty}^{x} dy\, r \\ -2q \int_{-\infty}^{x} dy\, q & -\frac{\partial}{\partial x} + 2q \int_{-\infty}^{x} dy\, r \end{pmatrix}$$

$$\text{(2.5b)} \qquad \cdot \begin{pmatrix} \phi_2^2 \\ -\phi_1^2 \end{pmatrix} = \zeta \begin{pmatrix} \phi_2^2 \\ -\phi_1^2 \end{pmatrix},$$

respectively. We note that the relation expressing the preservation of the two forms (2.3) is found from (2.4a) and (2.4b) by taking the inner product for two different δ's and using the orthogonality relations. We note further that in the small q, r limit, the equations (2.4) reduce exactly to the expansions for the ordinary Fourier transforms with kernels $e^{\pm 2i\zeta x}$.

From (2.4) and (2.5) it is straightforward to obtain the equations of motion which lead to separable equations in scattering space and which are thereby integrable. We simply identify those operators B which operate on each ζ-component in the expansion for $(r, q)^T$ separately. For these operators, the equation

$$\delta\binom{r}{-q} = B\binom{r}{q} \tag{2.6}$$

simply expresses the variation of the scattering data (which are the coefficients in the expansion $(\delta r, -\delta q)^T$) in terms of the scattering data *at the same* ζ (which are the coefficients in the expansion for $B\binom{r}{q}$). The complete class of operators B with this property is not known (we conjecture that it is the closure of all ratios of polynomials of L^A) but the largest known class is discussed in [38, 39]. We may note straightaway the simplest example which occurs when B is equal to any entire function of L^A. In fact, for $B = 4i(L^A)^2$, the equation (2.6) becomes the generalized nonlinear Schrödinger equation; for $B = 8i(L^A)^3$, the modified Korteweg-deVries equation.

Calogero has pointed out that by simply writing δ as the pseudo-directional derivative

$$\delta = F(L^A)\frac{\partial}{\partial t} + \vec{G}(L^A)\cdot\nabla_{\vec{y}}, \tag{2.7}$$

fully three dimensional equations may be obtained. The equations for the scattering data are given in [39].

One new contribution of the notes to the Arizona meeting (written up in [39]) is the analogous expressions for the potential $q(x, t)$ of the Schrödinger equation

$$V_{xx} + (\zeta^2 + q(x, t))V = 0 \tag{2.8}$$

in terms of appropriate squared eigenfunctions and their derivatives. These expressions are somewhat more complicated than (2.4) for a number of reasons, one of which is that the dual and adjoint eigenfunctions are no longer the same. Nevertheless, the same ideas go through.

In summary, the whole method closely resembles, and is indeed a natural extension of, the ideas of Fourier analysis and normal mode expansions. In order to complete the analogy we present the nonlinear analogue of Parseval's relations (see (58]) or what are also called the trace formulae. They are found by identifying the terms of two asymptotic expansions in ζ for that ubiquitous scattering function $\ln a(\xi, t)$, one written in terms of the original potentials and the other in terms of the scattering data. We write down these expressions for the case $r = -q^*$:

$$C_m = \sum_{n=1}^{N} \frac{\zeta_k{}^m - \zeta_k{}^{*m}}{m} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \xi^{m-1} \ln aa^* d\xi, \quad m = 1, 2, \cdots, \tag{2.9}$$

where the first three members of the set $\{C_m\}_{m=1}^{\infty}$ are:

$$\begin{aligned} C_1 &= \frac{1}{2i} \int_{-\infty}^{\infty} qr\, dx, \quad C_2 = \frac{1}{(2i)^2} \int_{-\infty}^{\infty} qr_x dx, \\ C_3 &= \frac{1}{(2i)^3} \int_{-\infty}^{\infty} (qr_{xx} - q^2r^2)\, dx \end{aligned} \tag{2.10}$$

and the rest can be found from recurrence formulae [58] (following the ideas of Zakharov, Faddeev and Shabat [10], [60]). Indeed for most integrable flows, the $\{C_m\}_{m=1}^{\infty}$ are the conserved quantities; in fact, they are also the flow generators. Namely, each C_m, taken in turn as a Hamiltonian, generates that flow which can be identified with the dispersion relation $\Omega(\zeta) \propto \zeta^{m-1}$. For example, the Hamiltonian for the nonlinear Schrödinger equation is simply $-i \int_{-\infty}^{\infty} (qq^*_{xx} + q^2q^{*2})\, dx$. In each such flow, all of the other conserved quantities and potential Hamiltonians $\{C_n\}_{n=1}^{\infty}$ are also conserved.

