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SOME FINITE DIMENSIONAL INTEGRABLE SYSTEMS

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1. We consider Hamiltonian systems of n particles on a line interacting with each other where the Hamiltonian is of the form:

$$(1)\qquad \begin{aligned} H(x, y) = \frac{1}{2} \sum_{i=1}^{n} y_i^2 + \sum_{1 \leqq i < j \leqq n} V_1(x_i - x_j) \\ + \sum_{i=1}^{n} V_2(x_i),\ V = (V_1, V_2). \end{aligned}$$

Examples are

$$V^{(1)}(x) = \left(x^{-2},\ -\frac{\alpha^2}{2} x^2 \right),$$

$$V^{(2)}(x) = \left(\frac{1}{2} \coth \frac{x}{2},\ \alpha e^x \right).$$

Calogero and Marchioro [1] and Sutherland [2] have studied some of these systems in the context of quantum mechanics, and their work suggested looking at the classical systems. For the case $\alpha = 0$, J. Moser [3] has shown that both of the above examples are integrable systems, i.e., possess n integrals whose associated Hamiltonian flows commute, and in addition the integrals are rational in (x_i, y_i), (e^{x_i}, y_i) respectively. The method he used was based on the isospectral technique of Lax [4] as first applied by H. Flaschka [5] to the Toda lattice. This consists in the construction of a matrix function of (x, y) whose spectrum remains fixed in t if $x = x(t)$, $y = y(t)$ are solutions of the above Hamiltonian system. We then take functions on the spectrum to be the desired integrals and study systems whose Hamiltonians are functions of these integrals.

We extend this method to some new systems. Moreover we construct a second matrix function of (x, y, t) whose spectrum is invariant under the Hamiltonian flow, which allows us to describe the solutions more or less explicitly. In this way it turns out that all solutions of the system with $V = V^{(1)}$, α purely imaginary, are periodic.

It is then easy to discuss the scatttering behavior of the above systems quite explicitly in the case the particles ultimately disperse, like in

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the case $\alpha > 0$, and to construct scattering maps. These scattering maps are canonical, given by polynomial relations, and lead to surprising algebraic transformations between the above systems. Moreover, they are found to be their own inverses.

It is surprising that the scattering map for the system with $V = V^{(1)}$, $\alpha > 0$, which relates data at $t = -\infty$ with data at $t = +\infty$, is precisely equal to the scattering map for the system with $V = V^{(1)}$, $\alpha = 0$ which relates data at $t = 0$ with data at $t = \infty$. Another interesting fact is that the scattering map for the system with $V = V^{(1)}$, $\alpha > 0$, which relates data at $t = 0$ with data at $t = \infty$, transforms the Hamiltonian of that system into an integral of the system with $V = V^{(2)}$, $\alpha = 0$, after a trivial change of coordinates.

2. **Methods of Solution and Results.** We describe the method for one example, namely (1) with $V = V^{(1)}$.

(A) To solve the equations of motion $\dot{x}_i = \partial H/\partial y_i$, $\dot{y}_i = -\partial H/\partial x_i$, $(x, y) \in \Omega = \{(x, y), x, y \in R^n, x_i < x_{i+1}$, for all $i\}$ with $V = V^{(1)}$, we write the differential equations in matrix form. Following J. Moser [3] we define the matrix functions $L(x, y)$, $B(x)$ on Ω by

$$Y_{jk} = \{L(x, y)\}_{jk} = \delta_{jk} y_j + \frac{i(1 - \delta_{jk})}{(x_j - x_k)}, \tag{2}$$

$$B_{jk} = \{B(x)\}_{jk} = i \left\{ \delta_{jk} \left(- \sum_{s \neq j} B_{js} \right) + \frac{i(1 - \delta_{jk})}{(x_j - x_k)^2} \right\}. \tag{3}$$

