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SOLITONS AS APPROXIMATE DESCRIPTIONS OF PHYSICAL PHENOMENA

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ABSTRACT. Those partial differential equations that exhibit soliton solutions often arise as approximate models of physical systems. These approximations restrict the range of experiments in which one might hope to identify and observe solitons in the original system. In two typical examples, it is shown which restrictions are implied by the approximation scheme, and what experimental data would be required to identify a soliton in the physical system.

1. **Introduction.** Solitons are special solutions of certain models of physical systems. (For the purpose of this discussion, we may define a soliton to be a localized travelling wave solution of a nonlinear partial differential equation that asymptotically preserves its shape and velocity upon interaction with any other localized disturbance.) The review paper by Scott, Chu and McLaughlin [10] gives an extensive list of the physical systems in which solitons arise, and the list has grown considerably since the time (1973) of that publication. However, every wave is not a soliton, and it might be appropriate to reiterate in these Proceedings the limitations on the initial data and the time scales required to observe the behavior of solitons. It is hoped that such a summary might be especially helpful in the interpretation of experimental observations of wave interactions, from which the existence of solitons is inferred (sometimes incorrectly).

The two main points of this paper are the following.

(1) Partial differential equations that exhibit solitons arise as approximate models of certain physical systems. The time scale over which the model is valid is dictated by these approximations in terms of the initial data. Only observations made within this time scale provide information about solitons.

(2) Partial differential equations that exhibit solitons (on $-\infty < x < \infty$) also exhibit decaying oscillations. Even though these oscillations vanish as $t \to \infty$, at any finite time (when experimental data is taken) it may be difficult to determine experimentally what part of the data should be attributed to solitons and what part to the oscillations.

These considerations apply to any problem which is modelled by an

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equation with solitons. However, in order not to confuse the discussion, we restrict our attention to two examples, the Korteweg-deVries equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

as a model of the Fermi-Posta-Ulam problem (FPU, 1955), and the nonlinear Schrödinger equation

$$iu_t + u_{xx} + 2|u|^2u = 0 \tag{2}$$

as a model of the evolution of almost monochromatic waves of moderate amplitude in a weakly nonlinear, dispersive system.

In either of these systems, the solution consists of waves which propagate as *linear* waves on a short time scale; the emergence of (nonlinear) solitons occurs only on a long time scale. That these waves retain their identities despite interactions that occur quickly (i.e., on a short time scale) suggests not that the waves are solitons, but that they are linear waves whose interaction time is too short for them to affect each other significantly. It is only when the interactions occur slowly (i.e., on a long time scale) and the waves still retain their identities that solitons are suggested. Consequently, in physical or numerical experiments which are intended to observe solitons, observations must be made over this long time scale in order to be conclusive. (Precise meaning of "long time" and "short time" depends on the particular problem, as the examples below will show.)

On the long time scale, solitons separate in space, and each acquires its characteristic shape. However, the envelope of the oscillations also acquires a characteristic shape, the amplitude of which decays slowly (algebraically) in time. Depending on the problem, these two shapes may or may not differ significantly. In solutions of (1), there is no confusion between the solitons and the oscillations. On the other hand, depending on the initial conditions, it may be impossible to examine the solution of (2) at any single time and determine experimentally which wave packets are solitons and which are not. Obviously, this problem can complicate any experimental search for solitons.

It should be stated at the outset that the main points of this paper are not new. The have been stated or implied in previous papers by Zabusky and Kruskal [16], Su and Gardner [14], Hammack and Segur [3], Kruskal [7], Newell [8], Rogers and Mei [9], Segur and Ablowitz [12], and Segur [13]. However, because of their implications in relating solitons to observations of physical systems, perhaps they are worth noting again.