The trace formulae serve many useful purposes beyond identifying the form of the Hamiltonian in scattering space. One particular use is in a discussion of the solution behavior, particularly the component arising from the continuous spectrum. Another is the determination of the slow rate of change of the action variables in terms of the slow rate of change of the conserved quantities in problems which are close to being integrable.

3. **A Singular Perturbation Theory.** Given the background of the previous section, it now becomes possible to investigate the behavior of solutions of equations which are perturbations of a member of the integrable class. Kaup [62] was the first to exploit these perturbations and he has calculated the initial effect of damping in the nonlinear Schrödinger equation and in coherent pulse propagation systems. The author of the present paper has been engaged in developing a singular perturbation theory whereby one develops uniform asymptotic expansions for the scattering functions over time scales inversely proportional to the small coupling coefficient multiplying the "nonintegrable" term in the equation. Of particular importance is the development of the concept of "resonance" or at least the extension of this concept from weakly nonlinear to strongly nonlinear systems.

A "weakly" nonlinear system will correspond to the situation when no discrete eigenvalues are present and the system is analyzed in terms of the various functions connected with the continuous spectrum. We know that if we attempt to perform a regular perturbation expansion in the coupling coefficient, it becomes nonuniform due to resonances between waves obeying the conservation of momentum and energy conditions $\Sigma_{i=1}^N k_i = 0$, $\Sigma_{i=1}^N \omega_i = 0$, $(N = 3, 4, 5, \cdots)$. The expansions are renormalized and the nonuniformities are removed by allowing the scattering functions (in linear problems, the Fourier coefficients) to be slowly varying functions of time. (Small insert: the strength of the nonuniformity is a function of the model and one obtains different answers if one assumes a discrete model, a continuous model or a continuous random model. See [63]).

What we are principally after in this discussion is the mechanisms for sustaining "resonances" between soliton solutions, with the ultimate idea of describing interactions between strongly nonlinear "particles." Now the word resonance does not really apply because it is primarily a linear concept and amplitude independent. Namely, one can excite a linear oscillation by tuning to its natural frequency, and its amplitude grows without bound. In a nonlinear system, the oscillator or soliton will detune after its amplitude is changed. Therefore, what is more apt to happen is that nonlinear normal modes *synchronize,* much in thesame way that two coupled Van der Pol oscillators do. Below we briefly describe some interesting results we have obtained in connection with the phase locking of a nonlinear Schrödinger soliton to an external applied field. One might also expect that, if two solitons travel at approximately the same speed, then under certain conditions they can become phase locked due to the effect of the nonintegrable term. We have succeeded in showing that this happens in what we feel to be a typical model-the "double" sine-Gordon equation,

$$u_{xt} = \sin u - \epsilon\mu_0 \sin u/2, \tag{3.1}$$

and the synchronized states bear a close resemblance to the wobblers which Bullough and Caudrey [64] have discovered numerically and which are described in this volume.

We are going to discuss briefly the following six examples. More details are given in [23] and [65].

$$\text{1. a. } q_t - iq_{xx} - 2iq^2q^* = -\epsilon\Gamma q + \epsilon E e^{i\sigma}, \tag{3.2}$$

$$\text{b. } q_t + q_x = -\epsilon\Gamma q, \tag{3.3}$$

$$\text{c. } q_t - iq_{xx} - 2iq^2q^* = -2i\epsilon\delta\nu(x)q, \tag{3.4}$$

$$\text{d. } q_t - iq_{xx} - 2iq^2q^* = \epsilon(\chi q - \beta q^2q^* + \gamma q_{xx}), \tag{3.5}$$

$$\text{2. a. } u_{XX} - u_{TT} = u_{xt} = \sin u + \epsilon g(X),\ x = \frac{X+T}{2},\ t = \frac{X-T}{2}, \tag{3.6}$$

$$\text{b. } u_{XX} - u_{TT} = u_{xt} = \sin u - \epsilon\mu_0 \sin u/2. \tag{3.7}$$