In addition we define $X = \operatorname{diag}(x_1, \cdots, x_n)$, and denote the Hamiltonian vector field $D_H = \sum_{i=1}^n (H_{y_i} \partial_{x_i} - H_{x_i} \partial_{y_i})$, which acts on matrices componentwise, and then compute

$$\dot{X} = D_H X = Y + [B, X], \quad \dot{Y} = D_H Y = \alpha^2 X + [B, Y] \tag{4}$$

or equivalently

$$D_H(Y \pm \alpha X) \mp \alpha (Y \pm \alpha X) = [B, Y] \pm \alpha [B, X] = [B, Y \pm \alpha X], \tag{5}$$

which, upon letting $M^{\pm} = (Y \pm \alpha X) e^{\mp \alpha t}$, is equivalent to

$$\dot{M}^{\pm} = [B, M^{\pm}]. \tag{6}$$

Equation (6) expresses the equations of motion in isospectral from, i.e., the spectrum of $M^{\pm}$ is preserved in time. Observing that all products of

M^+, M^- satisfy (6), we form the time independent matrix $E = (1/2)\, M^+M^-$, and the traces of its powers. They are clearly rational functions of (x, y), and being functions of the eigenvalues they are the desired integrals. It turns out that their Poisson bracket vanishes. To compute the explicit solution, observe that

$$X = \frac{1}{2\alpha}\,(e^{\alpha t}M^+ - M^-e^{-\alpha t}) \tag{7}$$

and raise both sides of (7) to the ν^{th} power. Taking the trace and using the observation that all products of M^+, M^- undergo an isospectral deformation we arrive at

$$\sum_{i=1}^{n} x_i^{\,\nu} = \sum_{s=0}^{\nu} c_\nu^{\,s} e^{\alpha t(\nu - 2s)}. \tag{8}$$

Here the $c_\nu^{\,s}$ are rational functions of the initial conditions and independent of t. In particular for purely imaginary α we find that all solutions are indeed periodic, verifying a conjecture of F. Calogero [6].

(B) The integrals found above were of the form $\operatorname{tr} E^p$. More generally, $\operatorname{tr} f(E) \equiv H^f$ for any polynomial f gives rise to an integral. It is thus natural to consider H^f as the Hamiltonian of a flow, just as $\operatorname{tr} E$ is the Hamiltonian for the given system. For these more general flows $\operatorname{tr} E^p$ are, of course, integrals, and one may ask for isospectral matrices for these flows, and their explicit solutions, in analogy to A). We would find in this case that

$$E,\ M^+ = e^{-\alpha t f'(E)}(Y + \alpha X),\ M^- = (Y - \alpha X)e^{\alpha t f'(E)}$$

are isospectral during this flow, and we obtain the explicit behavior of the solutions from

$$X = \frac{1}{2\alpha}\,(e^{\alpha t f'(E)}M^+ - M^-e^{-\alpha t f'(E)}). \tag{9}$$

The equations of motion are expressed in the form

$$\dot E = [B_f, E],\ \dot M^{\pm} = [B_f, M^{\pm}], \tag{10}$$

with some matrix B_f, quite analogous to (3). The determination of B_f has been carried out, which we describe only for $\alpha = 0$. Thus (A) is completely generalized.

(C) The case (B) for $\alpha = 0$ is itself of interest and we mention the result. With X, Y defined as above, $H = H^f = \operatorname{tr} f(Y)$, we have the beautiful formula

$$\{B_f\}_{jk} = \{B_f(x,y)\}_{jk} = \delta_{jk}\left(-\sum_{s\neq j} B_{js}\right) + \frac{(1-\delta_{jk})}{(x_j - x_k)}[f'(Y)]_{jk}. \tag{11}$$

Now the matrix form of the Hamiltonian equations of motion is

$$D_H X = f'(Y) + [B, X],\ D_H Y = [B, Y], \tag{12}$$

and defining $M = X - tf'(Y)$ we have

$$\dot{M} = [B_f, M],\ \dot{Y} = [B_f, Y]. \tag{13}$$

Thus M, Y have fixed spectrum and we obtain the explicit behavior of the solution via:

$$X = \mathrm{t}f'(Y) + M. \tag{14}$$

For $f(s) = (1/2)\,s^2$, J. Moser [3] has solved the above system, and obtained the above result.