2. **The KdV model.** Let us consider first an FPU problem, which describes the motion of a one-dimensional chain of identical masses, con-

nected by weakly nonlinear springs. The governing equations are

$$\begin{aligned} my_{i_{tt}} &= k(y_{i+1} - 2y_i + y_{i-1})(1 + \alpha(y_{i+1} - y_{i-1})) \\ &\text{for } i = 1, 2, \cdots, N-1, \text{ with } y_0, y_N \text{ given.} \end{aligned} \tag{3}$$

In the numerical experiments by FPU, the nonlinear effects were relatively small and the energy remained largely in the lowest modes (i.e., the long waves).

Zabusky and Kruskal [16] replaced the original discrete model with a continuum model, by expanding y_{i+1} and y_{i-1} in terms of y_i by means of a Taylor series (c.f. Kruskal [7]). The continuous analogue of (3) is

$$y_{tt} = \frac{kh^2}{m}\left(y_{xx} + \frac{h^2}{12} y_{xxxx} \right)(1 + 2\alpha h y_x). \tag{4}$$

This approximation can be made either by letting $N \to \infty$, h fixed, or by letting $N \to \infty$, $h \to 0$, (Nh) fixed. In the first limit, (4) is defined on $(-\infty < x < \infty)$, and one ultimately obtains the KdV equation in the form solved by Gardner, Greene, Kruskal and Miura [2]. We restrict our attention to this problem.

For definiteness, we assume that the initial data has a typical amplitude, a, and typical wavelength, D. In terms of these parameters, the following assumptions have been made in deriving (4):

(i) long waves (to justify truncating the Taylor series),

$$h \ll D;$$

(ii) weakly nonlinear springs,

$$\alpha h a \ll D.$$

Neglecting both of these small effects, yields the linear wave equation:

$$y_{0_{tt}} = c^2 y_{0_{xx}}, \tag{5}$$

where $c^2 = kh^2/m$. The solution is well known:

$$y_0(x, t) = a\, f\left(\frac{x - ct}{D}\right) + ag\left(\frac{x + ct}{D}\right), \tag{6}$$

where f, $g = O(1)$. This solution is also an approximate solution of (4) for a limited time:

$$\frac{ct}{D} = \frac{h}{D}\left(\frac{k}{m}\right)^{1/2} t = O(1). \tag{7}$$

During this time, the right-running (f) and left-running (g) waves do not interact because the nonlinear effects are simply too weak to influence the motion significantly over such a short time. This defines the "short time scale" mentioned above.

In order to determine the long time scale, we formalize the approximation procedure used by Zabusky and Kruskal by introducing a small parameter

$$\epsilon = \frac{\alpha h a}{D} \ll 1, \tag{8}$$

and by making the additional assumption

$$\left(\frac{h}{D}\right)^2 = O(\epsilon) = 24\epsilon, \tag{9}$$

so that the two small effects are of comparable size. We need two (dimensionless) time scales:

$$t_1 = \frac{ct}{D}, \quad \tau = \frac{c\epsilon}{D}\, t, \tag{10}$$

and a perturbation expansion of the solution of (4):

$$\begin{aligned} y(x, t; \epsilon) &= a f\left(\frac{x - ct}{D}\ ;\ \frac{c\epsilon}{D}\ t\right) \\ &+ ag\left(\frac{x + ct}{D}\ ;\ \frac{c\epsilon}{d}\ t\right) + \epsilon y_1 + O(\epsilon^2). \end{aligned} \tag{11}$$

Substituting (11) back into (4) and eliminating secular terms; one obtains necessary conditions for (11) to remain uniformly valid on a "long-time scale":

$$\frac{\epsilon c}{D}\ t = \alpha\, a \left(\frac{h}{D}\right)^2 \left(\frac{k}{m}\right)^{1/2} t = O(1). \tag{12}$$

These conditions are as follows.