The integrable ($\epsilon = 0$) parts of these six equations are all members of the class (2.6) obtained by setting the operator $B = -2\Omega(L^A)$, where the function $\Omega(\zeta)$ is closely related to the dispersion relation of each of the equations. For example, if $r = -q^*$ and $\Omega = -2i\zeta^2$ we obtain the nonlinear Schrödinger equation; for $r = -q$ and $\Omega = i\zeta$, we obtain 1.b with $\epsilon = 0$; for $r = -q$ and $\Omega = i/4\zeta$ we obtain 2.a,b with $\epsilon = 0$. The method of solving the nonintegrable systems (ϵ small but nonzero) is to map the solutions of each into the scattering space associated with the integrable system. We will then find that the scattering data

$$(3.8) \qquad S = S\left\{\left(\zeta_k(t), \beta_k(t) = \frac{1}{b_k a'_k}\right)_{k=1}^{N}, b^*/a\ (\xi, t)\right\}$$

evolve in time according to the formulae,

$$(3.9) \qquad \zeta_{kt} = 0 + \epsilon P,$$

$$(3.10) \qquad \beta_{kt} + \frac{a''_k}{a'_k}\zeta_{kt}\beta_k = 2\Omega(\zeta_k)\beta_k + \epsilon Q,$$

$$(3.11) \qquad \left(\frac{b^*}{a}\right)_t = 2\Omega(\xi)\frac{b^*}{a} + \epsilon R,$$

where P, Q and R are nonlocal expressions involving infinite integrals over products of the perturbation terms and the squared eigenfunctions [see [38] or [39] for exact formulae]. The method then is to solve (3.9, 11) iteratively, with the zeroth order solution consisting of the normal modes (solitons, breathers, radiation) of the integrable system. It is to be expected that in most of the interesting cases, the resulting asymptotic expansions in powers of ϵ will be nonuniform in time. Using well known ideas of singular perturbation theory, multiple scaling, the WKBJ method, matched asymptotic expansions, etc., we can make these expansions uniform by choosing (a) the slow time ($\epsilon^{-\alpha}$ time scale, $\alpha > 0$) behavior of those parameters (such as the eigenvalues) which in the integrable system were fixed and (b) the appropriate asymptotic expansion sequence in ϵ. The advantage of our approach is that the scattering space of the exactly integrable problem provides the most natural framework in which to describe the motion of the perturbed system. If one solves (3.2-7) by a perturbation approach directly using the integrable techniques to find the zeroth order solution, one is often led at the next order to partial differential equations in which it is difficult to interpret the meaning of secular terms. Of course, if the resulting partial differential equation is synthesized using the natural basis of the integrable system, the results obtained will be precisely the same as those obtained by the direct method described above.

For purposes of explicit calculation, it is usually practical only to include in the zeroth order state a single or multisoliton solution. The single soliton solution of the system (2.6) is given by

$$q(x,t) = 2\eta_1 \operatorname{sech} 2\theta\, e^{-2i(\phi+\pi/4)} \tag{3.12}$$

where $\zeta_1 = \xi_1 + i\eta_1$, $\beta_1 = -2\eta_1 e^{-2\bar\theta - 2i\bar\phi}$, $\theta = -\eta_1 x + \bar\theta$, $\phi = \xi_1 x + \bar\phi$ and to leading order the time dependence of the parameters ξ_1, η_1, $\bar\theta$ and $\bar\phi$ are

$$\xi_{1t} = \eta_{1t} = 0,\ \bar\theta_t = -\Omega_r,\ \bar\phi_t = -\Omega_i \tag{3.13}$$