(D) In a similar manner we can discuss the system with the potential $V = V^{(2)}$, $\alpha = 0$. We define the matrices

$$\begin{aligned} Y_{jk} &= \delta_{jk} y_j + \frac{i}{2}(1-\delta_{jk}) \coth \frac{(x_j - x_k)}{2}, \\ [B(x)]_{jk} &= i\left\{-\left(\sum_{s\neq j} B_{js}\right)\delta_{jk} + (1-\delta_{jk})\left(\frac{1}{2}\sinh^{-1}\frac{(x_j-x_k)}{2}\right)^2\right\} \end{aligned} \tag{15}$$

and in addition $e^X = \mathrm{diag}\,(e^{x_1}, e^{x_2}, \cdots, e^{x_n})$, $Y^{\pm} = Y \pm (1/2)C$, where $C_{jk} = i(1 - \delta_{jk})$. For this case the differential equations in matrix form are

$$\begin{aligned} D_H Y^{\pm} = [B, Y^{\pm}],\ D_H e^X &= [B, e^X] + e^X Y^- \\ &= [B, e^X] + Y^+ e^X. \end{aligned} \tag{16}$$

Letting $M^- = e^X e^{-tY^-}$, $M^+ = e^{-tY^+} e^X$, the isospectral equations of motion become

$$\dot{M}^{\pm} = [B, M^{\pm}],\ \dot{Y}^{\pm} = [B, Y^{\pm}], \tag{17}$$

and we find the explicit behavior of the solutions from

$$(18) \qquad e^X = M^- e^{tY^-} = e^{tY^+} M^+.$$

The above can be shown to be equivalent to a result of Olshanetzky and Perelomov.

(E) In an obvious generalization of (B), we consider the flows generated by the integrals constructed, such as $H = H^f = \operatorname{tr} f(Y^+) = \operatorname{tr} f(Y^-)$. In order to get these equations in matrix form, one merely replaces $Y^\pm$ by $f'(Y^\pm)$ in (16), (17), (18). Determination of the matrix B_f, with which the equations take the form

$$\dot{Y} = [B_f, Y], \ \dot{M}^\pm = [B_f, M^\pm],$$

is easily accomplished.

(F) We now consider the system with $V = V^{(2)}$, where $\alpha \neq 0$. Then with the same definition of Y, $Y^\pm$, e^X, B as in D), and with $E = (1/2)Y^+Y^- + \alpha e^X$, the equations of motion imply that

$$(19) \qquad \dot{E} = [B, E],$$

and the eigenvalues of E are the n commuting integrals. For this system and $\alpha > 0$, the symmetric polynomials in e^{x_k} turn out to be rational functions of n exponentials $e^{\lambda_1 t}, \cdots, e^{\lambda_n t}$. To be more precise, this can be recovered from the formula for $\alpha \neq 0$:

$$(20) \qquad e^X \sim (2\alpha^{-1}\dot{P}P^{-1})^{\dot{}}, \ P = A_1 e^{\Lambda t} A_2 + A_3 e^{-\Lambda t} A_4.$$

where $\sim$ indicates unitary equivalence, and A_i, Λ are constant matrices. For $\alpha > 0$, the eigenvalues of 2Λ are distinct and positive, in fact they are the $-\lambda_i$'s $\equiv v_i$'s in (26). For the case $\alpha < 0$, the eigenvalues need not be distinct, or even real.

3. **Scattering Theory.** The scattering theory of these systems is intimately bound up with the above procedure. The construction of a scattering map for a set of differential equations is really a method of linearizing the equation, and since the maps we shall construct are canonical, they provide an alternate route for deriving some of the results in the previous section. However, as the formal properties of the systems in the previous section have been discussed, we shall instead concentrate on properties of the maps themselves, and on the light they throw on interrelations among the above systems.

For all the following cases, the particles cannot collide so that we can order them according to $x_i < x_{i+1}$, for all i, and all t.