(i) In terms of the variables

$$\frac{x - ct}{D} = \zeta,\ \tau = \frac{\epsilon\, ct}{D},\ u = f_\zeta,$$

the right-running wave must satisfy the KdV equation:

$$u_\tau + uu_\xi + u_{\xi\xi\xi} = 0. \tag{1a}$$

(ii) In terms of

$$\frac{x + ct}{D} = \epsilon, \qquad \tau = \frac{\epsilon c}{D} t,\ V = g_\epsilon,$$

the left-running wave must also satisfy the KdV equation:

$$-V_\tau + VV_\eta + V_{\eta\eta\eta} = 0. \tag{1b}$$

(iii) In this same notation,

$$f,\ f_\xi,\ g,\ g_\eta = O(1).$$

The last condition assures that the two wavetrains, f and g, are too weak and too localized to affect each other, even on this long time scale. The first two conditions then determine how each wave evolves (independent of the other) as it interacts with itself.

Solitons emerge only on this long time scale, a point that is not always observed in interpreting experimental results. For example, in the numerical experiments by K. Miura, discussed by Jackson (these Proceedings), waves traveling in opposite directions are observed to pass through each other (on a short-time scale). This observation does not necessarily suggest that solitons exist in this system, but only that the system is weakly nonlinear and hyperbolic. The observed interaction is virtually linear.

The "Boussinesq equation"

$$\phi_{xx} - \phi_{tt} + 6(\phi^2)_{xx} + \phi_{xxxx} = 0, \tag{13}$$

is sometimes proposed as a preferable alternative to the *KdV* equation, because it not only possesses soliton solutions but also allows waves to travel in either direction. As a model of either (3) or (4), it is accurate only to $O(\epsilon^2)$. Consequently, the time scale over which it is a valid model is given by (12), the same as for the *KdV* equation. Thus, if one interprets the "*KdV* model" to mean not just (1), but the entire perturbation scheme, starting with (5), that ultimately yields (1) then the *KdV* and Boussinesq equations are equally valid approximate models of (3), or (4).

Typically, the *KdV* model arises in an autonomous system that is non-dissipative, weakly nonlinear and weakly dispersive. In such a system, the solution can be described in terms of:

(i) a short time scale, during which the appropriate model is the wave equation, (5); and

(ii) a long-time scale, during which the appropriate model is the *KdV* equation, (1), for each of the two solutions of (5). This point was first noted explicitly by Su and Gardner [14].

3. **The nonlinear Schrödinger model.** The nonlinear Schrödinger equation, (2), is comparable to the *KdV* equation, (1), in that it also describes the evolution on a long time scale of a wave that satisfies a linear equation on a short time scale. In either case, the original system must be nondissipative. The difference between the two models is as follows.

(i) If the problem is weakly dispersive and weakly nonlinear (as was (4)), then the zeroth order approximation is the linear wave equation

(5), and one can expect the KdV equation (1) to appear at the next order.

(ii) If the problem is fully dispersive and weakly nonlinear, then the zeroth order approximation is linear and dispersive, and one can expect the nonlinear Schrödinger equation to appear at higher order.

The derivation of (2) is well documented in the literature (e.g., Newell [8], Hasimoto and Ono [5], or any of the appropriate references in Scott, et al. [10]. We mention here only the main points of the derivation. The basic problem has the form

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) \psi = \epsilon N(\psi), \tag{13}$$

where L is a linear differential operator with constant coefficients, N is a nonlinear operator, and $\epsilon \ll 1$. The small parameter, ϵ, might be exhibited explicitly in the equation or it might be implied by the initial data, as in (4).

The zeroth approximation of (13) is

$$L\psi_0 = 0, \tag{14}$$

a solution of which is

$$\psi_0 = Ae^{i(kx-\omega t)} + (*). \tag{15}$$

Here $(*)$ denotes complex conjugate, and

$$\omega = \omega(k) \tag{16}$$

is the dispersion relation corresponding to the operator L. We require that ω be real for real k (nondissipative), and that $d^2\omega/dk^2 \neq 0$ (dispersive). The general solution of (14) is obtained by integrating over all solutions of the form (15). However, since the problem is dispersive, usual group velocity arguments show that for large times the wave number (k_0) will be the dominant wavenumber in the region given by

$$x \sim \frac{d\omega}{dk}(k_0)t, \tag{17}$$

where $d\omega/dk$ is the group velocity of that wavenumber. Thus, for each wavenumber, k_0, this problem will have three time scales:

(i) $t_i = 1/\omega(k_0)$, the period of a wave;

(ii) $t_{ii} = L/(d\omega/dk(k_0) - d\omega/dk(k_1))$, the time required for the stationary phase points of two different wave numbers, k_0 and k_1, to separate by a distance L; and

(iii) t_{iii}, the time required for the small nonlinear effects to produce a significant cumulative effect.