where $\Omega(\zeta_1) = \Omega_r(\xi_1, \eta_1) + i\,\Omega_i(\xi_1, \eta_1)$. If the soliton solution is to continue to dominate over long times (namely if the energy in the continuous spectrum measured by $b^*/a(\xi, t)$ is to remain small), we must first ensure that the perturbation terms ϵR in (3.11) do not lead to an ϵt or larger algebraic growth in b^*/a. The cause of such a growth will be the time dependence of ϵR. If the perturbation terms are $-\Gamma q$ or $Ee^{i\sigma}$, then we can show that the (fast) time dependence of R is contained in the factors $e^{-2i(\bar\phi+(\xi_1-\xi/\eta_1)\bar\theta)}$ and $e^{i\sigma}$, $e^{-i\sigma-4i\bar\phi}$, respectively. Secular growths in b^*/a will only occur if these phases resonate with the fundamental frequency $2\Omega(\xi)$. Thus the criterion that the energy in the continuous spectrum remains small is that for any ξ, $-\infty < \xi < \infty$, to within $O(\epsilon)$,

$$2\Omega(\xi) \neq -2i(\bar\phi + \frac{\xi_1 - \xi}{\eta_1}\bar\theta)_t, \tag{3.14a}$$

$$2\Omega(\xi) \neq i\sigma_t, \tag{3.14b}$$

$$2\Omega(\xi) \neq -i\sigma_t - 4i\bar\phi_t. \tag{3.14c}$$

In example 1, these criteria hold for 1a, c and d. However, in 1b., $\Omega(\xi) = i\xi$, $\Omega(\zeta_1) = i\xi_1 - \eta_1$ and hence $-2i(\bar\phi + (\xi_1 - \xi/\eta_1)\bar\theta)_t = -2i(-\xi + \xi - \xi) = 2i\xi$ and the resonance occurs for all ξ. Hence the initial condition $q = 2\eta_0 \operatorname{sech} 2\eta_0 x$ will not propagate like $2\eta_0 e^{-2\epsilon\Gamma t} \operatorname{sech} 2\eta_0 \epsilon^{-2\epsilon\Gamma t}(x - t)$ as we would predict if the soliton remained dominant. Rather, the continuous spectrum is excited to an order one magnitude in order that the solution can propagate as $2\eta_0{}^{-\epsilon\Gamma t} \operatorname{sech} 2\eta_0(x - t)$, the exact solution. I am grateful to Herman Flaschka who suggested this example.

In each of the other examples (in 1a., we insist $\sigma_t > 0$), the resonance (3.14) does not occur. In 1a., the time evolution of the soliton parameters η, ξ, $\bar\theta$ and $\bar\phi$ (we will now omit the subscripts 1) is given by ($\tau = \epsilon t$) (see reference [23])

$$\eta_\tau = -2\Gamma\eta - \frac{\pi E}{2} \sin\chi \operatorname{sech} \frac{\xi\pi}{2\eta}$$

$$\xi_\tau = \frac{\pi E}{2\eta} \xi \sin\chi \operatorname{sech} \frac{\xi\pi}{2\eta}$$

(3.15)
$$\chi_\tau = \Omega - 4(\xi^2 + \eta^2) + \frac{\pi E}{2\eta} \cos\chi \operatorname{sech} \frac{\xi\pi}{2\eta} \left[- \frac{\pi^2 \xi}{2\eta} \frac{e^{\xi\pi/2\eta} - \operatorname{sech} \xi\pi/2\eta}{1 + e^{\xi\pi/2\eta}} \right]$$

$$(\bar{\theta}/\eta)_\tau = -4\xi - \frac{\pi E}{4\eta^2} \operatorname{sech} \frac{\xi\pi}{2\eta} \left[\frac{\pi^2}{4} \left(\frac{e^{-\xi\pi/2\eta} - \operatorname{sech} \xi\pi/2\eta}{1 + e^{-\xi\pi/\eta}} \right) \right].$$

where $\chi = \sigma + 2\bar{\phi} + 2\xi/\eta\ \bar{\theta}$ and $\sigma_t = \bar{\Omega} > 0$. We immediately see that $(\xi\eta)_\tau = -2\Gamma(\xi\eta)$ which shows that, depending on initial conditions, either ξ or η tends to zero. Assuming that $\xi = 0$ we find that,

(3.16a)
$$\eta_\tau = -2\Gamma\eta - \frac{\pi E}{2} \sin\chi$$

(3.16b)
$$\chi_\tau = \bar{\Omega} - 4\eta^2$$

(3.16c)
$$(\bar{\theta}/\eta)_\tau = 0.$$

Note, since $\theta = -\eta(x - \bar{\theta}/\eta)$, the equation for $\bar{\theta}$ is really an equation for the position of the envelope. Here we have taken $\xi = 0$ and (3.16c) shows that the central position of the soliton remains stationary, whereas (3.16a), (3.16b) admit the solution $\eta = (\bar{\Omega}/2)^{1/2}$, $\sin\chi_0 = -2\Gamma(\bar{\Omega})^{1/2}/\pi E$ as a stable node as long as the applied frequency $\bar{\Omega}$ lies in the window $0 < \bar{\Omega} < (\pi^2 E^2)/4\Gamma^2$.