(A) For the case $V = V^{(1)}$, α real, all solutions behave asymptotically as

$$x_i = \frac{1}{2^{1/2}}\left(q_i{}^{\pm} e^{\pm\alpha t} + \frac{p_i{}^{\pm}}{\alpha} e^{\mp\alpha t} \right) + \mathcal{O}(e^{-2|\alpha t|}), \quad t \to \pm\infty, \tag{21}$$

where $q_i{}^{\pm} < q_{i+1}^{\pm}$ for all i. We are thus able to define the canonical self-maps of Ω, $\phi_{\pm}$, ϕ by

$$\phi_+(x, y) = (q^+, p^+), \quad \phi_-(x, y) = (q^-, p^-).$$

Then if

$$\rho(x, y) = (x, -y) \text{ then } \phi_- = \phi_+ \circ \rho,$$

and if

$$\phi(q^-, p^-) = (q^+, p^+) \text{ then } \phi = \phi_+ \circ \phi_-{}^{-1} = \phi_+ \circ \rho \circ \phi_+{}^{-1}.$$

If one lets L be defined as in § 2 (A), $L_\alpha \equiv 1/(2^{1/2}\alpha)\, L$, and if $\sim$ indicates unitary equivalence, then ϕ_+ is given implicitly by the following algebraic relations:

$$\begin{aligned} X &\sim L_\alpha(q^+, p^+ + \alpha q^+), \\ L_\alpha(x, y + \alpha x) &\sim Q, \end{aligned} \tag{22}$$

where the same unitary equivalence occurs in both formulas and $X = \operatorname{diag}(x_1, \cdots, x_n)$, $Q = \operatorname{diag}(q_1, \cdots, q_n)$. We read off at once that $\phi_+ \circ \phi_+ =$ indentity. One also shows that ϕ, for $\alpha = 1$, is given implicitly by the algebraic relations:

$$\begin{aligned} &Q^- \sim L^T(q^+, p^+), \text{ (where } T \text{ indicates transpose)} \\ &L(q^-, p^-) \sim Q^+, \end{aligned} \tag{23}$$

where we use the same L as in § 2 (A), $Q^{\pm} = \operatorname{diag}(q_1{}^{\pm}, \cdots q_n{}^{\pm})$, and the same unitary equivalence occurs in both formulas.

(B) For the case $V = V^{(1)}$, $\alpha = 0$, J. Moser [3] has shown that all solutions behave asymptotically as:

$$x_i(t) = v_i t + w_i + \mathcal{O}(1/t), \text{ where } t \to +\infty,\ v_i < v_{i+1} \tag{24}$$

for all i. So if we define the canonical self-map of Ω, ψ by $\psi(x, y) = (v, w)$, we find the intriguing formula:

$$\phi = \psi, \tag{25}$$

i.e., the scattering map relating data at $t = -\infty$ to data at $t = +\infty$ for $V = V^{(1)}$, $\alpha > 0$ is equal to the scattering map relating data at $t = 0$

to data at $t = \infty$ for the case $V = V^{(1)}$, $\alpha = 0$. Since $\phi = \phi_+ \circ \rho \circ \phi_+^{-1}$, $\phi = \psi$, we find that the involution ψ is in fact conjugate to a linear involution.

(C) The two systems discussed here both have a set of integrals associated with them. The two sets of integrals can be transformed into each other by the canonical map ϕ_+ combined with a simple change of variables $q_k = e^{x_k}$, $-p_k q_k = y_k$. Thus the two systems are intimately related.

(D) We conclude by mentioning the case $V = V^{(2)}$, $\alpha > 0$. There we have for all solutions the following asymptotic behavior:

$$x_i = \pm v_i t + w_i^{\pm} + \mathcal{O}(1/t),\ v_i < 0,\ v_i < v_{i+1} \quad \text{for all } i,\ t \to \pm\infty, \tag{26}$$

$$w_i^+ + w_i^- = 2\ln(\alpha v_i^2). \tag{27}$$

These results indicate the many interrelations and surprising connections of these integrable systems, but the real theory underlying this phenomenon is still not clear. I wish to thank J. Moser for his many suggestions in both formulation and technical specifics.

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