The nonlinear Schrödinger equation ordinarily is derived by assuming

$t_i \ll t_{ii} \ll t_{iii}$ and asking how a slowly-varying plane wave evolves in time. Thus, the analysis will incorporate the assumption that we have focused on a single wavenumber, k_0, that is well-separated in space from other wavenumbers. We introduce the multiple scales

$$x = x,\ x_1 = \epsilon x, \qquad t = t,\ t_1 = \epsilon t,\ \tau = \epsilon^2 t, \tag{18}$$

and an expansion of the solution of (13):

$$\psi(x, t; \epsilon) = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots \tag{19}$$

where

$$\psi_0 = A(x_1, t_1, \tau)e^{i(k_0 x - \omega_0 t)} + (*). \tag{15a}$$

Notice that use of the variables x and x_1 effectively restricts the range of wavenumbers present to

$$|k - k_0| = O(\epsilon k_0). \tag{20}$$

Eliminating secular terms at $O(\epsilon)$ yields

$$A = A(\xi, \tau), \tag{21}$$

where the variable

$$\xi = x_1 - \frac{d\omega}{dk}(k_0)t_1 \tag{17a}$$

represents simply the effect of linear group velocity. Nonlinear effects first arise at $O(\epsilon^2)$, where the elimination of secular terms yields the nonlinear Schrödinger equation:

$$iA_\tau + \frac{1}{2}\frac{d^2\omega}{dk^2}(k_0)A_{\xi\xi} + C|A|^2 A = 0, \tag{2a}$$

where C is a constant depending on $N(\psi)$ in (13).

Thus, the nonlinear Schrödinger model, like the *KdV* model, is not simply the differential equation (2) but an entire perturbation procedure for a certain class of problems. In applying the model to a physical problem, the entire procedure must be utilized in order to make the results meaningful. Failure to include any of the restrictions can lead to erroneous conclusions.

We close this section with an example of the type of difficulties that one encounters by separating the equation (2) from the perturbation procedure that leads up to it. Hasimoto and Ono [5] derived (2) as a model of deep water waves, and Yuen and Lake [15] verified the model experimentally. One of their experiments shows the interaction of two wave packets (which appear to be solitons) traveling in opposite directions. The interaction is observed to be very weak. This observation, however, neither supports nor denies the hypothesis that the wave packets are solitons. These two waves interact on such a short time scale that no significant nonlinear effects can occur; the observed interaction is simply linear. The observed interaction of two wave packets can test the model only if the wavenumbers of the two packets, k_0 and k, satisfy (20). Otherwise, there is no need for a nonlinear theory, because the time required for the interaction is too short.

4. **Solitons and oscillations.** The outstanding feature of the solution of either (1) or (2) on $-\infty < x < \infty$ is that the initial data (which must be smooth and vanish rapidly as $|x| \to \infty$) evolve into a definite combination of permanent waves and decaying oscillations. The amplitude of the oscillation decays algebraically in time, so that after a sufficiently long time only the permanent waves remain. Consequently, and justifiably, most workers have concentrated on these permanent waves. However, the results of a physical or numerical experiment contain some combination of permanent waves and oscillations, and one may not have the option of concentrating on the permanent waves. Therefore, the purpose of this section is to emphasize the differences between solitons and oscillations which would permit identification of each in a wave record or computer print-out. As above, in order to keep the discussion simple, we restrict our attention to the two cases, (1) and (2), although the same question would arise in any problem in which one attempts to identify solitons experimentally.