We discuss these results and several pertinent and interesting applications in references [23, 65]. For now, we simply point out that the theory has allowed us to describe the long time (ϵ^{-1}) behavior of any soliton belonging to any member of the family (2.6) which is nonresonant in the sense of (3.16). Further we have seen that the different frequency ranges excite different normal modes. If $\bar{\Omega} < 0$, then the continuous spectrum and in particular the modes $\pm(1/2)(-\bar{\Omega})^{1/2}$ are excited. If $\bar{\Omega} > 0$ and fixed, then there is no soliton excitation until the amplitude E is large enough so that $\bar{\Omega}$ lies in the frequency window at which point the soliton synchronizes and acquires an amplitude independent of the forcing amplitude E.

If the external field E is neglected altogether, the soliton slowly decays according to $\eta_{1\tau} = -2\Gamma\eta$, $\xi_{1\tau} = 0$, with its height and speed changing. However, as η approaches zero, which occurs only when

$\tau(=\epsilon t)$ is very large, the continuous spectrum at the wavenumber ξ_1 is strongly excited. This reflects the fact that the NLS soliton requires a nonzero area ($\int_{-\infty}^{\infty} |q| dx$) in order to be generated. Thus, when the eigenvalue approaches zero, an almost monochromatic wavepacket, which itself is described by the same NLS equation, is formed; gradually, it too disperses and decays. The mathematics describing this transition requires some elementary ideas of matched asymptotic expansions although this exercise has not yet been carried out.

Example 1c. takes account of medium inhomogeneities (such as density changes in a plasma, depth changes in deep water) which are contained in $\delta\nu(x)$. If $\delta\nu(x) = \alpha x$, then Chen and Liu [66] have shown that by a simple transformation $q(x, t) = \phi(x + 2\alpha t^2, t)e^{-2i\alpha xt-(4i/3)\alpha^2 t^3}$, ϕ satisfies the unperturbed nonlinear Schrödinger equation. Thus, solitons behave like linear waves in traversing spatial inhomogeneities; namely, they are turned around by a denser ($\alpha > 0$) medium. On the other hand, if $\delta\nu(x) = \beta x^2$, the equation cannot be handled exactly but a perturbation analysis shows that

$$\begin{aligned} \eta_\tau = 0,\ \xi_\tau = \frac{2}{\eta}\beta\bar{\theta},\ \bar{\theta}_\tau = -4\xi\eta,\ \bar{\phi}_\tau \\ = 2i(\xi^2 - \eta^2) - \beta\left(\frac{\pi^2}{32\eta^2} + \frac{\bar{\theta}}{4\eta} + \frac{3\bar{\theta}^2}{2\eta^2}\right). \end{aligned} \tag{3.17}$$

Thus, the NLS soliton (or indeed the soliton of *any* member of the family (2.6)) is trapped in a density minimum ($\beta > 0$), and repelled from a density maximum.

Case 1d. is of importance in examining the post bifurcation behavior of the envelope q of the most unstable wave in a system which supports a continuum of wavenumbers. The equation is very general and was first derived by Newell and Whitehead [67] and used by Stuart, Stewartson and Hocking [68] in their investigation of parallel flow instabilities. Suppose that the following situation prevails: (1) the critical parameter R (Reynolds number) marginally exceeds critical, i.e., $R = R_c(1 + \epsilon^2\chi)$ (2) the dispersion parameter $d^2\omega/dk^2$ is much larger than the diffusion parameter d^2R/dk^2 (the curvature of the dispersion relation exceeds the curvature of the R vs. k stability curve at the critical wavenumber) (3) the imaginary part of the nonlinear coupling coefficient is larger than the real part. In this case we may treat the Newell-Whitehead equation as a perturbation of the nonlinear Schrödinger equation and obtain the following *uniform* description for the motion of the eigenvalue of a single soliton. It is

$$\eta_T = \eta\left(2\chi - \left(\frac{32}{3}\beta + \frac{8}{3}\gamma\right)\eta^2 - 8\gamma\xi^2\right) \tag{3.18a}$$

(3.18b) $$\xi_T = -\frac{32\gamma}{6}\eta^2\xi, \text{ with } R = \epsilon t.$$

This is a most interesting result for it shows that on the time scale $1/\epsilon$, the real part of the eigenvalue decays to zero, and in fact suggests that all isolated eigenvalues converge to the single point (at which time a new description would be necessary),

(3.19) $$\xi = 0, \ \eta = (3\chi/16\beta + 4\gamma)^{1/2}.$$

This means that in the initial value problem, whereas the original profile splits up into pulses (solitons) with different velocities $c_g - 4\xi_j$ (the equation itself is written in the frame of reference of the group velocity c_g of the critical wave), the dispersion is halted by the effects of diffusion (d^2R/dk^2) and eventually all the solitons phase lock in momentum space. This has important consequences for long wave-short wave interactions and the mechanism of Benney [54] for the excitation of long waves, for it suggests that the short wave envelope continues to stay locked with the phase velocity of the long. It is not an unexpected result, as in some sense one expects the wave-number k with the fastest growth rate to be dominant over the sidebands $k - 2\xi$. [see [69]].

In example 2, the unperturbed equation is the integrable sine-Gordon equation. In 2a., we examine the effect of an impurity described by $g(X)$ on a kink (2π pulse)

(3.20) $$u(x, t) = 4\tan^{-1}e^{-2\theta}$$

when $\theta = -\eta x - (1/4\eta)t + \theta_0 = -(1 - V^2)^{-1/2}(X - VT) + \theta_0$, $V = -1 + 2/(4\eta^2 + 1)$. By setting $\bar{\phi} = \pi/4$, we find $\beta = 2i\eta\, e^{-2\bar{\theta}}$ and equations (3.9), (3.10) become

(3.21a) $$\eta_t = -\epsilon/8 \int_{-\infty}^{\infty} g(X)u_x\,dx,$$

(3.21b) $$\bar{\theta}_t = -\frac{1}{4\eta} - \epsilon/8 \int_{-\infty}^{\infty} xg(X)u_x\,dx.$$

There are two cases of interest. The first involves a kink which moves slowly with respect to the fixed impurity and thereby undergoes a *strong* interaction (an order one change over long time). Since the kink velocity is slow, V is small and we write $\eta = 1/2(1 + \mu\psi)$, $0 < \mu \ll 1$. Setting the time scale $\tau = \nu t$, $0 < \nu \ll 1$ and writing $\bar{\theta} + \eta t = \chi$, we find that a balance is achieved when $\mu = \nu = \epsilon^{1/2}$. Then

(3.22a) $$\psi_\tau = -1/4 \int_{-\infty}^{\infty} g(X)u_x\,dx,$$

(3.22b) $$\chi_T = \psi.$$

Finally we note that (3.22a) can be written in the form

(3.23) $$\psi_T = -\frac{\partial U}{\partial \chi}, \quad U(\chi) = -\frac{1}{8}\int_{-\infty}^{\infty} g(X)u(X, \chi)\, dX,$$

since the phase of u is approximately $-(1/2)X + \chi$. Hence, the kink simply acts as a Newtonian particle under the influence of the potential U. For example, if $g(X) = \alpha\delta(X - \bar{X}) - \alpha\delta(X + \bar{X})$, the potential U is

(3.24) $$U(x) = \frac{\alpha}{2}\tan^{-1}e^{\bar{X}-2x} - \frac{\alpha}{2}\tan^{-1}e^{-\bar{X}-2x}.$$

The kink is trapped in a periodic motion if $\alpha < 0$ and is repelled if $\alpha > 0$.

These results were first obtained by Fogel, Bishop, Krumhansl and Trullinger [70]. Their method was to perturb the sine-Gordon equation about the kink solution and to solve the resulting inhomogeneous Schrödinger equation (with sech^2 potential) in terms of the complete set of eigenfunctions which are known for that particular operator. Their method is restricted by the fact that perturbations about other types of multisoliton solutions would not be possible. In addition, their analysis was algebraically complicated, and the particular form of the impurity (used in our example above) was crucial. On the other hand, by expanding in the correct basis (the squared eigenfunctions), the analysis becomes relatively straightforward.