In the case of the *KdV* model, (1), the question is relatively simple. The only permanent waves that can arise are solitons, each of which is a positive wave traveling with positive velocity. (In the light of the derivation of (1) given above, this statement should be interpreted to mean "positive velocity, relative to the linearized wave velocity, c.") The oscillations, on the other hand, take on both positive and negative values and travel with a negative group velocity (Segur, [11]). For any initial data, therefore, the solitons can eventually be identified purely by location: $x > 0$, with the oscillations located at $x < 0$.

A second test is that the solitons must be entirely positive, and a third test is actually to compare the shape of the observed wave with that required:

$$u(x, t) = 3c\,\operatorname{sech}^2\{(c/4)^{1/2}(x - ct + x_0)\}.$$

All of these tests require that one wait until the solitons and the oscillations have separated in space. At earlier times, it is much more difficult to make this identification (cf., Hammack and Segur [3]).

For the nonlinear Schrödinger model, (2), the problem is more difficult, even if all the permanent waves are solitons. The solution for an isolated soliton is

$$u_s(x, t) = 2\eta \operatorname{sech}\{2\eta(x + 4\xi t)\}$$
$$* \exp\{-2i(\xi x + 2(\xi^2 - \eta^2)t)\}.$$

Thus, the soliton is an envelope of oscillations, with amplitude, 2η, and speed, -4ξ. To a good approximation, one can replace $\xi \sim x/4t$ in the phase, because the envelope vanishes elsewhere. Thus, (23) becomes approximately

$$u_s(x, t) \sim 2\eta \operatorname{sech}\{2\eta(x + 4\xi t)\} \tag{24}$$
$$* \exp\left\{\frac{it}{4}\left(\left(\frac{x}{t}\right)^2 + (4\eta)^2\right)\right\}.$$

By way of contrast, in the absence of any permanent waves, the decaying oscillations take the form

$$u_0(x, t) = t^{-1/2}R(x/t, t) \tag{25}$$
$$* \exp\left\{\frac{it}{4}\left(\left(\frac{x}{t}\right)^2 + O\left(\frac{\ln t}{t}\right)\right)\right\},$$

where

$$R^2\left(\frac{x}{t}, t\right) = -\frac{1}{4\pi} \ln\left[1 - \left|\frac{b}{a}\left(-\frac{x}{4t}\right)\right|^2\right] + O\left(\frac{\ln t}{t}\right),$$

and $b/a(k)$ depends on the initial data (Segur and Ablowitz, [12]). One can show that $b/a(k) \to 0$ as $|k| \to \infty$; as a special case, let us assume that

$$\frac{b}{a}(k) = 0 \qquad |k| > K$$

$$\frac{b}{a}(k) \neq 0 \qquad |k| < K$$

At any particular time, t_0, it would be almost impossible to tell the soliton solution, (24), from the oscillations, (25). Each consists of a single packet of waves, oscillating at nearly the same frequency. The envelope in (24) has a definite shape, but nothing prevents the packet in (25) from having the same shape. Thus, without comparing two or more wave records, at different times, one cannot conclusively identify a soliton in this problem.

By comparing wave records at two or more times, one obtains two complementary methods to differentiate (24) from (25):

(i) The amplitude of the oscillations decreases in time whereas the amplitude of the soliton does not;

(ii) The physical region of support of the oscillations, $|x| < 4Kt$, increases linearly in time, whereas the region of support of the soliton is constant.

In a physical problem with slight dissipation and which is only approximately modelled by (2), even these two tests may fail (e.g., Yuen and Lake [15]).

The main point of this section is that, even in cases where the exact solution is known, it may be difficult to identify solitons experimentally. When one seeks to determine experimentally whether a given physical system does or does not have solitons, without knowing an appropriate analytical model, the problem may well be impossible. In such a case, it would seem that a blend of theory and experiment is essential for further progress.

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