The second case involves a fast-moving kink. Here after a small amount of calculation (see [65] for details), we can show that the kink behaves like a relativistic particle under the influence of a scalar potential. In fact,

$$\frac{d}{dT}\frac{V}{(1 - V^2)^{1/2}} = -\frac{\partial}{\partial X_0}U$$

$$\frac{d}{dT}\frac{1}{(1 - V^2)^{1/2}} = -\frac{\partial}{\partial T}U$$

where

(3.25) $$U(X_0, T) = -\frac{\epsilon}{8}\int_{-\infty}^{\infty} g(X)u(X - VT - X_0)\, dX.$$

Finally we examine example 2b., which is perhaps the most novel result. Let us suppose that in the $\epsilon = 0$ limit, there are two 2π pulses given by the double soliton (4π pulse) solution,

$$u(x, t) = 4 \tan^{-1} \frac{1 - \left(\dfrac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \right)^2 e^{2\theta_1 + 2\theta_2}}{e^{2\theta_1} + e^{2\theta_2}} \tag{3.26}$$

where $\theta_j = -\eta_j x + \overline{\theta}_j$, $\overline{\theta}_j = -t/4\eta_j + \overline{\theta}_{j0}$. In laboratory coordinates $\theta_j = -(1/2)(\eta_j + 1/4\eta_j)\,(X - V_j T - X_j)$. It is fairly clear that a strong interaction will only take place when η_1 and η_2 are close; i.e., the kinks remain together long enough for a strong interaction to take place. Treating the parameter $\beta = (\eta_1 - \eta_2)/(\eta_1 + \eta_2)$ as small, we obtain the following perturbation series for the energies (the eigenvalues η_j) and the positions $\beta_j = (1/2i\eta_j a_j^2)\, e^{-2\overline{\theta}_j}$ of the two pulses:

$$\eta_{1t} = -\eta_{2t} = \frac{\epsilon\mu_0}{2} \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} + O(\epsilon\beta), \tag{3.27a}$$

$$\beta_{1t} = \frac{1}{2\eta_1}\beta_1 + \frac{\epsilon\mu_0}{2\eta\beta}\beta_1 - \frac{1}{\eta\beta}\eta_{1t}\beta_1 + O(\epsilon \ln \beta), \tag{3.27b}$$

$$\beta_{2t} = \frac{1}{2\eta_2}\beta_2 - \frac{\epsilon\mu_0}{2\eta\beta}\beta_2 + \frac{1}{\eta\beta}\eta_{2t}\beta_2 + O(\epsilon \ln \beta). \tag{3.27c}$$

More details can be found in [**71**]. We note that the average $\eta = (\eta_1 + \eta_2)/2$, which is proportional to the total energy of the system, is conserved in agreement with the basic equation (3.7). We observe that if the two pulses are widely separated, with the η_2 pulse a transition from -2π to 0 to the left (in X, T space) of the η_1 pulse, sending u from 0 to 2π, then

$$\eta_{1t} = \frac{-\epsilon\mu_0}{2} \text{ and } \eta_{2t} = \frac{+\epsilon\mu_0}{2}. \tag{3.28}$$

This agrees precisely with the equation (3.7), since

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} u_x^2 = 16\, \frac{\partial \eta}{\partial t} = +4\mu_0\epsilon \left[\cos \frac{u}{2} \right]_{u=a}^{u=b} \tag{3.29}$$

and if the jump in u is from 0 to 2π we obtain the opposite sign from when it makes the transition from -2π to 0. Recalling that $\eta_{1t} = -2\eta_{1T}$ where T is real time, we note that for $\mu_0 > 0$, the right-most pulse increases (with T) its value of η and thereby decreases its velocity $V = -1 + 2/1 + 4\eta^2$ whereby the left-most pulse increases its velocity. Hence, we would expect a possible phase locking only when $\mu_0 > 0$ which as we shall see is exactly what happens.

Returning to a perturbation analysis of (3.27) we see that for

$1 \gg \beta \gg 0(\epsilon^{1/2})$, a WKBJ type analysis will show that the pulses interact without a strong exchange. On the other hand, when $\beta = 0(\epsilon^{1/2})$, we can develop singular perturbation solutions by introducing the following changes of variables:

$$\beta = \epsilon^{1/2} b(\tau),\ \tau = \epsilon^{1/2} t,\ \beta_j = b_j(\tau) e^{(1/2\eta)t}. \tag{3.30}$$

For $0 < \beta \ll O(\epsilon^{1/2})$, the expansions break down and a further inner expansion is needed which will use for a basic description (3.26), the solution corresponding to a double zero of $a(\zeta)$. However, unless the initial speeds of the two pulses are this close, this description is not required.

One obtains the following result:

$$\begin{aligned} b_1 - b_2 &= + \frac{2\eta}{\mu_0} (b_1 + b_2)\, b_\tau \\ b_1 + b_2 &= \text{const. } e^{-(1/2\mu_0)b^2} \end{aligned} \tag{3.31}$$

and $z = b^2$ satisfies the following equation,

$$z_{\tau\tau} - \frac{1}{2\mu_0} z_\tau^{\ 2} + \frac{\mu_0}{2\eta^2}(z - \mu_0) = 0. \tag{3.32}$$

Equation (3.32) may be integrated once,

$$\frac{\eta^2}{\mu_0^{\ 2}} z_\tau^{\ 2} = z - (\mu_0 - C) e^{1/\mu_0(z - \mu_0)}. \tag{3.33}$$

We can write C for the present model in terms of $z_0 = b^2(0)$, which describes the initial difference in velocities, and $\nu = b_1(0) - b_2(0)/b_1(0) + b_2(0)$, $-1 < \nu < 1$ which gives the initial separation. Then (3.33) becomes

$$\frac{\eta^2}{\mu_0^{\ 2}} z_\tau^{\ 2} = z - p e^{z/\mu_0},\ p = (1 - \nu^2) z_0 e^{-z_0/\mu_0}$$

and the solution oscillates between the two real positive roots of the right hand side z_1 and z_2. The point $z = \mu_0$, $z_\tau = 0$ is a stable center and in general for $0 < C < \mu_0$, the solution orbit is a stable periodic one about this center. The largest stable orbit occurs for C just less than μ_0, for which value z_1 is very small and z_2 large. In the present model in which only two pulses are included, C is always less than μ_0 (equality occurs only when $\nu = -1$) and the two pulses always synchronize for $\mu_0 > 0$. (Whereas the solution behavior depends considerably on the sign of μ_0, the sign can be readily changed in the equation by letting $u \to u + 2\pi$.) We can define the stability of the synchronized state or the binding energy of the two pulses to be the minimal distance between the given orbit and the largest stable orbit.

Given the existence of these phase-locked pulses, we can play the following game. Imagine a perfect universe with field equation (3.7) with $\epsilon = 0$ whose solutions interact only by shifting position but not character and identity (the solitons). Now suppose the perfect inverse is perturbed into a "real world" described by (3.7) with $\epsilon \neq 0$ and the particles of the real world consist of synchronized collections of noninteracting particles of the perfect universe. However, the particles of the real world can interact strongly. One can well imagine that a solution consisting of four synchronized pulses can interact with a solution consisting of four synchronized pulses to split into a number of pulses with different identities. Such a decomposition will naturally depend on the binding strength (or related stability) of each particle. Studies along these lines are continuing.

There has been also some progress made on the original question posed in this section. Miles [74, 75] has found that there is a critical angle at which the interaction between two almost parallel solitons [solitary waves!] of the water wave equation breaks down. Indeed if one looks at the Hirota solution [76] for the Kadomtsev-Petviashvili [77] equation, it is clear that something dramatic occurs when the criterion for resonance associated with linear waves is satisfied! We are presently attempting to describe the meaning of these singularities in the Hirota solution in terms of changes in the momenta and positions of the three solitons. One expects that the nonlinear concepts of synchronization and phase locking are also valid here.

Whereas it is evident at the present time that many of the latter suggestions are pure conjecture and highly speculative, it is also clear that the singular perturbation approach is a powerful way in which to obtain information about systems which are close to being integrable.

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